Recap

9. Non-Abelian Gauge Theories

9.1 The Geometry of Gauge Invariance

1| Local $U(1) = G$ symmetry of Dirac field.

$$\tilde{\psi}(x) = e^{i\alpha(x)}\,\psi(x)$$

$\alpha: \mathbb{R}^{1,3} \to \mathbb{R}$ arbitrary

2| G-invariant Lagrangian

Example: $\bar{\psi}\psi$

3| No problem without derivatives:

global $U(1)$ sym $\to$ local $U(1)$-sym.

4| Directional derivative.

$$n^\mu \partial_\mu \psi := \lim_{\varepsilon \to 0} \frac{\psi(x+\varepsilon n) - \psi(x)}{\varepsilon}$$

$\to$ $n^\mu \partial_\mu \psi$ has no simple transformation under G

(e.g. $n^\mu \partial_\mu \tilde{\psi} \neq e^{i\alpha(x)}\, n^\mu \partial_\mu \psi$)

5| "Comparator": $U: \mathbb{R}^{1,3} \times \mathbb{R}^{1,3} \to \mathbb{C}$

- $\tilde{U}(y,x) = e^{i\alpha(y)}\, U(y,x)\, e^{-i\alpha(x)}$
- $U(y,y) = 1$
- $U(y,x) = e^{i\phi(y,x)}$

$\Rightarrow$ $\underline{\tilde{U}(y,x)\tilde{\psi}(x)} = e^{i\alpha(y)}\, \underline{U(y,x)\psi(x)}$

$\underline{\tilde{\psi}(y)} = e^{i\alpha(y)}\, \underline{\psi(y)}$

6| Covariant derivative

$$\boxed{n^\mu D_\mu \psi := \lim_{\varepsilon \to 0} \frac{\psi(x+\varepsilon n) - U(x+\varepsilon n, x)\,\psi(x)}{\varepsilon}}$$

7| $U(y,x)$: continuous

$$U(x+\varepsilon n, x) = 1 - ie\varepsilon n^\mu A_\mu(x) + O(\varepsilon^2)$$

e: arbitrary constant

A_μ: new vector field = (gauge) field

$$\psi(x+\varepsilon n) = \psi(x) + \varepsilon n^\mu \partial_\mu \psi + O(\varepsilon^2)$$

8| $\boxed{D_\mu \psi(x) = \partial_\mu \psi(x) + ie A_\mu \psi(x)}$

9| $\boxed{\tilde{A}_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)}$

10| $\boxed{\hat{D}_\mu \hat{\psi}(x) = e^{i\alpha(x)} D_\mu \psi(x)}$

→ $D\psi$ transforms ψ

→ global U(1) term (∂) → local U(1) term ($\partial \to D$)

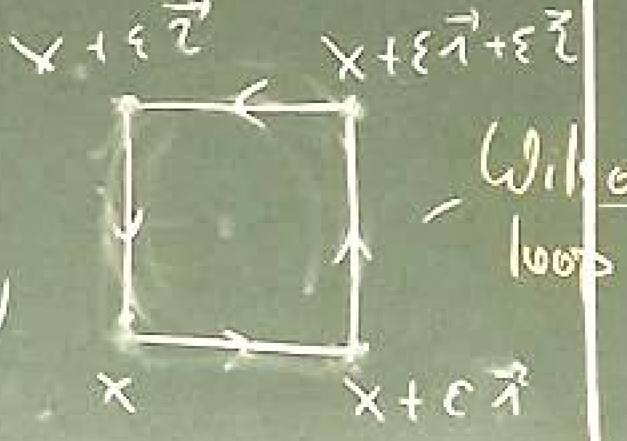
11| Conclusion

Local symmetry → Gauge field A_μ for (covariant derivative)

12| Kinetic energy term for A_μ?

i) ∃ Locally invariant loop

$$U(x) = U(x, x+\epsilon\vec{2})\, U(x+\epsilon\vec{2}, x+\epsilon\vec{1}+\epsilon\vec{2})\, U(x+\epsilon\vec{1}+\epsilon\vec{2}, x+\epsilon\vec{1})\, U(x+\epsilon\vec{1}, x)$$



