Recap:

3. The Dirac Field

3.1 The Dirac Equation

$$\boxed{(i\gamma^\mu \partial_\mu - m)\Psi = 0}$$

Dirac equation

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

γ-matrices in Weyl rep.

$$\begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \\ \Psi_3(x) \\ \Psi_4(x) \end{pmatrix}$$

Bispinor field

$$\mathbb{R}^{1,3} \to \mathbb{C}^2 \oplus \mathbb{C}^2$$

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \mathbb{1}_{4\times 4}}$$

Dirac algebra

$$\Downarrow$$

$$\boxed{S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$$

Generators of bispinor rep. of $SO^+(1,3)$

$$\Downarrow$$

$$\boxed{\Lambda_{\frac{1}{2}} = \exp\left[-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\right]}$$

Bispinor rep. of $SO^+(1,3)$

$$\Downarrow$$

$$\boxed{\Lambda_{\frac{1}{2}}^{-1} \gamma^\mu \Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu \gamma^\nu}$$

Non-unitary representations

$$\Rightarrow \boxed{\Psi'(x) = \Lambda_{\frac{1}{2}} \Psi(\Lambda^{-1} x)}$$

Lorentz trafo of Bispinor field

4-vector rep. of $SO^+(1,3)$

- Dirac adjoint:

$$\boxed{\bar{\Psi} := \Psi^\dagger \gamma_0}$$

$$\Rightarrow \bar{\Psi}' = \bar{\Psi} \Lambda_{\frac{1}{2}}^{-1} \qquad \Psi'^\dagger = \Psi \Lambda_{\frac{1}{2}}^{\dagger} \neq \Lambda_{\frac{1}{2}}^{-1}$$

$$\Rightarrow \bar{\Psi}' \Psi' = \bar{\Psi} \Psi$$

Lorentz scalar

- Dirac Lagrangian:

$$\boxed{\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi}$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \bar{\Psi}_a} - \partial_\mu \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi}_a)}}_{=0} = 0 \Rightarrow \boxed{(i\gamma^\mu \partial_\mu - m)\Psi_a = 0}$$

3.1 Free particle solutions of the Dirac equation

(Details: Problemset 3)

1] $(\partial^2+m^2)\psi=0$ (KG equation)

$\Rightarrow \Psi^\pm(x) = \psi^\pm(p)\, e^{\mp ipx}$, $\psi^\pm(p) \in \mathbb{C}^4$, $p^0=\sqrt{\vec p^{\,2}+m^2}$

$p^2=m^2$, $p^0=E_{\vec p}>0$

2] $(\pm\gamma^\mu p_\mu - m)\,\psi^\pm(p) = 0$

$$\Leftrightarrow \begin{pmatrix} -m & \pm p\sigma \\ \pm p\bar\sigma & -m \end{pmatrix}\begin{pmatrix} \psi_L^\pm \\ \psi_R^\pm \end{pmatrix} = 0$$

3] Note:

- $(p\sigma)(p\bar\sigma) = p^2 = m^2$ $\quad\sqrt{|\vec p|^2+m^2}$
- EV $p\sigma$, $p\bar\sigma$: $p^0 \pm |\vec p| > 0$

$\Rightarrow p^0>0,\ m>0 \rightarrow$ spectrum positive

4] $\psi_L^\pm = \sqrt{p\sigma}\,\xi^\pm$, $\xi^\pm \in \mathbb{C}^2$, $\xi^\dagger\xi = 1$

Spinors

$-m\sqrt{p\sigma}\,\xi^\pm \pm p\sigma\,\psi_R^\pm = 0$

$\sqrt{p\sigma}\cdot\sqrt{p\bar\sigma}=m$ $\Leftrightarrow$ $\psi_R^\pm = \pm\dfrac{m}{\sqrt{p\sigma}}\,\xi^\pm = \pm\sqrt{p\bar\sigma}\,\xi^\pm$

$$\boxed{\begin{gathered}\bar v^r u^s = \bar u^r v^s = 0\\ u^{r\dagger}(\vec p)\,v^s(-\vec p)\\ v^{r\dagger}(p)\,u^s(-p) = 0\end{gathered}}$$

