Recap

9. Non-Abelian Gauge Theories

9.2. The Yang-Mills Lagrangian

1| Lie Group G, Unitary rep. V ($n \times n$ matrix).

2| Fermion fields: $\psi = (\psi_1, \ldots, \psi_n)^T$ ← Dirac bispinor

Gauge trans.: $\boxed{\tilde{\psi}(x) = V(x)\,\psi(x)}$

3| Lie Algebra:

structure constant, generators (Hermitian $n \times n$ matrices)

$$\boxed{[t^a, t^b] = i f^{abc}\, t^c}$$

Exponential map: $a = 1, \ldots, N$

$$V(x) = \exp\left(i\alpha^a(x)\, t^a\right) = \mathbb{1} + i\alpha^a(x)\, t^a + O(\alpha^2)$$

4| Comparator

$$\tilde{U}(y,x) = V(y)\, U(y,x)\, V^\dagger(x)$$

• $U(x+\epsilon n, x) = \mathbb{1} + i g \epsilon n^\mu \underbrace{A^a_\mu(x)\, t^a}_{\equiv A_\mu(x)} + O(\epsilon^2)$

g: arbitrary constant

$A_\mu(x)$: Lie-algebra valued gauge field

5| Covariant derivative:

$$\boxed{D_\mu = \partial_\mu - i g A^a_\mu t^a}$$

6| Gauge transformation of A_μ:

$$\boxed{\tilde{A}^a_\mu t^a = V(x)\left[A^a_\mu(x)\, t^a + \frac{i}{g}\partial_\mu\right] V^\dagger(x)}$$ (valid for all $V \in G$)

$$\boxed{\tilde{A}^a_\mu \approx A^a_\mu + \frac{1}{g}\partial_\mu \alpha^a + \underbrace{f^{abc} A^b_\mu \alpha^c}_{\text{new for non-abelian } (f^{abc} \neq 0) \text{ gauge groups!}}}$$ (valid for $V \approx \mathbb{1}$)

7| $\boxed{\tilde{D}_\mu \tilde{\psi} \overset{!}{=} V D_\mu \psi}$

→ $D_\mu \psi$ transforms like ψ

Example: $\bar{\psi} D_\mu \psi$ is gauge invariant

$\bar{\psi} \not{D} \psi$ — " — Lorentz & gauge — " —

8) Kinetic term for A^a_μ?

i) $\tilde{D}_\mu \tilde{D}_\nu \tilde{\psi} = V D_\mu D_\nu \psi$

→ $\underbrace{[\tilde{D}_\mu, \tilde{D}_\nu]}\tilde{\psi} = V [D_\mu, D_\nu] \psi$

$= \underline{V [D_\mu, D_\nu] V^\dagger} \tilde{\psi}$

→ $[\tilde{D}_\mu, \tilde{D}_\nu] = V [D_\mu, D_\nu] V^\dagger$

ii) $$\boxed{\begin{aligned} &[D_\mu, D_\nu] = -ig F^a_{\mu\nu} t^a \\ &F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \end{aligned}}$$ ← Field-strength tensor

iii) $$\boxed{\tilde{F}_{\mu\nu} = \tilde{F}^a_{\mu\nu} t^a = V F_{\mu\nu} V^\dagger}$$

→ $F_{\mu\nu}$ is no longer gauge invariant!

iv) $$\boxed{\begin{aligned} \mathcal{L}_{YM} &= -\frac{1}{2} \mathrm{Tr}[F^2] \quad \text{Yang-Mills Lagrangian} \\ &= -\frac{1}{2} \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \\ &= -\frac{1}{2} \mathrm{Tr}[(F^a_{\mu\nu} t^a)(F^{\mu\nu b} t^b)] \\ &= -\frac{1}{2} F^a_{\mu\nu} F^{\mu\nu b} \underbrace{\mathrm{Tr}[t^a t^b]}_{\frac{1}{2}\delta^{ab}} = -\frac{1}{4} (F^a_{\mu\nu})^2 \end{aligned}}$$

Note 9.1:

$F^2 \sim (\partial A)^2 + f (\partial A) AA + f^2 AAAA$

→ Interacting QFT ($f \neq 0$: non-abelian)

→ Gauge bosons can scatter off each other

Example: QCD: $G = SU(3)$

Gauge bosons: Gluons. Bound states: Glue-balls

9) Couple Dirac field to YM gauge field:

$$\boxed{\mathcal{L}_{YM+D} = \bar{\psi}(i\cancel{D} - \cancel{m})\psi - \frac{1}{4}(F^a_{\mu\nu})^2}$$

- m: Fermion mass $\qquad + A^2$
- g: Coupling constant

$\Rightarrow$ Yang-Mills theories describe <u>all</u> fundamental forces in the SM.