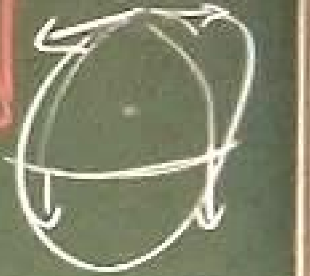
$\tilde{U}(x) = U(x)$ by construction

ii) $$U(x) \doteq 1 - i\epsilon^2 e \overbrace{\left[\partial_1 A_2(x) - \partial_2 A_1(x)\right]}^{F_{12}} + O(\epsilon^3)$$

→ $\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}$

Field strength tensor

is locally gauge invariant



$$R^a{}_{b\mu\nu} = \partial_\mu \Gamma - \partial_\nu \Gamma + \dots$$

13) Most general gauge (and Lorentz) inv. Lagrangian in $D = 3+1$.

- Gauge inv. → ψ, $D_\mu \psi$, $F_{\mu\nu}$, $\partial_\mu F_{\mu\nu}$
  (globally $U(1)$ symmetric terms)
- Relativistic → Lorentz scalars
- Renormalizable → Term of mass dimension at most 4

$$\boxed{\mathcal{L} = \bar{\psi} i \not{D} \psi - m \bar{\psi}\psi - \frac{1}{4} F_{\mu\nu}^2 \Big(\underbrace{- c_1 \, \varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}}_{\substack{SO(1,3)\text{ inv.} \\ \text{but breaks P, T symmetry}}} \Big) + \underbrace{c_2 (\bar{\psi}\psi)^2}_{\text{non-renormalizable}} + \dots}$$

→ Most general P/T symmetric Lagrangian. Minimally coupled Max-Dirac Lagrangian
→ QED

9.2. The Yang-Mills Lagrangian

Goal: Replace local $U(1)$ symmetry by non-abelian Lie group G

1) $\forall$ Lie group G represented by $n \times n$ unitary matrices V

2) Fields $\psi = (\psi_1, \dots, \psi_n)^T$ are n-plet of Dirac fields ψ_i.

$$\psi : \mathbb{R}^{1,3} \to \mathbb{C}^4 \otimes \mathbb{C}^n \simeq \mathbb{C}^{4n}$$

$$\boxed{\tilde{\psi}(x) = V(x)\, \psi(x)}$$

$V : \mathbb{R}^{1,3} \to G$ arbitrary

3) G Lie group $\to$ Lie algebra $\mathfrak{g}$ with N Hermitian generators t^a ($n \times n$ matrices, $a = 1, \dots, N$)

