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Information on lecture and tutorials

Here a few infos on the modalities of the course ***"Quantum Field Theory"***:

- The `LV-Nr` of this course is 045570002.
- This course can be used for the following modules:
  - **107280**: Schwerpunktmodul MSc. Physik (12 ECTS)
  - **68030**: Ergänzungsmodul MSc. Physik (9 ECTS)
  - **68030**: Semicompulsory module MSc. Physics (9 ECTS)
- If you want to use the course as "Schwerpunktmodul", you have to attend the block course *Quantum Field Theory - Advanced Topics* (`LV-Nr 045575002`) which takes place during the first week after the lecture period and comprises five additional lectures on the Standard Model.
- You can find detailed information on lecture and tutorials on the website of our institute:

  `https://itp3.info/qft`
- You can find video recordings and blackboard images on `ILIAS`:

  `https://ilias3.uni-stuttgart.de/goto_Uni_Stuttgart_crs_3216936.html`
- The lecture notes can be downloaded here:

  `https://download.itp3.info/scripts/QFT.pdf`
- **Written** problems have to be handed in and will be corrected by the tutors. You must earn at least **80 %** of the written points to be admitted to the exam.
- **Oral** problems have to be prepared for the exercise session and will be presented by a student at the blackboard. You must earn at least **66 %** of the oral points to be admitted to the exam.
- Every student is required to **present** at least **2** of the oral problems at the blackboard to be admitted to the exam.
- Problems marked with an asterisk ($*$) are optional and can earn you bonus points.
- If you have questions regarding the problem sets, feel free to contact your tutor at any time.

Problem 1.1: Functional derivatives

[**Written** | 2 pt(s)]

ID: ex_functional_derivatives:qft23

Learning objective

In this problem, you should familiarize yourself with functional derivatives which are an important tool in (quantum) field theory, for example in the context of calculating correlation functions of operators.

For a given compact manifold $\mathcal{M}$ of smooth functions ϕ, which are vanishing at the boundary, and a functional F with $F: \mathcal{M} \to \mathbb{R}$ or $\mathbb{C}$, the functional derivative $\frac{\delta F[\phi]}{\delta\phi}$ is defined as

$$\int \mathrm{d}x' \, \frac{\delta F[\phi]}{\delta\phi(x')} f(x') = \lim_{\varepsilon\to 0} \frac{F[\phi(x) + \varepsilon f(x)] - F[\phi(x)]}{\varepsilon} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F[\phi + \varepsilon f]\bigg|_{\varepsilon=0} \tag{1}$$

for all test functions $f \in \mathcal{M}$.

a) Show that for two functionals F and G **1**pt(s)

$$\frac{\delta(F+\lambda G)[\phi]}{\delta\phi(x)} = \frac{\delta F[\phi]}{\delta\phi(x)} + \lambda\frac{\delta G[\phi]}{\delta\phi(x)} \quad \text{for } \lambda \in \mathbb{R} \tag{2}$$

$$\frac{\delta(FG)[\phi]}{\delta\phi(x)} = \frac{\delta F[\phi]}{\delta\phi(x)} G[\phi] + F[\phi]\frac{\delta G[\phi]}{\delta\phi(x)}, \text{ with } (FG)[\phi] = F[\phi]G[\phi]. \tag{3}$$

If $G[\phi]$ is a function-like functional, i.e. can be treated as a function itself

$$\frac{\delta F[G[\phi]]}{\delta\phi(y)} = \int \mathrm{d}x \, \frac{\delta F[G]}{\delta G(x)} \frac{\delta G[\phi](x)}{\delta\phi(y)}. \tag{4}$$

b) Calculate the functional derivative $\frac{\delta F[\phi]}{\delta\phi(x)}$ for the following functionals: **1**pt(s)

$$F[\phi] = \int \mathrm{d}x' \, K(y,x')\phi(x'), \quad \text{where } K(y,x') \text{ is a so-called } \textit{integral kernel} \tag{5}$$

$$F[\phi] = \phi(y) \tag{6}$$

$$F[\phi] = \phi'(y) \tag{7}$$

$$F[\phi] = \int \mathrm{d}y \, f(\phi(y), \phi'(y)) \;\; \text{for a differentiable function } f: \mathbb{R}\times\mathbb{R} \to \mathbb{R} \tag{8}$$

Problem 1.2: Lorentz covariance

[**Written** | 5 pt(s)]

ID: ex_lorentz_covariance:qft23

Learning objective

This exercise serves as a **brief** revision of tensor calculus and the covariant formulation of classical electromagnetism.

