Recap

3.4 Quantization of the Dirac Field

- Continuous symmetries & Conserved charges
- Excitations = Particles (Fermions + Anti)
- Unitary rep. of Lorentz transformations $U(\Lambda)$

  ↑ proper orthochronous $SO^+(1,3)$

$$\boxed{U(\Lambda)\,\psi(x)\,U^{-1}(\Lambda) = \Lambda_{\frac{1}{2}}^{-1}\,\psi(\Lambda x)}$$

rep on Hilbert space — bispinor rep. — 4-vector rep.

- Spin-statistics theorem
- Feynman propagator of the Dirac field

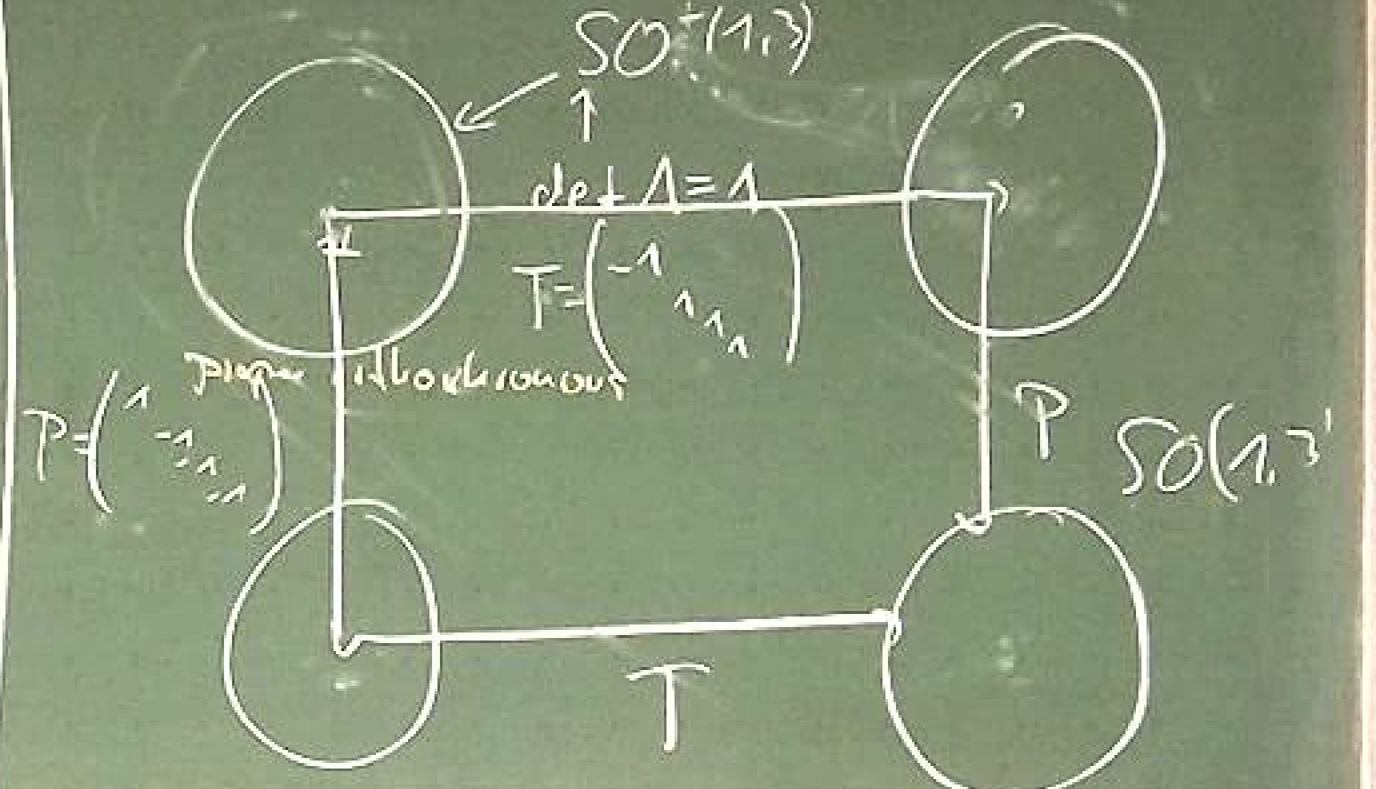
$$\boxed{S_F^{ab}(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\frac{i(\not{p}+m)_{ab}}{p^2-m^2+i\epsilon}\,e^{-ip(x-y)} = \langle 0|\,T\,\psi_a(x)\,\bar{\psi}_b(y)\,|0\rangle}$$