$$\boxed{[t^a, t^b] = i f^{abc}\, t^c}$$

Einstein summation

f^{abc}: structure constant $\in \mathbb{C}$

$SU(2)$, $n = 2$

$$V(x) = e^{i \hat{\omega}(x) \vec{\sigma}}$$

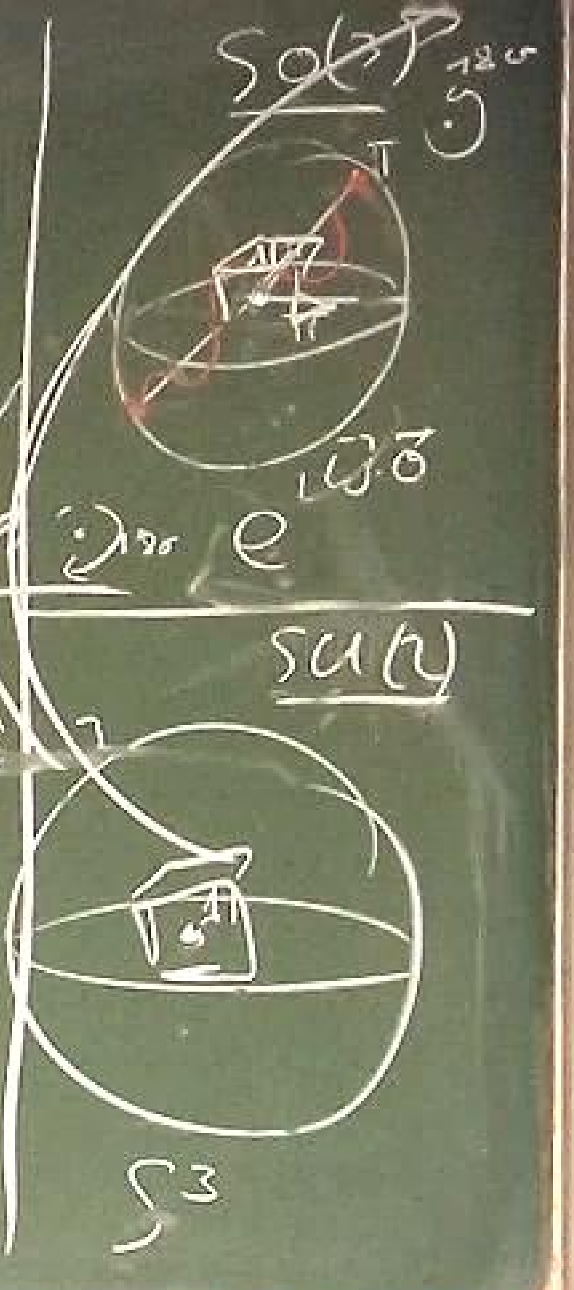
$$[\sigma_i, \sigma_j] = i \underbrace{\varepsilon_{ijk}}_{f} \sigma_k$$

$N = 3$

$$\underbrace{e^A}_{V} \underbrace{e^B}_{V'} = e^{A+B+\frac{1}{2}[A,B] + \dots}$$

hint

$SO(4)$ $\quad SO(1,3)$



3| Lie algebra g:

$$[t^a, t^b] = i f^{abc} t^b$$

$$\rightarrow V(x) = e^{i\alpha^a(x) t^a} = 1 + i\alpha^a(x) t^a + O(\alpha^2)$$

4| Comparator: $n \times n$ unitary matrix:

- $\tilde{U}(y,x) = V(y)\, U(y,x)\, V^\dagger(x)$
- $U(y,y) = \mathbb{1}$

$$\rightarrow U(x+\varepsilon n, x) = 1 + ig\,\varepsilon n^\mu \underbrace{A^a_\mu(x)\, t^a}_{A_\mu(x)} + O(\varepsilon^2)$$

- g: arbitrary constant
- A^a_μ: N vector fields

5| Covariant derivative.

$$\boxed{D_\mu \stackrel{\circ}{=} \partial_\mu - ig A^a_\mu t^a} \quad (**)$$

$e^{-i\alpha(x)}$

6| Transformation of A^a_μ:

i| –

ii|

$$\boxed{\tilde{A}^a_\mu t^a = V(x)\left[A^a_\mu t^a + \frac{i}{g}\partial_\mu\right] V^\dagger(x) \quad (*)}$$

(valid for all V)

$\partial_x e^{V(x)} \neq (\partial_x V)\, e^{V(x)}$

iii| α $V^\dagger(x) \approx \mathbb{1}$

$$\boxed{\tilde{A}^a_\mu \stackrel{\circ}{=} A^a_\mu + \frac{1}{g}\partial_\mu \alpha^a + \boxed{f^{abc} A^b_\mu \alpha^c}}$$

(valid for small α)

New term for non-abelian Lie groups!

$V(x) = x\sigma^x + \sigma^z$

$[V, V] = 0$

$\partial_x V = 0^\times$

7| (*) and (**)

$$\boxed{\tilde{D}_\mu \tilde{\Psi} \stackrel{\circ}{=} V D_\mu \Psi}$$

- $D_\mu \Psi$ transforms like Ψ under G
- $\bar{\Psi} \cancel{D}_\mu \Psi$ gauge invariant + Lorentz invariant.