<u>Note 9.2.</u>

- $\cancel{D} = \gamma^\mu D_\mu = \partial_\mu \gamma^\mu \mathbb{1}_n - ig A^a_\mu \gamma^\mu t^a$
- $\gamma^\mu t^a = \gamma^\mu \otimes t^a = \gamma^\mu_{\alpha\beta}\, t^a_{mn}$
- $\psi : \mathbb{R}^{1,3} \to \underline{\mathbb{C}^4 \otimes \mathbb{C}^n} \simeq \mathbb{C}^{4n}$
- $\tilde{\bar{\psi}} = (V\psi)^\dagger \gamma^0$

$$= \psi^\dagger \underbrace{V^\dagger}_{\mathbb{1}\otimes V^\dagger} \underbrace{\gamma^0}_{\gamma^0 \otimes \mathbb{1}} = \psi^\dagger \gamma^0 V^\dagger = \bar{\psi} V^\dagger$$

<u>Note 9.3</u>

- Mass term A^2 is <u>not</u> allowed (not gauge inv.)

$\hookrightarrow$ YM gauge bosons are massless

<u>Problem.</u>

Weak interaction: $W^\pm, Z$ bosons have mass

<u>Solution:</u> Higgs mechanism

10. Excursions

10.1 Higgs mechanism

Problem 1. How to give gauge bosons a mass?

Problem 2. $m\bar{\Psi}\Psi$ forbidden in SM

→ How do leptons get a mass?

Solution to both problems: Higgs mechanism

10.1.1 Abelian Example

• Abelian gauge theory

1) • Maxwell theory coupled to complex scalar field.

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + |D_\mu\phi|^2 - V(\phi)$$

with potential $V(\phi) = \mu^2 |\phi|^2 + \lambda |\phi|^4$

• $D_\mu = \partial_\mu + ieA_\mu$

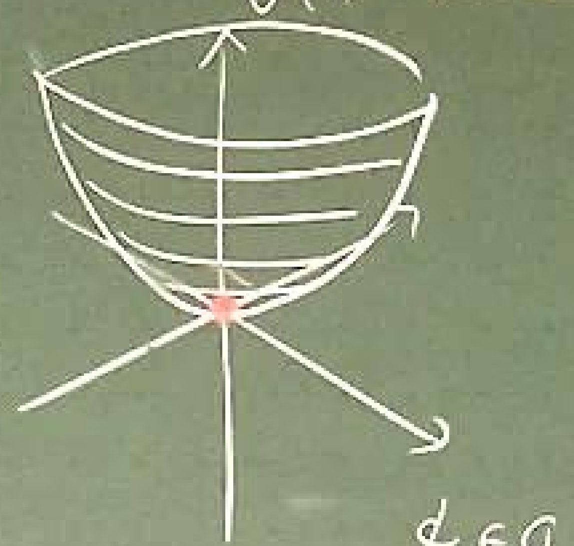
2) U(1) gauge theory

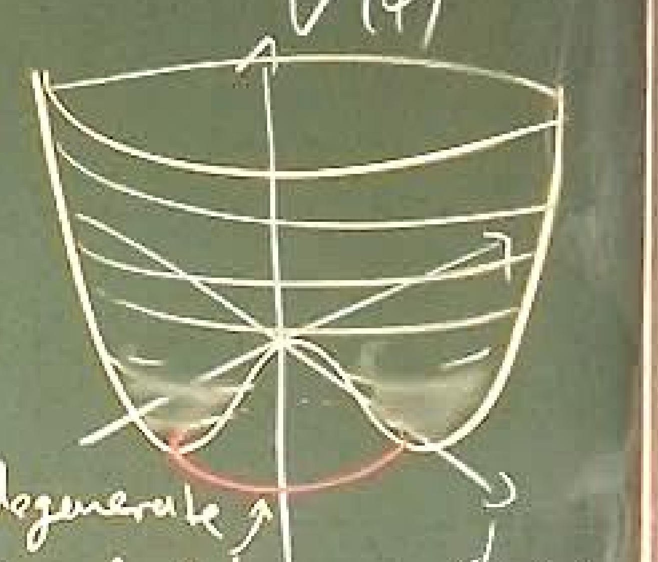
$$\tilde{\phi}(x) = e^{i\alpha(x)}\phi$$

$$\tilde{A}_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$$

3) • $V(\phi)$

Mexican-Hat potential





$\mu^2 > 0$: Unique minimum

$\langle \phi \rangle = 0$

$\mu^2 < 0$: degenerate minima

$\downarrow$

Ground states are not symmetric under global phase rotations

$\downarrow$

Spontaneous symmetry breaking (SSB) of global $U(1)$ symmetry.

