Recap:

2. The Klein-Gordon Field

2.1 Canonical Quantization

· $\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$

$$\boxed{\begin{aligned} [\phi(\vec{x}), \pi(\vec{x})] &= i\delta^{(3)}(\vec{x}-\vec{y}) \\ [\phi(\vec{x}), \phi(\vec{y})] &= 0 \\ [\pi(\vec{x}), \pi(\vec{y})] &= 0 \end{aligned}}$$

$$\boxed{\begin{aligned} \phi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}}\, e^{\boxed{i\vec{p}\vec{x}}} + a^\dagger_{\vec{p}}\, e^{-i\vec{p}\vec{x}}\right) \\ \pi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\vec{p}}}{2}}\left(a_{\vec{p}}\, e^{i\vec{p}\vec{x}} - a^\dagger_{\vec{p}}\, e^{-i\vec{p}\vec{x}}\right) \end{aligned}}$$

Momentum modes.

$$\boxed{[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})}$$

$$\Rightarrow H = \int d^3x\, \mathcal{H} = \int \frac{d^3p}{(2\pi)^3}\, \overset{\sqrt{\dots}}{\omega_{\vec{p}}}\left(a^\dagger_{\vec{p}} a_{\vec{p}} + \frac{1}{2}\underbrace{\cancel{[a_{\vec{p}}, a^\dagger_{\vec{p}}]}}_{\infty}\right)$$

energy for each particle with momentum $\vec{p}$

Eigenstates. $a^\dagger_{\vec{p}}\, a^\dagger_{\vec{q}} \dots |0\rangle$

Vacuum = Groundstate

Normalization:

$|\vec{p}\rangle = \sqrt{2E_p}\, a^\dagger_{\vec{p}}|0\rangle$

Lorentz invariant measure: $\begin{pmatrix} E_{p'} \\ \vec{p}' \end{pmatrix}$ $\quad p' = \Lambda p \quad$ $\begin{pmatrix} E_p \\ \vec{p} \end{pmatrix}$

$$\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}} = \frac{d^3p'}{(2\pi)^3}\frac{1}{\cancel{2E_{\vec{p}}}}\underbrace{\left|\frac{\partial\vec{p}}{\partial\vec{p}'}\right|}_{\cancel{E_p}/E_{p'}} = \frac{d^3p'}{(2\pi)^3}\frac{1}{2E_{p'}}$$

2.2. The Klein-Gordon Field in Space-Time

1] Heisenberg operator:

$$\boxed{\phi(x) = e^{iHt}\,\phi(\vec{x})\,e^{-iHt}}$$

$(\vec{x}, t)$ "Schrödinger field"

2] Heisenberg equation:

$$i\partial_t O = [O, H], \quad O = \phi, \pi$$

$$\Rightarrow \boxed{\begin{aligned} &(\partial_t^2 - \nabla^2 + m^2)\phi = 0 \\ &\pi = \partial_t \phi \end{aligned}}$$

Klein-Gordon equation

3] Time-evolution of modes

$$\boxed{\begin{aligned} e^{iHt} a_{\vec{p}} e^{-iHt} &= a_{\vec{p}}\, e^{\boxed{-iE_{\vec{p}}t}} \\ e^{iHt} a_{\vec{p}}^\dagger e^{-iHt} &= a_{\vec{p}}^\dagger\, e^{+iE_{\vec{p}}t} \end{aligned}}$$

phase

always Schrödinger picture

4] Field operators

$$p_\mu x^\mu = \boxed{p_0 t} \boxed{- \vec{p}\vec{x}}, \quad E_{\vec{p}}$$

$$\boxed{\begin{aligned} \phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[a_{\vec{p}} e^{-ipx} + a_{\vec{p}}^\dagger e^{ipx} \right] \Bigg|_{p^0 = E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}} \\ \pi(x) &= \partial_t \phi(x) \end{aligned}}$$

"on shell momenta"