In the following, we will work in units where $c = 1$. Further, we will make use of *Einstein notation* where summation over indices appearing twice is assumed.

We first introduce the four-vector

$$x^\mu = (t, \mathbf{r}), \quad \mu = 0, 1, 2, 3, \tag{9}$$

which we will call a **contravariant** vector or tensor of first order. The vector x_μ is called **covariant** vector. In special relativity, the metric tensor is given by

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{10}$$

and the relationship between co- and contravariant vectors is given by

$$x_\mu = g_{\mu\nu} x^\nu. \tag{11}$$

A **Lorentz vector** is an object that under a Lorentz transformation $\Lambda^\mu{}_\nu$ transforms like

$$\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu. \tag{12}$$

In tensors of higher order, each index transforms as a Lorentz vector, e.g.

$$\tilde{A}^{\alpha\beta\gamma}{}_{\delta\varepsilon} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \Lambda^\gamma{}_\xi \Lambda_\delta{}^\rho \Lambda_\varepsilon{}^\sigma A^{\mu\nu\xi}{}_{\rho\sigma}. \tag{13}$$

A **Lorentz scalar** is a quantity that is invariant under Lorentz transformations.

a) Show that $x^\mu x_\mu$ is a Lorentz scalar, i.e. show that $x^\mu x_\mu = \tilde{x}^\sigma \tilde{x}_\sigma$. **1**pt(s)

Another important object is the four-gradient

$$\frac{\partial}{\partial x^\mu} = \partial_\mu = (\partial_t, \nabla). \tag{14}$$

b) Compute the d'Alembert operator $\partial^\mu \partial_\mu$. Is this quantity a Lorentz scalar? **1**pt(s)

In a covariant formulation of electromagnetism, the electric and magnetic field **E** and **B**, respectively, are given by the antisymmetric field tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \qquad F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{15}$$

The fields can also be described by the four-potential $A_\mu = (\Phi, -\mathbf{A})$, where Φ is a scalar potential and $\mathbf{A}$ is a vector potential.

c) Show that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ reproduces the fields **E** and **B**. **1**pt(s)

d) Show that $F_{\mu\nu}$ is invariant under the gauge transformation $A_\mu \to A_\mu - \partial_\mu f$, where f is an arbitrary function. **1**pt(s)

e) Show that in Lorenz gauge, $\partial_\nu A^\nu = 0$, and for no external sources, the Maxwell equations $\partial_\mu F^{\mu\nu} = 0$ reduce to $\partial^\mu \partial_\mu A^\nu = 0$. **1**pt(s)

Problem 1.3: Maxwell equations

[**Oral** | 4 pt(s)]

ID: ex_maxwell_equations:qft23

Learning objective

In this problem, you use the methods from Problem 1.1 and Problem 1.2 to derive the famous Maxwell equations from a covariant formulation of classical electrodynamics. You calculate the energy-momentum tensor of the electromagnetic field which contains information about the flow of energy and momentum.

The Maxwell equations for classical electromagnetism in vacuum can be derived from the action

$$S = \int \mathrm{d}^4 x \mathcal{L} = \int \mathrm{d}^4 x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \tag{16}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $c = 1$.

a) Derive the Maxwell equations from the action (16). Use the Euler-Lagrange equations and treat the components $A_\mu(x)$ as the dynamic variables. Write the two "inhomogeneous" Maxwell equations in their standard form by using $F_{0i} = -F^{0i} = E_i$ and $F_{ij} = F^{ij} = -\varepsilon_{ijk} B_k$, $i = x, y, z$. **1** pt(s)

   What happens with the two homogeneous equations?

   Note: The electric and magnetic field are not four-vectors! Therefore, there are neither co- nor contravariant and their index is really just a label, which for consistency we will always write as lower index.

b) Show, that the energy-momentum tensor $T^{\mu\nu}$ for the electromagnetic field is not symmetric **1** pt(s)

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda - g^{\mu\nu} \mathcal{L}. \tag{17}$$

c) However, we can make the tensor symmetric by adding a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. **1** pt(s)

   Calculate the symmetric energy-momentum tensor

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \tag{18}$$

   with

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu , \tag{19}$$

   and show that it is indeed symmetric.

d) Show that the symmetrized tensor yields the standard form of the electromagnetic energy density and the momentum density (Poynting vector) **1** pt(s)

$$\mathcal{E} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \quad \text{and} \quad \mathbf{S} = \mathbf{E} \times \mathbf{B}. \tag{20}$$