Recap:

4. Interacting fields and Feynman diagrams

4.1. Preliminaries

ϕ^4-Theory

dimensionless coupling constant

$$\boxed{\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \underbrace{\frac{\lambda}{4!}\phi^4}_{\text{Interaction}}}$$

Yukawa Theory:

$$\boxed{\mathcal{L}_{\text{Yukawa}} = \underbrace{\bar{\Psi}(i\not{\partial} - m)\Psi}_{\text{Dirac}} + \underbrace{\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2}_{\text{KG}} - \underbrace{g\bar{\Psi}\Psi\phi}_{\text{Yukawa coupling}}}$$

Quantum Electrodynamics (QED)

$$\mathcal{L}_{\text{QED}} = \bar{\Psi}(i\not{\partial} - m)\Psi - \frac{1}{4}(F_{\mu\nu})^2 - e\underbrace{\bar{\Psi}\gamma^\mu\Psi}_{j^\mu} A_\mu$$

$$= \bar{\Psi}(i\not{D} - m)\Psi - \frac{1}{4}(F_{\mu\nu})^2$$

covariant derivative $\rightarrow$ minimal coupling

$$D_\mu = \partial_\mu + ieA_\mu$$

- No known exactly solvable QFTs in $D > 1+1$
- Assume that coupling constants are small

$\Downarrow$

$$\boxed{\text{Perturbation Theory}}$$

4.2 Perturbation Expansion of Correlation Functions

1| **Goal**: Green's function

$$\langle\Omega| T\, \phi(x)\phi(y)|\Omega\rangle$$

$|\Omega\rangle$: Ground state of interacting theory

$|0\rangle$: — // — free

2| **Remember**: $\underline{\lambda = 0}$

$$\langle 0| T\, \phi(x)\phi(y)|0\rangle = D_F(x-y)$$

3| **Now**:

$$H = H_0 + \int d^3x \, \frac{\lambda}{4!}\phi^4(\vec{x})$$

4| **Todo**:

Express $\left\{\begin{matrix}\phi(x) \\ |\Omega\rangle\end{matrix}\right\}$ in terms of $\left\{\begin{matrix}\text{free field } \phi_I(x) \\ \text{free vacuum } |0\rangle\end{matrix}\right\}$

5| ✗ Reference time t_0

$$\phi(t_0,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_{\vec{p}}\, e^{i\vec{p}\vec{x}} + a^\dagger_{\vec{p}}\, e^{-i\vec{p}\vec{x}}\right)$$

6| **Def**

$$\phi(t,x) := e^{iH(t-t_0)}\phi(t_0,\vec{x})\, e^{-iH(t-t_0)}$$

Heisenberg field / picture

$$\phi_I(t,x) := e^{iH_0(t-t_0)}\phi(t_0,\vec{x})\, e^{-iH_0(t-t_0)}$$

$$\Rightarrow \phi_I(t,x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p\, e^{\boxed{-ipx}} + a^\dagger_p\, e^{\boxed{ipx}}\right)$$

Interaction picture field

$$\phi(t,\vec{x}) = U^\dagger(t,t_0)\,\phi_I(t,\vec{x})\,U(t,t_0)$$

$$U(t,t_0) = e^{i\underline{H_0}(t-t_0)}\, e^{-i\underline{H}(t-t_0)}$$

7] $U(t,t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$

- $U(t_0,t_0) = \mathbb{1}$

- $\boxed{i\partial_t U(t,t_0) \overset{!}{=} H_I(t)\, U(t,t_0)}$

- $H_I(t) = e^{iH_0(t-t_0)} H_{int}\, e^{-iH_0(t-t_0)}$

$$= \int d^3x\, \frac{\lambda}{4!}\, \phi_I^4(t,\vec{x})$$

$$\phi(t,\vec{x}) = U^\dagger(t,t_0)\, \phi_I(t,\vec{x})\, U(t,t_0) \quad (3)$$

8] Dyson series

$$\boxed{U(t,t_0) = \left\{ \mathbb{1} + (-i)\int^t dt_1\, H_I(t_1) + \frac{(-i)^2}{2!}\int_0^{t_0} dt_1 \int_0^t dt_2\, \mathcal{T}\{H_I(t_1)H_I(t_2)\} + \dots \right. = \mathcal{T}\exp\left[-i\int_{t_0}^t ds\, H_I(s)\right]}$$

9] $t \geq t'$

- $U(t,t') = e^{iH_0(t-t_0)}\, e^{-iH(t-t')}\, e^{-iH_0(t'-t_0)}$
- $U^{-1}(t,t') = U^\dagger(t,t')$
- $t_1 > t_2 > t_3$: $U(t_1,t_2)\,U(t_2,t_3) = U(t_1,t_3)$

10] Ground state $|\Omega\rangle$:

$\lambda \ll 1 \longrightarrow \langle\Omega|0\rangle \neq 0$ Eigenstates of H

$$e^{-iHT}|0\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle$$
$$= e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle$$

Now:
$$\frac{e^{iE_0T}\, e^{-iE_nT}}{e^{i\overbrace{(E_0-E_n)}^{<0}T}} \left\{ -\Re \right. \quad T \to -i\varepsilon$$