Note 2.2

1] $\phi(\vec{x}, t) = e^{iHt}\,\phi(\vec{x}, 0)\,e^{-iHt}$

2] $\phi(\vec{x}) = e^{-i\vec{p}\vec{x}}\,\phi(\vec{0})\,e^{i\vec{p}\vec{x}}$

3] $\phi(x) = e^{iPx}\,\phi(0)\,e^{-iPx}$

$$P = \begin{pmatrix} H \\ \vec{P} \end{pmatrix}$$

$$\boxed{i\hbar\partial_t \psi = H\psi}$$

$$i\partial_t e^{-i\omega t} = \hbar\omega\, \phi$$

Note 2.3

$e^{-ipx} \leftrightarrow$ positive frequency solution $\leftrightarrow a_{\vec{p}}^\dagger$

$e^{+ipx} \leftrightarrow$ negative — " — $\leftrightarrow a_{\vec{p}}^{-1}$

Causality:

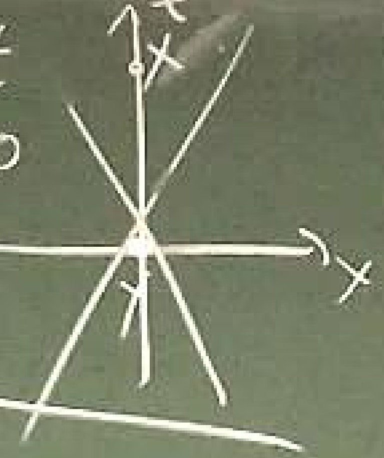
* Amplitude for a particle to propagate from y to x:

$$\boxed{\begin{aligned} D(x-y) &= \langle 0|\phi(x)\phi(y)|0\rangle \\ &\stackrel{\circ}{=} \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\, e^{-ip(x-y)} \end{aligned}}$$

Is LI for $\Lambda \in SO^+(1,3)$:

$$D(\Lambda(x-y)) = D(x-y)$$

1) * Time-like distance: $x^0 - y^0 = t$, $\vec{x}-\vec{y} = 0$



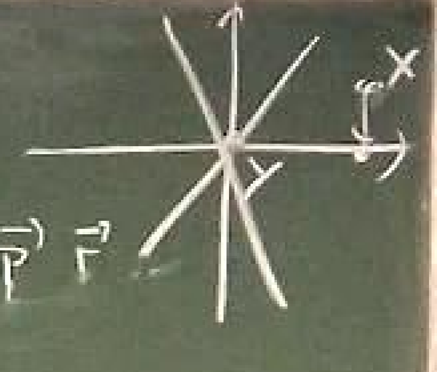
$$D(x-y) = \frac{4\pi}{(2\pi)^3}\int_0^\infty dp\, \frac{p^2}{2\sqrt{p^2+m^2}}\, e^{-i\underbrace{\sqrt{p^2+m^2}}_{E}t}$$

