Recap:

3. The Dirac Field

3.4. Quantization of the DF

$$\boxed{\bar\psi(\vec x)=\sum_{s=1,2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[a_{\vec p}^{s\dagger}\bar u^s(p)e^{-i\vec p\vec x}+b_{\vec p}^{s}\bar v^s(p)e^{+i\vec p\vec x}\right]}$$

with modes

$$\{a_{\vec p}^r,a_{\vec q}^{s\dagger}\}=(2\pi)^3\delta^{rs}\delta(\vec p-\vec q)\quad (a\to b)$$

$$\{a,b\}=0$$

$$\Leftrightarrow\ \boxed{\{\psi_a(\vec x),\psi_b^\dagger(\vec y)\}=\delta_{ab}\delta^{(3)}(\vec x-\vec y)}$$

Hamiltonian:

$$\boxed{H=\int d^3x\,\mathcal{H}\doteq\sum_s\int\frac{d^3p}{(2\pi)^3}E_p\left[a_{\vec p}^{s\dagger}a_{\vec p}^s+b_{\vec p}^{s\dagger}b_{\vec p}^s\right]}$$

$E_p=\sqrt{\vec p^2+m^2}$ Fermionic Number operator

↳ positive energies, Vacuum = Ground state.

$$H|0\rangle=0$$

Hilbert space: Fermionic Fock space

(Modes: $a_{\vec p}^s,\ b_{\vec p}^s$)

11| Heisenberg Picture:

$$\psi(x)=e^{iHt}\psi(\vec x)e^{-iHt}$$

$(t,\vec x)$

Heisenberg field   Schrödinger field

$$\Rightarrow e^{iHt}a_{\vec p}^s e^{-iHt}\doteq a_{\vec p}^s e^{-iE_pt}$$

$$e^{iHt}b_{\vec p}^s e^{-iHt}\doteq b_{\vec p}^s e^{-iE_pt}$$

$$\boxed{\Rightarrow\ \bar\psi(x)=\sum_s\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[a_p^{s\dagger}\bar u^s(p)e^{+ipx}+b_p^{s}\bar v^s(p)e^{-ipx}\right]}$$

Continuous Symmetries & Conserved Charges

- Time translation $\rightarrow$ H
- Spatial translations
  $\rightarrow$ Momentum operator

$$\boxed{\vec{P} \stackrel{\circ}{=} \int d^3x\, \psi^\dagger(-i\nabla)\psi \stackrel{\circ}{=} \sum_s \int \frac{d^3p}{(2\pi)^3}\, \vec{p}\left[a_p^{s\dagger} a_p^s + b_p^{s\dagger} b_p^s\right]}$$

- Rotations in space
  $\rightarrow$ Angular momentum operator $\vec{J}$

- Global phase rotation: $e^{i\alpha}\psi$

$\stackrel{\circ}{\rightarrow}$ Conserved current $j^\mu = \bar{\psi}\gamma^\mu\psi$

$\rightarrow$ Conserved charge

$$\boxed{\begin{aligned} Q = \int d^3x\, \psi^\dagger\psi &\stackrel{\circ}{=} \sum_s \int \frac{d^3p}{(2\pi)^3}\left[a_p^{s\dagger} a_p^s + b_{-p}^s b_{-p}^{s\dagger}\right] \\ &= \sum_s \int \frac{d^3p}{(2\pi)^3}\left[a_p^{s\dagger} a_p^s - b_p^{s\dagger} b_p^s\right] + \cancel{(\infty)} \end{aligned}}$$

Excitations = Particles

$a_{\vec{p}}^{s\dagger}|0\rangle$: Fermion with
- energy $E_{\vec{p}}$
- momentum $\vec{p}$
- spin $J = \frac{1}{2}$ (polarization s)
- charge $Q = +1$

$\uparrow$ QED: Electrons

$b_{\vec{p}}^{s\dagger}|0\rangle$: Antifermion with
- energy $E_{\vec{p}}$
- momentum $\vec{p}$
- spin $J = \frac{1}{2}$ (polarization opposite to s)
- Charge $Q = -1$

$\uparrow$ QED: Positrons

Lorentz transformations

1] $\measuredangle$ LT $\Lambda \in SO^+(1,3)$
on single particle state.

$$|\vec{p}, s\rangle_a := \sqrt{2E_{\vec p}}\, a^{s+}_{\vec p} |0\rangle .$$

$$|\vec{p}, s\rangle \longmapsto U(\Lambda)\, |\vec{p}, s\rangle$$

$U(\Lambda)$: representation of $SO^+(1,3)$
on Fock space.