fermionic time ordering

- Causality

3.5 Discrete Symmetries of the Dirac theory

Lorentz group: $O(1,3)$



Parity: $P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ $L = \vec{r} \times \vec{p}$

1) Unitary rep. on Fock space:

$U(P)\, a^s_{\vec{p}}\, U^{-1}(P) = \eta_a^{+1} \cdot a^s_{-\vec{p}}$

$U(P)\, b^s_{\vec{p}}\, U^{-1}(P) = \underbrace{\eta_b}_{-1} b^s_{-\vec{p}}$

$\gamma^0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

2) $\overset{\circ}{\rightarrow}$

$$\boxed{\begin{aligned} &U(P)\, \psi(t,\vec{x})\, U^{-1}(P) \\ &= \underbrace{\gamma^0}_{P_{\frac{1}{2}}} \psi\underbrace{(t,-\vec{x})}_{P_x} \end{aligned}}$$

3) Example

$\cdot\, U(P)\, \bar{\psi}\psi\, U^{-1}(P) \overset{\circ}{=} \bar{\psi}\psi(t,-\vec{x})$

$\cdot\, U(P)\, \bar{\psi}\gamma^5\psi\, U^{-1}(P) = -\bar{\psi}\gamma^5\psi(t,-\vec{x})$

Time Reversal:

1) Time reversal should ...

- $U(T)\, \psi(t,\vec{x})\, U^{-1}(T) = T_{\frac{1}{2}}\, \psi(-t,\vec{x})$
- $U(T)\, a^s_{\vec{p}}\, U^{-1}(T) = a^{\tilde{s}}_{-\vec{p}}$ flip spins
- Symmetry: $[U(T), H] = 0$
- $U^{-1}(T) = U^{\dagger}(T)$

2) Problem: $\psi(t,\vec{x}) = e^{iHt}\, \psi(\vec{x})\, e^{-iHt}$

$\Rightarrow U\psi(t,\vec{x})U^{-1} = e^{iHt}\, U\psi(\vec{x})U^{-1}\, e^{-iHt}$

$\Rightarrow T_{\frac{1}{2}}\, \psi(-t,\vec{x})|0\rangle = e^{iHt}\, T_{\frac{1}{2}}\, \psi(\vec{x})\, |0\rangle$

$\Rightarrow T_{\frac{1}{2}}\, e^{-iHt}\, \psi(\vec{x})\, |0\rangle = e^{iHt}\, T_{\frac{1}{2}}\, \psi(\vec{x})|0\rangle$

$\Rightarrow \underbrace{e^{-2iHt}}_{\text{time dependent}}\, T_{\frac{1}{2}}\, \psi(\vec{x})|0\rangle = T_{\frac{1}{2}}\, \psi(\vec{x})\, |0\rangle$

3] Solution:

$U(T)$ must be antilinear / antiunitary

$$\boxed{U(T)\, c = c^* U(T)} \qquad c \in \mathbb{C}$$

$\ulcorner \langle Ux | Uy \rangle = \overline{\langle x | y \rangle} \lrcorner$ $\quad \ulcorner \mathcal{U} = U \circ K \lrcorner$

4] Transformation of Spin.

