2. The Klein Gordon Field

2.1. Canonical Quantization

1) Theory

- Real field $\phi(x) : \mathbb{R}^4 \to \mathbb{R}$
- Lagrangian: $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$ (free scalar field)
- EOM: $(\partial^2 + m^2)\phi = 0$
- Hamiltonian: $\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$

2) Canonical quantization:

$$[q_i, p_j] = i\delta_{ij} \longrightarrow \boxed{\begin{aligned} [\phi(\vec{x}), \pi(\vec{y})] &= i\delta^{(3)}(\vec{x}-\vec{y}) \\ [\phi(\vec{x}), \phi(\vec{y})] &= 0 \\ [\pi(\vec{x}), \pi(\vec{y})] &= 0 \end{aligned}} \quad (*)$$

with $\phi^\dagger = \phi$, $\pi^\dagger = \pi$, $\vec{x} \in \mathbb{R}^3$

3) Goals:

- Representation ✓
- Spectrum of $\mathcal{H}$ (✓)
- Time evolution of ϕ, π

4) Motivation:

- Fourier transform KG equation in space.

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\vec{x}} \tilde{\phi}(\vec{p}, t)$$

$$\Rightarrow \left[\partial_t^2 + \underbrace{(\vec{p}^2 + m^2)}_{\omega_p^2}\right] \tilde{\phi}(\vec{p}, t) = 0$$

$\Rightarrow$ Decoupled Harmonic oscillators with frequency $\omega_p = \sqrt{\vec{p}^2 + m^2}$ and $\tilde{\phi}^*(\vec{p}, t) = \tilde{\phi}(-\vec{p}, t)$

- & $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$ $\quad [a_p, a_p^\dagger] = 1$

$\tilde{\phi}(p)$ — $q = \frac{1}{\sqrt{2\omega_p}}(a + a^\dagger)$, $\underset{\tilde{\pi}(p)}{p} = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$

$\Rightarrow H = \omega\left(a^\dagger a + \frac{1}{2}\right)$

5) Field operators:

$$\boxed{\begin{aligned}\phi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_{\vec{p}} + a^\dagger_{-\vec{p}}\right) e^{i\vec{p}\vec{x}} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{i\vec{p}\vec{x}} + a^\dagger_{\vec{p}} e^{-i\vec{p}\vec{x}}\right) \\ \pi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{\omega_p}{2}} \left(a_p - a^\dagger_{-\vec{p}}\right) e^{i\vec{p}\vec{x}}\end{aligned}}$$

with momentum modes

$$\boxed{[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3 \delta^3(\vec{p} - \vec{q})}$$

$\Rightarrow$ Check $(*)$ is true

6) Hamiltonian:

$$\boxed{H \overset{\circ}{=} \int \frac{d^3p}{(2\pi)^3} \omega_p \left(a^\dagger_{\vec{p}} a_{\vec{p}} + \frac{1}{2}\underbrace{[a_{\vec{p}}, a^\dagger_{\vec{p}}]}_{\delta^{(3)}(0) = \infty}\right)}$$

drop it!

$\rightarrow$ Excitations $a^\dagger_{\vec{p}}$ commute and carry additive energy + momentum

$\rightarrow$ Bosonic particles

7) Eigenstates + Spectrum:

- $[H, a^\dagger_{\vec{p}}] \overset{\circ}{=} \omega_p a^\dagger_{\vec{p}}$

$\Rightarrow$ Vacuum $|0\rangle \rightarrow$ Eigenstates $(a^\dagger_{\vec{p}})^{n_p}(a^\dagger_{\vec{q}})^{n_q}|0\rangle$

- Energy: $\boxed{E_{\vec{p}} = \omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}}$ c^2

relativistic dispersion

- Kinetic momentum:

$$P^i = \int d^3x\, \pi(\vec{x})(\partial_i \phi(\vec{x})) \overset{\circ}{=} \int \frac{d^3p}{(2\pi)^3} p^i a^\dagger_{\vec{p}} a_{\vec{p}}$$

- Statistics: $a^\dagger_{\vec{p}} a^\dagger_{\vec{q}} |0\rangle = a^\dagger_{\vec{q}} a^\dagger_{\vec{p}} |0\rangle$

8] Normalization: $\frac{v}{c}$

i] $\forall\ \Lambda \in R' L(\cdot) R \quad SO^+(1,3)$

Rotation, Boost

$\rightarrow \quad p' = (E_{p'}, \vec{p}\,') = \Lambda p$

$E_{p'} = \sqrt{\vec{p}\,'^2 + m^2}$, $\quad (E_{\vec p}, \vec{p})$, $\quad \sqrt{\vec{p}^{\,2} + m^2}$