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1} e^{-iH(T+t_0)}|0\rangle$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1} U(t_0,-T)|0\rangle \quad (2)$$

Analog:

$$\langle\Omega| = \lim_{T\to\infty(1-i\varepsilon)} \langle 0|U(T,t_0)\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1} \quad (1)$$

1) Green's function. $x^0 > y^0 > t_0$

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = (1)\times \underset{x}{(3)} \times \underset{y}{(3)} \times (2)$$

$$= \lim_{T\to\infty(1-i\varepsilon)} N_T^{-1} \underbrace{\langle 0|U(T,t_0)U^\dagger(x_0,t_0)}_{(1)\; U(T,x^0)}\phi_I(x_0,\vec{x})\underbrace{U(x_0,t_0)U^\dagger(y_0,t_0)}_{(3)\; U(x^0,y^0)}\phi_I(y_0,\vec{y})\underbrace{U(y_0,t_0)U(t_0,-T)}_{(2)\; U(y^0,-T)}|0\rangle$$

$$N_T = \langle 0|U(T,t_0)U(t_0,-T)|0\rangle = \langle 0|U(T,-T)|0\rangle$$

$x^0 \gtrless y^0$ arbitrary

$$\boxed{\langle\Omega|T\,\phi(x)\phi(y)|\Omega\rangle = \frac{\langle 0|T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T}dt\, H_I(t)\right]\right\}|0\rangle}{\langle 0|T\left\{\exp\left[-i\int_{-T}^{T}dt\, H_I(t)\right]\right\}|0\rangle}}$$

4.3. Wick's Theorem

$$\langle \Omega | T\, \phi(x)\phi(y) | \Omega \rangle$$
$$= \sum \langle 0 | T\{\phi_I(x_1)\phi_I(x_2) \ldots\ldots\} | 0 \rangle$$

1) Def.

$$\phi_I(x) = \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}}\, a_p e^{-ipx}}_{\phi_I^+(x)} + \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}}\, a_p^\dagger e^{+ipx}}_{\phi_I^-(x)}$$

$$\phi_I^+ |0\rangle = 0$$
$$\langle 0 | \phi_I^- = 0$$

2) ϕ. $x^0 > y^0$, $y^0 > x^0$ $\qquad \phi_I \to \phi$

$$T\{\phi(x)\phi(y)\} = \phi^+(x)\phi^+(y) + \underline{\phi^+(x)\phi^-(y)} + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y)$$
$$= \phi^+(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) + [\phi^+(x), \phi^-(y)]$$

3) Definitions:

$$\underbrace{\overline{\phi_I(x)\phi_I(y)}}_{\text{Contraction}} := \begin{cases} [\phi^+(x), \phi^-(y)] & x^0 > y^0 \\ [\phi^+(y), \phi^-(x)] & y^0 > x^0 \end{cases} \overset{?}{=} D_F(x-y)$$

Normal order (linear meta-operator)

$$: a_1^{(\dagger)} \ldots a_n^{(\dagger)} : = (\text{creation}) \times (\text{annihilation})$$

Example: $:\phi^+(x)\phi^-(y): = \phi^-(y)\phi^+(x)$

$$\Downarrow$$

$$T\{\phi_I(x)\phi_I(y)\} = :\phi(x)\phi(y) + \overline{\phi(x)\phi(y)}:$$

$$\Downarrow$$

$$\langle 0 | T\{\phi_I(x)\phi_I(y)\} | 0 \rangle = D_F(x-y)$$

4] Wick's theorem:

$$T\{\phi_1(x_1)\,\phi(x_2)\ldots\phi(x_n)\} = :\phi(x_1)\ldots\phi(x_n) + \text{all possible contractions}:$$

where $:A\,\phi_i\,B\,\phi_j\,C: \;\equiv\; D_F(x_i - x_j)\;:ABC:$

5] Corollary:

$$\langle 0|\,T\{\phi_1\ldots\phi_n\}\,|0\rangle = \text{all possible full contractions}$$

$Q = \int : \;\;-\;\; :$ $\quad H = \int d^3x : \;\_\; \phi^2 :$

6] Example:

$$T\{\phi_1\phi_2\phi_3\phi_4\} = :\phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi\phi\phi\phi + \phi\phi\phi\phi$$
$$+\ \phi\phi\phi\phi + \phi\phi\phi\phi + \phi\phi\phi\phi$$
$$+\ \phi\phi\phi\phi + \phi\phi\phi\phi + \phi\phi\phi\phi :$$

$\phi(x_1)$

$$\langle 0|\,T\{\phi_1\phi_2\phi_3\phi_4\}\,|0\rangle = D_F(x_1-x_2)\,D_F(x_3-x_4) + D_F(x_1-x_3)\,D_F(x_2-x_4)$$
$$+\ D_F(x_1-x_4)\,D_F(x_2-x_3)$$

$$\cancel{a a^\dagger} = :a a^\dagger:$$
$$\times\ \overset{(CR)}{=} :a^\dagger a + 1:$$
$$\overset{(Lin)}{=} :a^\dagger a: + :1:$$
$$= \cancel{a^\dagger a} + 1$$

$=$