i] Spinor basis polarized in $\vec{n}$

$$\xi^1 = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi} \sin\frac{\theta}{2} \end{pmatrix}, \quad \xi^2 = \begin{pmatrix} -e^{-i\phi} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

"Time reversed" = flipp spinor

$$\boxed{\overline{\xi^s} := \underline{-i\sigma^2} \left(\xi^s\right)^*}$$

$$\overline{\xi^s} = U \circ K\, \xi^s$$

$$\left\{ \begin{pmatrix} \overline{\xi^1} \\ \overline{\xi^2} \end{pmatrix} \right. = \left\{ \begin{pmatrix} \xi^2 \\ -\xi^1 \end{pmatrix} \right.$$

$$\vec{\sigma} = \begin{pmatrix} \sigma^1 \\ \sigma^2 \\ \sigma^3 \end{pmatrix}$$

$$\vec{\sigma} \xrightarrow{T} -\vec{\sigma}$$

$$T \sigma_i T^{-1} = -\sigma_i$$

$$\sigma_x \sigma_z \sigma_x = -\sigma_z$$

σ_x, σ_y

- $U^{-1}(T) = U^\dagger(T)$

2] Problem:

$$\psi(t,\vec{x}) = e^{iHt} \psi(\vec{x}') e^{-iHt}$$

$$\Rightarrow U \psi(t,\vec{x}) U^{-1} = e^{-iHt} U \psi(\vec{x}') U^{-1} e^{\pm iHt}$$

$$\Rightarrow T_{\frac{1}{2}} \psi(-t,\vec{x}) |0\rangle = e^{-iHt} T_{\frac{1}{2}} \psi(\vec{x}') |0\rangle$$

$$\Rightarrow T_{\frac{1}{2}} e^{-iHt} \psi(\vec{x}') |0\rangle = e^{-iHt} T_{\frac{1}{2}} \psi(\vec{x}') |0\rangle$$

$$\Rightarrow \underbrace{e^{\cancel{-2iHt}}}_{\text{time dependent}} T_{\frac{1}{2}} \psi(\vec{x}') |0\rangle = T_{\frac{1}{2}} \psi(\vec{x}') |0\rangle$$

$\rightarrow$ not possible

ii] Bispinors

$$\overline{u^s(\tilde{p})} = \begin{pmatrix} \overline{\sqrt{p\sigma}}\ \xi^s \\ \overline{\sqrt{p\bar\sigma}}\ \xi^s \end{pmatrix}$$

$$\overline{v^s(\tilde{p})} = \begin{pmatrix} \overline{\sqrt{p\sigma}}\ \overline{\xi^s} \\ -\overline{\sqrt{p\bar\sigma}}\ \overline{\xi^s} \end{pmatrix} \quad -\xi^s$$

iii] Modes.

$$\begin{Bmatrix} \overline{a^1_{\vec p}} \\ b \\ \overline{a^2_{\vec p}} \\ b \end{Bmatrix} := \begin{Bmatrix} a^2_{\vec p} \\ b \\ a^1_{-\vec p} \\ b \end{Bmatrix}$$

iv] $\tilde{p} = (p^0, -\vec{p})$

$$\overline{u^s(\tilde p)} \doteq -\boxed{\gamma^1\gamma^3}\,[u^s(p)]^* \qquad \begin{pmatrix} i\sigma^2 & \\ & i\sigma^2 \end{pmatrix}$$

$$\overline{v^s(\tilde p)} \doteq -\gamma^1\gamma^3\,[v^s(p)]^*$$

5] $U(T)$: antilinear

$$U(T)\,\overset{}{\underset{b}{a}}{}^s_{\vec p}\,U^{-1}(T) := \overline{\underset{b}{a}{}^s_{-\vec p}}$$

$$\Downarrow$$

$$U(T)\,\psi(t,\vec x)\,U^{-1}(t,\vec x) = \gamma^1\gamma^3\cdot\psi(-t,\vec x)$$

