3. The Dirac Field

3.1. The Dirac Equation

1] Observation: Lorentz symmetry of KG equation

i] $\measuredangle$ $x' = \Lambda x$

$\phi'(x') = \phi(x)$

ii] $\measuredangle$ $(\partial^2 + m^2)\phi = 0 \quad \forall_x$

iii] $\rightarrow$ $\phi'(x) = \phi(\Lambda^{-1}x)$ is a new solution

$$(g^{\mu\nu}\partial_\mu\partial_\nu + m^2)\,\underline{\phi'(x)}$$

$$= \left[g^{\mu\nu}(\Lambda^{-1})^\sigma{}_\mu\,\partial_\sigma(\Lambda^{-1})^\rho{}_\nu\,\partial_\rho + m^2\right]\phi(\Lambda^{-1}x)$$

$$= \left[g^{\sigma\rho}\partial_\sigma\partial_\rho + m^2\right]\phi(\Lambda^{-1}x)$$

$$= (\partial^2 + m^2)\,\phi(\Lambda^{-1}x) = 0 \quad \forall_x$$

2] Observation: $\measuredangle$ Vector fields $\vec{\phi}' = R\,\vec{\phi}(\Lambda^{-1}x)$

$\rightarrow$ In general: $\phi'_a(x) = M_{ab}(\Lambda)\,\phi_b(\Lambda^{-1}x) \quad a = 1, \dots, n$

where

$$\underline{M(\Lambda')\cdot M(\Lambda)}\,\phi(\Lambda^{-1}\Lambda'^{-1}x)$$

$$\stackrel{!}{=} \underline{M(\Lambda'\Lambda)}\,\phi((\Lambda'\Lambda)^{-1}x)$$

$f'(5) = f'(x)\big|_{x=5}$

$\Lambda'\Lambda$

$\Rightarrow$ M is an n-dimensional representation of $SO^+(1,3)$

3] We want first-order relativistic field equation:

$$(\partial^\mu\partial_\mu + m^2)\phi = 0 \rightarrow (\square^\mu\partial_\mu + \text{const})\,\phi = 0$$

4) i) $x' = \Lambda x$

$$\phi'(x') = M(\Lambda)\,\phi(x)$$

ii) $\exists\, \phi$ $(i\,\square^\mu \partial_\mu + \text{const})\,\phi(x) = 0 \;\forall x$

iii) When is $\phi'(x) = M(\Lambda)\,\phi(\Lambda^{-1}x)$ a solution?

$$(i\,\square^\mu \partial_\mu + \text{const})\,\phi'(x)$$

$$= \left[i\,\square^\mu \underline{(\Lambda^{-1})^\nu{}_\mu}\,\partial_\nu + \text{const}\right] M(\Lambda)\,\phi(\underline{\Lambda^{-1}x}) \overset{!}{=} 0$$

$$\Longleftrightarrow \boxed{\left[i\,\underbrace{M^{-1}(\Lambda)\,\square^\mu M(\Lambda)(\Lambda^{-1})^\nu{}_\mu}_{\square^\nu}\,\partial_\nu + \text{const}\right]\phi(\Lambda^{-1}x) \overset{!}{=} 0}$$

$\rightarrow$ $\square^\mu = \gamma^\mu$ must be $n \times n$ matrix with

$$\boxed{M^{-1}(\Lambda)\,\gamma^\mu M(\Lambda) = \Lambda^\mu{}_\nu\,\gamma^\nu} \quad (*)$$

5) $SO^+(1,3)$ is Lie group ($\Rightarrow$ TS 3)

$$\Lambda_\omega = \exp\left[-\tfrac{i}{2}\,\omega_{\alpha\beta}\,\mathcal{J}^{\alpha\beta}\right]$$

$$\simeq \mathbb{1} - \tfrac{i}{2}\,\omega_{\alpha\beta}\,\mathcal{J}^{\alpha\beta}$$

$$M(\Lambda) = \exp\left[-\tfrac{i}{2}\,\omega_{\alpha\beta}\,S^{\alpha\beta}\right]$$

$$\simeq \mathbb{1} - \tfrac{i}{2}\,\omega_{\alpha\beta}\,S^{\alpha\beta}$$

$$(\mathcal{J}^{\alpha\beta})_{\mu\nu} = i\left(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu\right)$$

• Infinitesimal form of $(*)$:

$$\boxed{\begin{aligned}[\gamma^\mu, S^{\alpha\beta}] &\overset{!}{=} (\mathcal{J}^{\alpha\beta})^\mu{}_\nu\,\gamma^\nu \\ &= i\left(g^{\alpha\mu}\gamma^\beta - g^{\beta\mu}\gamma^\alpha\right)\end{aligned}}$$