5] Solutions: η^1, $\xi^1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\xi^2 = \begin{pmatrix}0\\1\end{pmatrix}$ $\Big\}\ \xi^s$

$$\Psi^+(x) = \underbrace{\begin{pmatrix}\sqrt{p\sigma}\,\xi^s\\ \sqrt{p\bar\sigma}\,\xi^s\end{pmatrix}}_{u^s(p)} e^{-ipx}$$

$$\Psi^-(x) = \underbrace{\begin{pmatrix}\sqrt{p\sigma}\,\eta^s\\ -\sqrt{p\bar\sigma}\,\eta^s\end{pmatrix}}_{v^s(p)} e^{+ipx}$$

$p^2=m^2$

$p^0>0$

$s=1,2$

6] Relations:

- Orthonormality
- $\bar u^s u^r = 2m\,\delta^{rs}$

$\bar v^s v^r = -2m\,\delta^{rs}$

$\bar u^s = u^{s\dagger}\gamma^0$

$\bar v^s = v^{s\dagger}\gamma^0$

$u^{r\dagger}u^s = 2E_{\vec p}\,\delta^{rs}$

$v^{r\dagger}v^s = 2E_p\,\delta^{rs}$

- **Spin sums:**

$$\boxed{\not{p} = \gamma^\mu p_\mu}$$

$$\boxed{\begin{aligned}&(i\underbrace{\gamma^\mu \partial_\mu}_{\not{\partial}} - m)\psi = 0\\ &\Rightarrow (i\not{\partial} - m)\psi = 0\end{aligned}}$$

$$\sum_{s=1,2} u^s(p)\bar{u}^s(p) = \not{p} + m\mathbb{1}$$

$$\sum_s v^s(p)\bar{v}^s(p) = \not{p} - m\mathbb{1}$$

3.3 Dirac Field Bilinears

1.) $$\boxed{\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1} & \\ & \mathbb{1}\end{pmatrix}}$$

$$\gamma^{5\dagger} = \gamma^5, \quad (\gamma^5)^2 = \mathbb{1}, \quad \{\gamma^5, \gamma^\mu\} = 0$$

2.) Bilinears. $\bar{\Psi}\Gamma\psi$ transform as

$$\boxed{\Gamma = \begin{cases} \mathbb{1} & \text{scalar} \\ \gamma^\mu & \text{vector} \\ \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] & \text{tensor} \\ \gamma^\mu\gamma^5 & \text{pseudo-vector} \\ \gamma^5 & \text{pseudo-scalar} \end{cases}}$$

$$\boxed{\begin{array}{c|c} O(3) & SO(3) \\ \vec{L} = \vec{r} \times \vec{p} & \end{array}}$$

Example:

$$j'^\mu = \bar{\Psi}'\gamma^\mu\psi' = \bar{\Psi}\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}\psi$$

$$= \bar{\Psi}\Lambda^\mu{}_\nu\gamma^\nu\psi$$

$$= \Lambda^\mu{}_\nu\bar{\Psi}\gamma^\nu\psi = \Lambda^\mu{}_\nu j^\nu$$

3.4 Quantization of the Dirac Field

1) Lagrangian. $\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$

2) Canonical mom.: $\Pi_a = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_a} = i\psi^*_a$

3) Hamiltonian:

$$H = \int d^3x\, \psi^\dagger \overbrace{[-i\vec{\alpha}\nabla + m\beta]}^{= H_D \text{ Dirac Hamiltonian}} \psi$$

with $\vec{\alpha} = \gamma^0\vec{\gamma}$, $\beta = \gamma^0$

→ Expand ψ in Eigenmodes of H_D to diagonalize H

4) Eigenmodes.