2] $\measuredangle$ Special case: quant. axis $\parallel$ boost/rotation axis

$\rightarrow$ Spin pol. do not mix

$$U(\Lambda)\, a^s_{\vec p}\, U(\Lambda)^{-1} = \sqrt{\frac{E_{\Lambda\vec p}}{E_{\vec p}}}\, a^s_{\Lambda\vec p} \quad \text{LI}$$

3]

$$\langle \vec{p}, s | \vec{q}, r\rangle = 2E_{\vec p}\, (2\pi)^3 \delta^{(3)}(\vec p - \vec q)\, \delta^{rs}$$

$$= \langle \vec p, s | \underbrace{U^+(\Lambda)\, U(\Lambda)}_{\mathbb{1}} | \vec q, r\rangle$$

$\Rightarrow$ <u>$U(\Lambda)$ unitary</u>

4] <u>3 representations:</u>

Λ acts on 4-vectors in $\mathbb{R}^{1,3}$, $D=4$, <u>not</u> unitary

$\Lambda_{\frac{1}{2}}$ acts on bispinors $\mathbb{C}^2 \oplus \mathbb{C}^2$, $D=4$, <u>not</u> unitary

$U(\Lambda)$ acts on ferm. Fock space, $D=\infty$, <u>unitary</u>

5] $\xrightarrow{\circ}$

$$\boxed{U(\Lambda)\, \psi(x)\, U^{-1}(\Lambda) = \Lambda^{-1}_{\frac{1}{2}}\, \psi(\Lambda x)}$$

Spin Statistics Theorem.

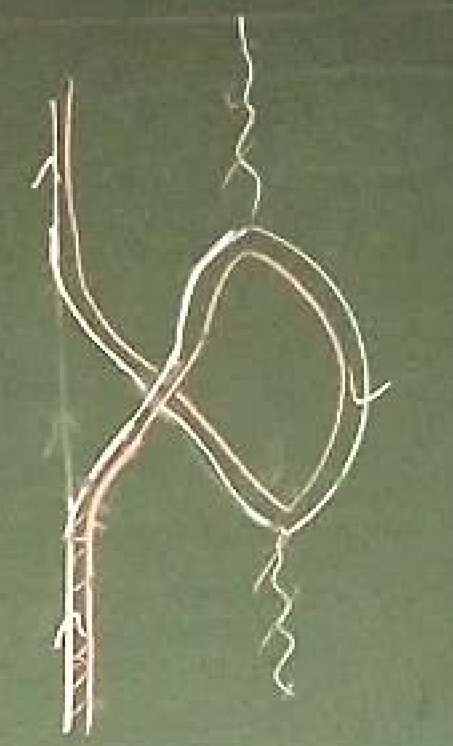
- Observation:

* KG ϕ: Spin 0 → commutator
  → bosonic ex.
* D ψ: Spin $\frac{1}{2}$ → anticommutators
  → fermionic ex.

- Spin-Statistics Theorem:

$$\boxed{\left.\begin{array}{l}\text{Lorentz inv.}\\ \text{Causality}\\ \text{Positive energies}\\ \text{Positive norms}\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\text{Integer Spin} \leftrightarrow \text{Boson}\\ \text{Half integer Spin} \leftrightarrow \text{Fermion}\end{array}\right.}$$

- Proof by picture:



Statistics

Spin

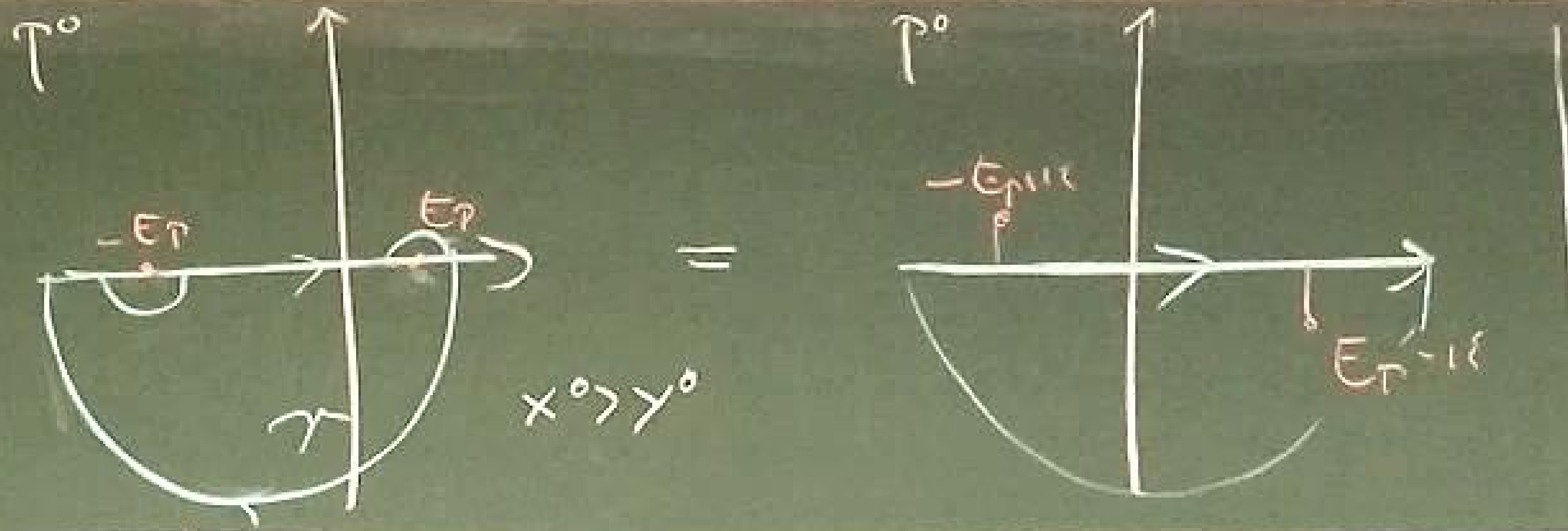
Dirac Propagator:

1) $\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle$ $\quad p^0=\sqrt{\vec{p}^2+m^2}$

$$= \underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\, e^{-ip(x-y)} \underbrace{\sum_s u_a^s \bar{u}_b^s(p)}_{(\not{p}+m)_{ab}}}_{(i\not{\partial}_x+m)\,D(x-y)}$$

$x^0>y^0$

$$= \int \frac{d^4p}{(2\pi)^3}\,\frac{i(\not{p}+m)_{ab}}{p^2-m^2+i\epsilon}\, e^{-ip(x-y)}$$



$$\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle = \underbrace{\int \cdots \underbrace{\sum_s v_a^s(p)\bar{v}_b^s(p)}_{(\not{p}-m)_{ab}}}_{-i(\not{\partial}_x+m)\,D(y-x)}$$

$x^0<y^0$

$$= -\int \frac{d^4p}{(2\pi)^4}\,\frac{i(\not{p}+m)_{ab}}{p^2-m^2+i\epsilon}\, e^{-ip(x-y)}$$

2) Feynman propagator:

$$S_F^{ab}(x-y)=\int\frac{d^4p}{(2\pi)^4}\,\frac{i(\not{p}+m)_{ab}}{p^2-m^2+i\epsilon}\, e^{-ip(x-y)}$$

$$=\begin{cases}\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle & x^0>y^0\\ -\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle & x^0<y^0\end{cases}$$

$$\equiv \langle 0|T\,\psi_a(x)\bar{\psi}_b(y)|0\rangle$$

$\uparrow$

Fermionic time ordering

$t_1>t_2$: $\quad T\,\psi(t_2)\psi(t_1) = -\,\psi(t_1)\psi(t_2)$

Causality

1) Measurable operators.

$$\hat{O}(x) = \sum_i \prod_{i=1}^{\text{even } N} \left(\psi_i^{(+)} \vee \partial\psi^{(+)} \vee \partial^2\psi^{(+)} \right)$$

Example: • $j^\mu = \bar{\psi}\gamma^\mu\psi$

($\text{check: } j^{\mu\dagger} = j^\mu$)

Counterex. $\psi_a + \psi_a^\dagger$ ✗

2) Causality: $\Leftrightarrow$

$$\{\psi_\alpha(x), \bar{\psi}_\beta(y)\} = 0 \quad \forall \ (x-y)^2 < 0$$

We find.

$$\{\psi_\alpha(x), \bar{\psi}_\beta(y)\} \overset{\circ}{=} (i\partial_x + m)_{\alpha\beta}\left[D(x-y) - D(y-x)\right]$$

$$\overset{(x-y)^2<0}{=} (i\partial_x + m)_{\alpha\beta}\left[\underbrace{D(x-y) - D(x-y)}_{=0}\right]$$

$$= \underline{\underline{0}}$$

$a^\dagger a$

$a^\dagger_{i+1} \ a_i$

$a_i^\dagger \ a^\dagger_{i+1} \ a_i \ a_{i+1}$

$a^\dagger$