$SO(3)$

$\downarrow$

$U = e^{-\frac{i}{2}\vec{\omega}\vec{L}}$

- $\mathcal{J}^{\alpha\beta} \to$ Lie algebra of LG.

$$[J^{\mu\nu}, J^{\rho\sigma}] \overset{!}{=} i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right) \quad (2)$$

$J \in \{\mathcal{J}, S\}$

6] Solution: Dirac's trick:

$\exists\ \gamma^\mu$ such Dirac algebra

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_{4\times 4}}$$

Then $\boxed{S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$ satisfies (1) and (2)

7] - At least 4-dimensional

- All 4-D reps. are unitarily equivalent.
- We use the Weyl representation:

$$\boxed{\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}}$$

Henceforth:

$$M(\Lambda) = \Lambda_{\frac{1}{2}}$$

$$[L_i, L_j] = i\varepsilon_{ijk}L_k$$

$$\Lambda$$

$$a_1 = \gamma_1 + i\gamma_2$$
$$a_2 = \gamma_3 + i\gamma_4$$

$$\boxed{[\gamma^\mu, S^{\alpha\beta}] \overset{!}{=} (\mathcal{J}^{\alpha\beta})^\mu{}_\nu \gamma^\nu = i(g^{\alpha\mu}\gamma^\beta - g^{\beta\mu}\gamma^\alpha)} \quad (1)$$

8] Setting const $= -m$

$$\boxed{(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0}$$

Dirac equation

$$\begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} \quad \mathbb{C}$$

ψ has values in $\mathbb{C}^4 = \underline{\mathbb{C}^2 \oplus \mathbb{C}^2}$

$\Downarrow$

bispinor field

$\psi : \mathbb{R}^{1,3} \to \mathbb{C}^4$

9] $\mathcal{L}$ component of Dirac bispinor ...

$$0 = \underline{(-i\gamma^\mu\partial_\mu - m)}(i\gamma^\mu\partial_\mu - m)\,\psi$$

$$\hat{=} \mathbb{1}_{4\times 4}(\partial^2 + m^2)\,\psi$$

KG operator

... satisfy the KG equation.

10] Dirac adjoint.

not unitary $\psi = \Lambda_{\frac{1}{2}}\psi$

$\psi'^{+} = \psi^{+}\Lambda^{+}_{\frac{1}{2}}$

$\Downarrow$ First try: $\psi'^{+}\psi' = \psi^{+}\underbrace{\Lambda^{+}_{\frac{1}{2}}\Lambda_{\frac{1}{2}}}_{\neq \mathbb{1}}\psi \neq \psi^{+}\psi$

11] Define:

$$\boxed{\overline{\psi} = \psi^{+}\gamma^0}$$ Dirac adjoint

$\overset{\circ}{\Rightarrow} \overline{\psi}'\psi' = \overline{\psi}\underbrace{\Lambda^{-1}_{\frac{1}{2}}\Lambda_{\frac{1}{2}}}_{\mathbb{1}}\psi = \overline{\psi}\psi$

$\Rightarrow$ Lorentz scalar

11] Lagrangian:

$$\boxed{\mathcal{L}_{Dirac} = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\,\psi}$$

$\rightarrow$ Euler Lagrange eq $=$ Dirac eq.

• Note 3.1 $\vec{\sigma} = \begin{pmatrix}\sigma^1\\ \sigma^2\\ \sigma^3\end{pmatrix}$

• $\sigma^\mu = \begin{pmatrix}\mathbb{1}\\ \vec{\sigma}\end{pmatrix}$, $\bar{\sigma}^\mu = \begin{pmatrix}\mathbb{1}\\ -\vec{\sigma}\end{pmatrix}$

$$\gamma^\mu = \begin{pmatrix}0 & \sigma^\mu\\ \bar{\sigma}^\mu & 0\end{pmatrix}$$

→ Dirac equation: $\sigma^\mu \partial_\mu$

$$\begin{pmatrix}-m & i\sigma\partial\\ i\bar{\sigma}\partial & -m\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = 0$$

$\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$

• ψ_L und ψ_R: left and right handed Weyl spinors.

•
$$S^{0i} = \frac{1}{4}[\gamma^0, \gamma^i] = -\frac{i}{2}\begin{pmatrix}\sigma^i & \\ & -\sigma^i\end{pmatrix} \quad \text{Boosts}$$

$$S^{ij} = \frac{1}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix}\sigma^k & \\ & \sigma^k\end{pmatrix} \quad \text{Rotations}$$

⇒ ψ_L, ψ_R do not mix under LT $\quad \Lambda_{\frac{1}{2}} = \begin{pmatrix}\Lambda_L & 0\\ 0 & \Lambda_R\end{pmatrix}$

• For $m=0$:

• $\boxed{i\bar{\sigma}\partial\psi_L = 0}$

• $\boxed{i\sigma\partial\psi_R = 0}$

Weyl equations

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