T_1

6] Example: $j^\mu = \bar\psi\gamma^\mu\psi$

$$U(T)\,j^\mu(t,x)\,U^{-1}(T) \doteq \begin{cases} +j^0(-t,\vec x) \\ -j^i(-t,\vec x) \end{cases}$$

Charge Conjugation

1] non-spacetime symmetry.

$$U(C)\,\underset{b}{a}{}^s_{\vec p}\,U^{-1}(C) = \underset{a}{b}{}^s_{\vec p}$$

2] $u^s(p) \overset{\circ}{=} -i\gamma^2[v^s(p)]^*$
$\quad\; v \qquad\qquad\quad u$

3] $U(C)\psi(x)U^{-1}(C)$

$\overset{\circ}{=} -i\gamma^2(\psi^\dagger)^T = -i(\bar{\psi}\gamma^0\gamma^2)^T$

$C_{\frac{1}{2}} = \psi^*$

4] $\boxed{U(C)\psi U^{-1}(C) = -i\gamma^2\psi^*}$

5] Example:

$U(C)\bar{\psi}\gamma^\mu\psi U^{-1}(C) \overset{\circ}{=} -\bar{\psi}\gamma^\mu\psi$

Note ??

- Relativistic QFT must be $SO^+(1,3)$
- The classical Dirac equation $(i\not{\partial}-m)\psi$ is invariant under $\{C,P,T\}$
- The quantized Dirac theory is $\{C,P,T\}$ invariant.

$[\underset{\uparrow}{H}, U(X)] = 0 \qquad X = C, P, T$

$\sum_s \int_p (a_p^{s\dagger}a_p^s + b_p^{s\dagger}b_p^s)$

- Weak interaction violates P and C (Preserves $\underline{PC}$) $\Rightarrow$ Wu experiment

$\rightarrow$ EoM

- $(i\not{\partial}-m)\psi = 0$
- $\partial_\mu F^{\mu\nu} = e j^\nu \leftarrow \bar{\psi}\gamma^\mu\psi$

Charge Conjugation

1] non-spacetime symmetry:

$U(C)\, a_{\vec{p}}^s\, U^{-1}(C) = b_{\vec{p}}^s$
$\qquad\;\; b \qquad\qquad\qquad a$

- Rare processes violate CP, T but preserve CPT
- CPT seems to be symmetry of nature
- CPT theorem:

- $SO^+(1,3)$ invariant
- Causality
- Stable vacuum
- Locality

$\Rightarrow$ CPT symmetric

4] Interacting Fields and Feynman Diagrams

4.1. Preliminaries

- $$H_{int} = \int d^3x \, \mathcal{H}_{int}(\phi(x)) = -\int d^3x \, \mathcal{L}_{int}(\phi)$$

- Examples:

1] ϕ^4-theory:

$$\boxed{\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4}$$

coupling constant

$\rightarrow$ EOM: $(\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3$

2] Yukawa theory

$$\boxed{\mathcal{L}_Y = \underbrace{\bar{\Psi}(i\not{\partial} - m)\Psi}_{\text{Dirac}} + \underbrace{\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2}_{\text{KG}} \underbrace{- g\bar{\Psi}\Psi\phi}_{\text{Interaction}}}$$

3. QED:

$$\boxed{\begin{aligned}\mathcal{L}_{QED} &= \bar{\Psi}(i\not{\partial} - m)\Psi - \frac{1}{4}(F_{\mu\nu})^2 \\ &\quad \underbrace{- e\bar{\Psi}\gamma^\mu\Psi A_\mu}_{\text{Interaction}} \\ &= \bar{\Psi}(i\not{D} - m)\Psi - \frac{1}{4}(F_{\mu\nu})^2\end{aligned}}$$

$D_\mu = \partial_\mu + ieA_\mu$: Gauge covariant derivative