$$= \frac{1}{4\pi^2}\int_m^\infty dE\,\sqrt{E^2-m^2}\; e^{-iEt}$$

$$\neq 0$$

2) * Space-like separation

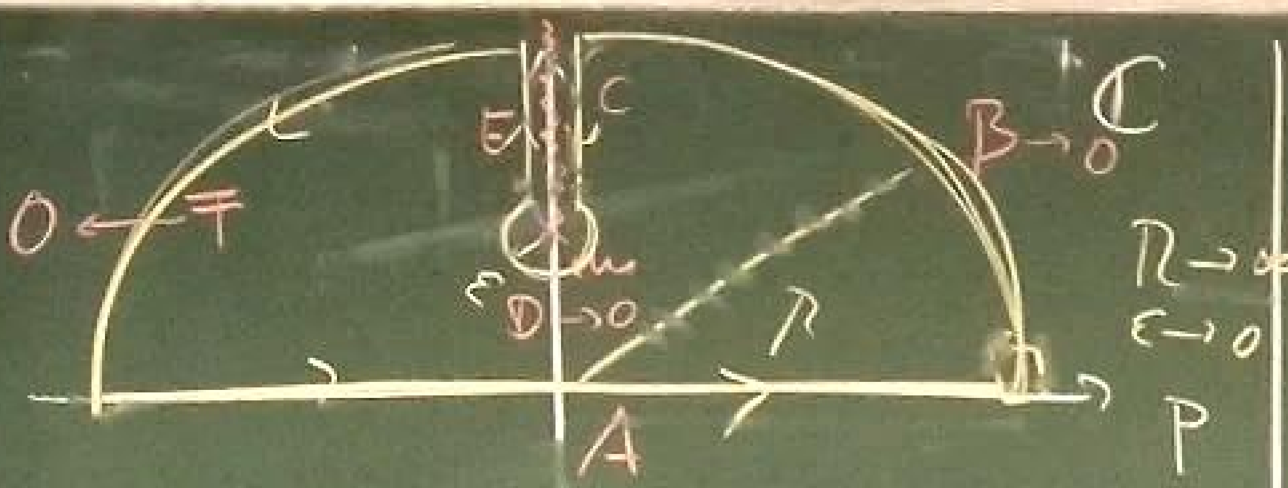
$x^0 - y^0 = 0$, $\vec{x}-\vec{y} = \vec{r}$



$$D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\, e^{i\vec{p}\vec{r}}$$

$$= \frac{2\pi}{(2\pi)^3}\int_0^\infty dp\, \frac{p^2}{2E_p}\, \frac{e^{ipr} - e^{-ipr}}{ipr}$$

$$= \frac{-i}{2(2\pi)^2 r}\int_{-\infty}^\infty dp\, \frac{p\, e^{ipr}}{\sqrt{p^2+m^2}}$$



Cauchy:

$$\oint \ldots = A + C + E = 0$$

$$D(x-y) = -C - E = -2C$$

$$= \frac{-i}{(2\pi)^2 r} \int_{im}^{i\infty} dp \, \frac{p\, e^{ipr}}{\sqrt{p^2+m^2}}$$

$$\overset{p=-i\rho}{=} \frac{1}{4\pi^2 r} \int_m^\infty d\rho \, \frac{\rho\, e^{-\rho r}}{\sqrt{\rho^2-m^2}}$$

$$\overset{r\to\infty}{\sim} e^{-mr}$$

$\Rightarrow$ Vanishes on length scale $\lambda_c = \frac{1}{m}$

3) Measurement: A and B cannot affect each other iff $[A,B]=0$

Simplest case: $A = \phi(x)$, $B = \phi(y)$

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p}}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec q}}}$$

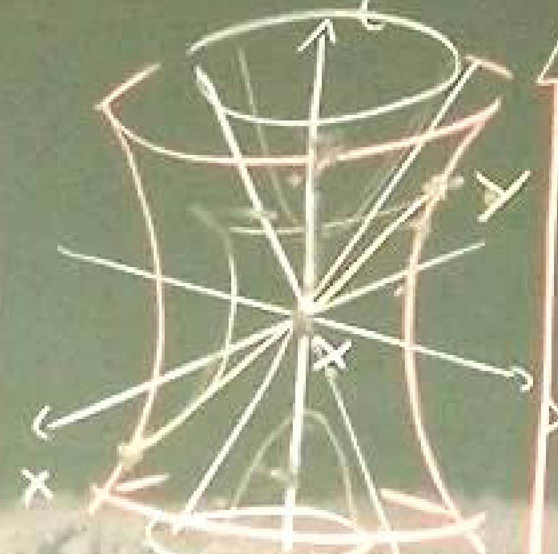
$$\times \left[a_p e^{-ipx} + a^\dagger_p e^{ipx}, a_q e^{-iqy} + a^\dagger_q e^{iqy} \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}} \left[e^{-ip(x-y)} - e^{ip(x-y)} \right]$$

$$= \left(D(x-y) - D(y-x) \right)$$

$(x-y)^2 < 0$ space like $\Rightarrow \exists\, \Lambda^* \in SO^+(1,3)$

$$\Lambda^*(x-y) = -(x-y)$$



$$\boxed{\begin{aligned} [\phi(x), \phi(y)] &= D(x-y) - D(\Lambda^*(y-x)) \\ &= D(x-y) - D(x-y) \\ &= 0 \end{aligned}}$$