$\phi_0 = \langle \phi \rangle$ and

$v = |\phi_0| = \sqrt{\frac{-\mu^2}{2\lambda}} \neq 0$

$\uparrow$

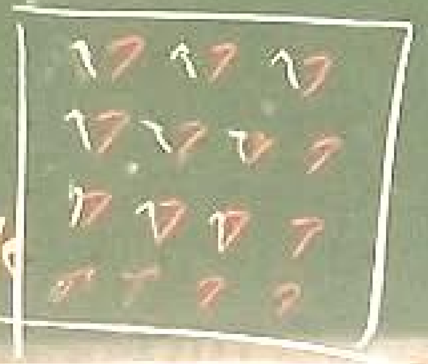
Vacuum expectation value (VEV)

4) Aside: Goldstone Theorem

$$\boxed{\text{If a global, continuous symmetry is spontaneously broken, there is one massless scalar (boson) particle for each broken symm. generator. These particles are called Nambu-Goldstone bosons.}}$$

"Proof by picture":

$\mathcal{H} = |\nabla \phi|^2$



Examples:

- Crystal breaks rotation sym $O(3)$ and translation symmetry ($\mathbb{R}^3$) to discrete symmetries
  $\longrightarrow$ longitudinal and transversal phonons

$G = O(3) \ltimes \mathbb{R}^3$

$\downarrow$ $\quad$ $\downarrow$

$L_x, L_y, L_z \quad P_x, P_y, P_z$

$[L_i, P_j] \neq 0$

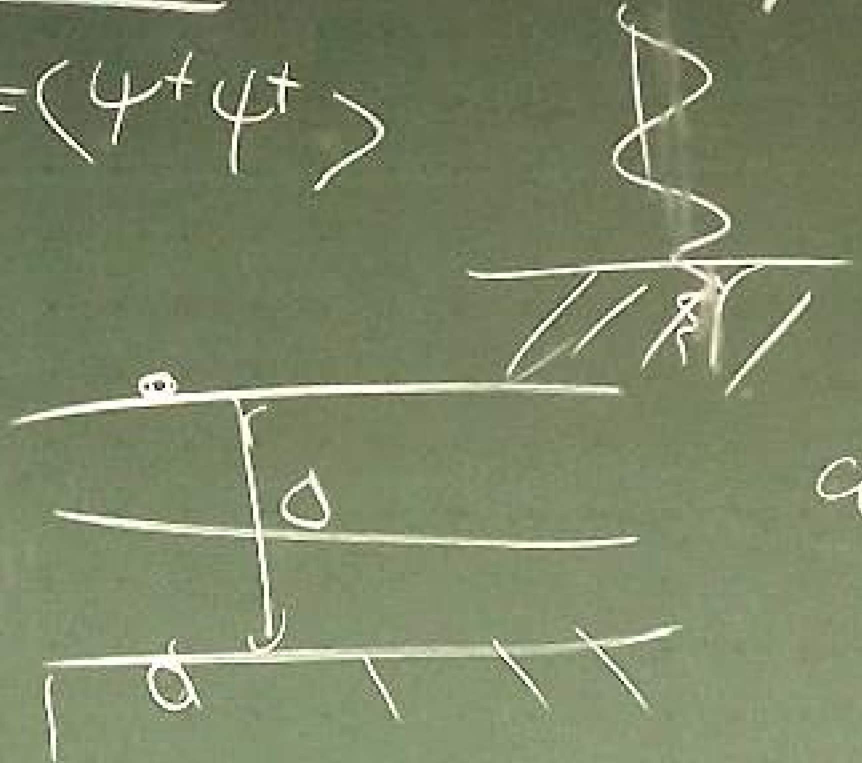
$-\sum_i o_i^2 o_{i+1}^2$

$\mathbb{Z}_2 \quad o^2 \to -o^2$

- Ferromagnet:
  $H = -\sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j \rightarrow SO(3)$ symmetric
  $\nearrow$ Spin waves (Magnons)

Exception: Superconductivity

$\psi = \langle \psi^{\dagger} \psi^{\dagger} \rangle$



$a + a^{\dagger}$