$$\underbrace{H_D\, u^s(\vec{p}) e^{i\vec{p}\vec{x}}} = E_{\vec{p}} \text{ " }$$

$$H_D\, v^s(\vec{p}) e^{-i\vec{p}\vec{x}} = -E_{\vec{p}} \text{ – " }$$

5) Mode expansion.

$$\psi(\vec{x}) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[a^s_{\vec{p}}\, u^s(\vec{p}) e^{i\vec{p}\vec{x}} + b^s_{\vec{p}}\, v^s(\vec{p}) e^{-i\vec{p}\vec{x}} \right]$$

$a^s_{\vec{p}}$ ↓ Operator, $b^s_{\vec{p}}$ ↓ Operator

$$\underbrace{H_D\psi} = \sum_s \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_p}{2}} \left[\_\_\_ - \_\_\_ \right]$$

$$\boxed{\begin{aligned} \Rightarrow H &= \int d^3x\, \psi^\dagger \underbrace{H_D \psi} \\ &\hat{=} \sum_s \int \frac{d^3p}{(2\pi)^3} E_p \left[a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right] \end{aligned}}$$

First try: Commutator

7] $[\psi_a(\vec{x}), \Pi_b(\vec{y})] = i\delta_{ab}\delta^{(3)}(\vec{x}-\vec{y})$

$\Leftrightarrow [\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})] = \delta_{ab}\delta^{(3)}(\vec{x}-\vec{y})$

8] $\Rightarrow$ Mode algebra:

$$\boxed{[a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}}] = [b^r_{\vec{p}}, b^{r\dagger}_{\vec{q}}] = (2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{q})}$$

$\rightarrow$ Irreducible Rep = Bosonic Fock Space

9] Problem: $(b_p^{s\dagger})^n|0\rangle$ has energy $-nE_p \xrightarrow{n\to\infty} -\infty$

$\rightarrow$ No stable vacuum

10] Fix (?): $b \leftrightarrow b^\dagger$

i] $\psi = \dots\ a + b^\dagger$

ii] $H = \dots\ a^\dagger a - \underline{b b^\dagger}$

iii] $[b, b^\dagger] = - \dots$

iv] $H = a^\dagger a - \underline{b^\dagger b}$ + const

v] $[H, b^\dagger] = E_p b^\dagger$

$\rightarrow$ $b^\dagger$ creates particles with positive energy!

$\rightarrow$ $H \geq 0$

vi] But:

$\| b^\dagger|0\rangle \|^2 = \langle 0| b b^\dagger |0\rangle$

$= \langle 0|[b, b^\dagger]|0\rangle = -\langle 0|0\rangle < 0$

6] Conclusion

- either: unstable vacuum
- or: loss of unitarity

→ No consistent quantization possible!

Second try: Anticommutator

7] $$\boxed{\begin{aligned} &\{\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})\} = \delta_{ab}\, \delta^{(3)}(\vec{x}-\vec{y}) \\ &\{\psi_a(\vec{x}), \psi_b(\vec{y})\} = 0 \end{aligned}}$$

8] $$\boxed{\{a^r_{\vec{p}}, a^{s\,\dagger}_{\vec{q}}\} = \{b^r_{\vec{p}}, b^{s\,\dagger}_{\vec{q}}\} = (2\pi)^3\, \delta^{rs}\, \delta^{(3)}(\vec{p}-\vec{q})}$$

→ Irrep = Fermionic Fock Space

9] Problem: $b^{s\dagger}_{\vec{p}}|0\rangle$ has energy $-E_{\vec{p}}$

→ Still no stable vacuum

10] Fix (?): $b \leftrightarrow b^\dagger$

i] Hamiltonian:

$$\boxed{H = \sum_s \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \left[a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right]} + \text{const}$$

ii] Mode algebra is invariant under $b \leftrightarrow b^\dagger$

→ Unitarity is preserved and Hamiltonian is positive