The Propagator

1) $\boxed{\langle 0|[\phi(x),\phi(y)]|0\rangle}$

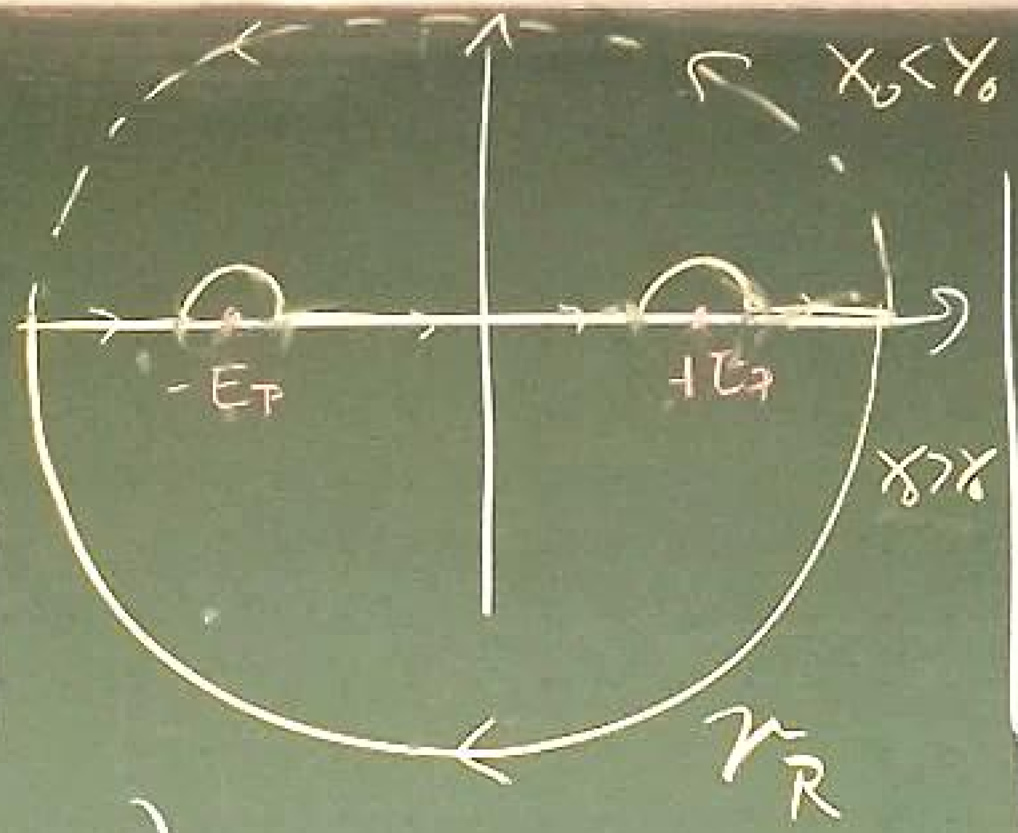
$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec p}}\Big[e^{-ip(x-y)} - \underbrace{e^{ip(x-y)}}_{\text{Subst. } \vec p' = -\vec p}\Big]$$

$$= \int \frac{d^3p}{(2\pi)^3}\left\{\frac{e^{-ip(x-y)}\big|_{p^0=E_p}}{2E_{\vec p}} + \frac{e^{-ip(x-y)}\big|_{p_0=-E_p}}{-2E_{\vec p}}\right\}$$