$p'_\mu p^{\mu'} = m^2$

ii] Jacobian in space:

$$\det\left(\frac{\partial \vec{p}\,'}{\partial \vec{p}}\right) \hat{=} \frac{dp'_3}{dp_3} \hat{=} \frac{E_{p'}}{E_p}$$

$$\rightarrow \delta^{(3)}(\vec{p} - \vec{q}) = \det\left(\frac{\partial \vec{p}\,'}{\partial \vec{p}}\right) \delta^{(3)}(\vec{p}\,' - \vec{q}\,') = \boxed{\frac{E_{p'}}{E_p}} \delta^{(3)}(\vec{p}\,' - \vec{q}\,')$$

$$\delta(\alpha x) = \frac{1}{|\alpha|}\delta(x)$$

$$\det\left(\frac{\partial p'}{\partial y}\right)$$

$\Downarrow$

$$\underbrace{E_p \delta^{(3)}(p-q) = E_{p'} \delta^{(3)}(p'-q')}_{\text{is Lorentz invariant}}$$

iii] Single particle eigenstates:

$$\boxed{|\vec{p}\rangle = \sqrt{2E_p}\, a_p^\dagger |0\rangle}$$

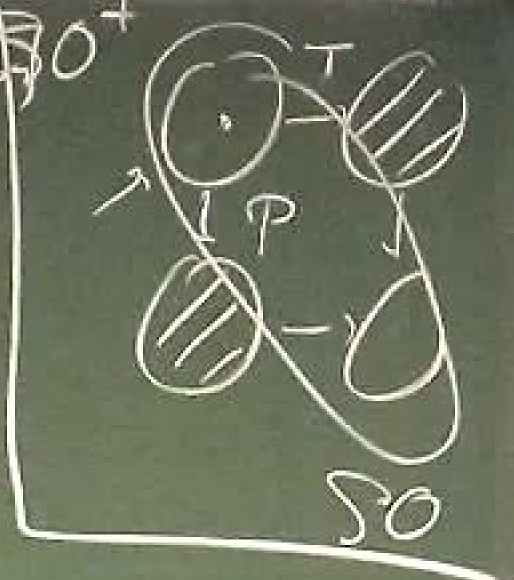
Normalization

$$\Rightarrow \langle \vec{p} | \vec{q} \rangle = (2\pi)^3 2E_p \delta^{(3)}(\vec{p} - \vec{q})$$

$\langle 0 | a_p a_q^\dagger | 0 \rangle$

9] Lorentz transformations

$\Lambda \in SO^+(1,3)$.



$$\boxed{U(\Lambda)|\vec{p}\rangle := |\Lambda \vec{p}\rangle}$$

$$p = \begin{pmatrix} E_p \\ \vec{p} \end{pmatrix} \rightarrow \Lambda p = \begin{pmatrix} E_{\vec p'} \\ \vec{p}\,' \end{pmatrix} \rightarrow \Lambda \vec{p}$$

Check $U^\dagger(\Lambda) U(\Lambda) = \mathbb{1}$

$$\Rightarrow \boxed{U(\Lambda) a_p^\dagger U^\dagger(\Lambda) = \sqrt{\frac{E_{\Lambda \vec{p}}}{E_{\vec{p}}}}\, a^\dagger_{\Lambda \vec{p}}}$$

10| Interpretation of $\phi(\vec x)$

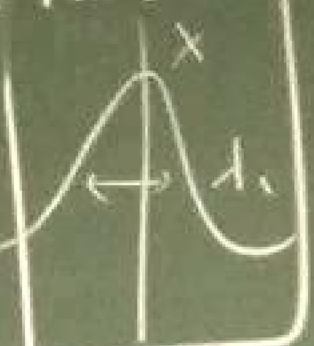
$$\phi(\vec x)\,|0\rangle = \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2E_{\vec p}}\right) e^{-i\vec p\vec x}\,|\vec p\rangle \;\text{"="}\; |x\rangle$$

• non-rel $|\vec p| \ll m$ $\quad\sqrt{p^2+m^2} \approx m$

→ $\phi(\vec x)$ creates a particle at $\vec x$

• Particle localized on length scale $\lambda_c = \frac{1}{m}$

$$\phi^\dagger = \phi$$



Note 2.1

• $\mathbb{1}_1 = \int \boxed{\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}}\, |\vec p\rangle\langle\vec p|$

LI measure

• $\langle f(p)$ LI → $\displaystyle\int \frac{d^3p'}{(2\pi)^3}\frac{1}{2E_{p'}} f(p')$ LI

"$f'(p')$"

2.2 The KG Field in Spacetime

1| Heisenberg operator

$$\phi(x) = \phi(\vec x, t) = e^{iHt}\phi(\vec x)e^{-iHt}$$

$$\frac{e^{ia^\dagger a t}\, a\, e^{-ia^\dagger a t}}{e^{Et}\, a}$$

2| Heisenberg equation

$$i\partial_t O = [O, H]\,,\quad O = \phi, \pi$$

• $i\partial_t\phi \doteq i\pi$

• $i\partial_t\pi \doteq -i(-\nabla^2+m^2)\phi$

$$\boxed{(\partial_t^2 - \nabla^2 + m^2)\phi = 0}$$

KG equation for field operator