$\boxed{x_0 > y_0}$

$$= \int \frac{d^3p}{(2\pi)^3}\int_{\gamma=\gamma_R}\frac{dp^0}{2\pi i}\,\underbrace{\frac{-1}{p^2-m^2}}_{(p^0-E_p)(p^0+E_p)}\,e^{-ip(x-y)} \quad \to p^0(x_0-y_0)$$

$$= \boxed{\int_{\gamma_R}\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-ip(x-y)} \quad (*)}$$



Residue Thm

$$\frac{1}{2\pi i}\oint_\gamma f(z)\,dz = \sum_a \operatorname{Res}(f,a)$$

$$\operatorname{Res}(f,a) = \lim_{z\to a}(z-a)f(z)$$

Therefore:

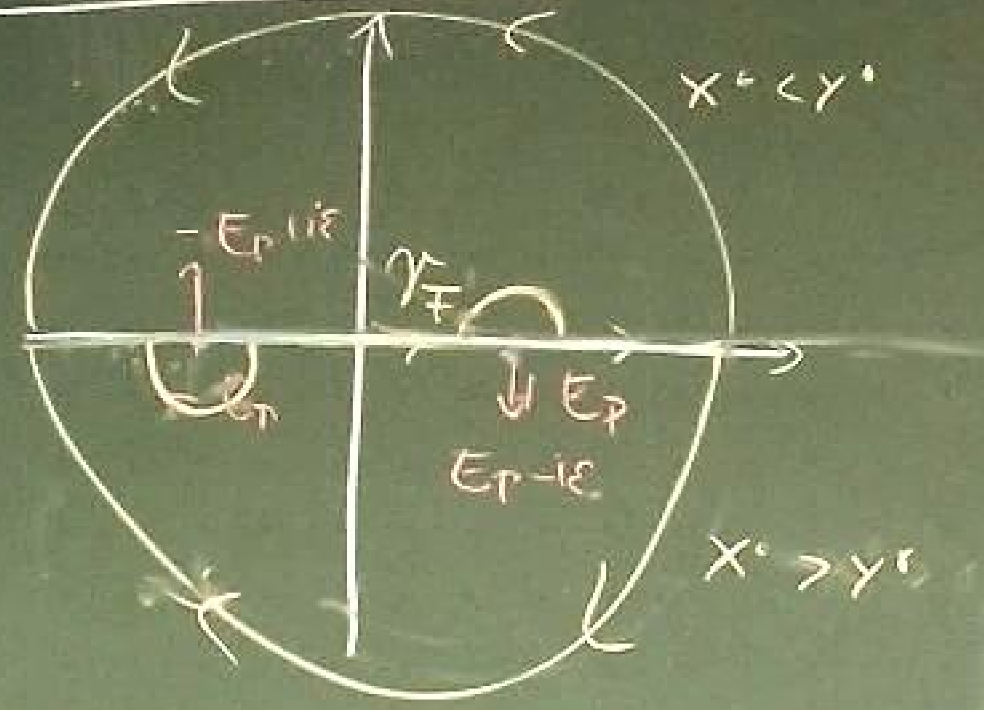
$$\boxed{\begin{aligned} D_R(x-y) &:= \langle 0|[\phi(x),\phi(y)]|0\rangle\,\Theta(x^0-y^0) \\ &= (*)\big|_{\gamma=\gamma_R}\end{aligned}}$$

2) Interpretation:

$$\boxed{(\partial^2+m^2)\,D_R(x-y) \overset{!}{=} -i\delta^{(4)}(x-y)}$$

$\Rightarrow$ Retarded Green's function of the Klein-Gordon operator

3| Alternative contours:



$$\boxed{\begin{aligned} D_F(x-y) &:= (*)\big|_{\gamma=\gamma_F} \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} e^{-ip(x-y)} \end{aligned}}$$

Feynman propagator

We find:

$$D_F(x-y) = \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & x^0 < y^0 \end{cases}$$

time-ordering operator

$$= \Theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \Theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle = \langle 0|T\phi(x)\phi(y)|0\rangle$$