Recap

8. Functional Methods

8.2 Path Integrals for scalar fields

$$\boxed{\begin{aligned}&\langle \Omega | T\, \hat\phi_H(x_1)\, \hat\phi_H(x_2) | \Omega \rangle \quad \text{Operators}\\ &= \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)\, e^{i\int_{-T}^{T} d^4x\, \mathcal{L}(\phi)}}{\int \mathcal{D}\phi\, e^{i\int_{-T}^{T} d^4x\, \mathcal{L}(\phi)}}\end{aligned}}$$

functions

Path Integral

Action

8.3 Application: Quantization of the EM field

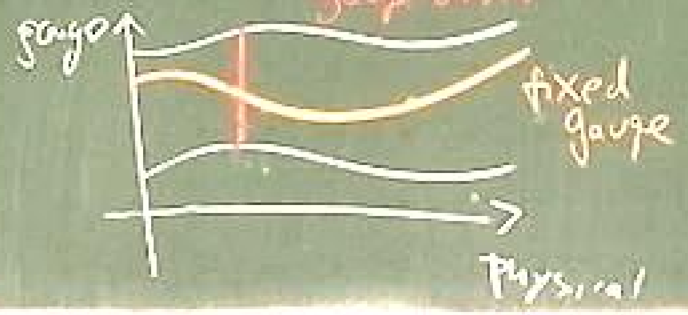
Goal: Propagator $\frac{-ig_{\mu\nu}}{k^2+i\epsilon}$

1) $S[A] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^2\right) \underset{\uparrow \text{FT, PI}}{=} \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde A_\mu(k)\left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right)\tilde A_\nu(-k)$

2) $\tilde A_\mu(k) = k_\mu\, \alpha(k) \longrightarrow S[A] = 0 \to \int \mathcal{D}A\, e^0 = \infty$

3) Problem: Gauge invariance. $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha$

($A_\mu = \partial_\mu \alpha \overset{\text{gauge}}{\longleftrightarrow} A_\mu \equiv 0$ → Overcounting of gauge-equivalent configurations!)



4) Solution: Count each physical configuration once → Faddeev & Popov procedure

i) Gauge fixing: $G(A) \doteq 0$ (Ex. $G(A) = \partial_\mu A^\mu$)

ii) $A_\mu^\alpha := A_\mu + \frac{1}{e}\partial_\mu \alpha$

$$1 = \int \mathcal{D}\alpha\, \delta\big(G(A^\alpha)\big) \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right)$$

$$1 = \left[\prod_i \int dg_i\right] \delta^{(n)}(\vec g) \underset{\uparrow\, \vec g = \vec g(\vec a)}{=} \left[\prod_i \int da_i\right] \delta^{(n)}\big(\vec g(\vec a)\big) \det\left(\frac{\partial \vec g}{\partial \vec a}\right)$$

iii) Assume $\frac{\delta G(A^\alpha)}{\delta\alpha}$ independent of A and α $\tilde{A} - \frac{1}{e}\partial\alpha$

iv) $$\int \mathcal{D}A\, e^{iS[A]}\, 1 = \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\int \mathcal{D}\alpha \int \mathcal{D}A\, \underbrace{e^{iS[A]}}_{O(A)}\, \delta(G(\underbrace{A^\alpha}_{\tilde{A}}))$$

$O(A)$

$\tilde{A} = A^\alpha = A + \frac{1}{e}\partial\alpha$, $\mathcal{D}\tilde{A} = \mathcal{D}A$

$(\tilde{x} = x + a \quad d\tilde{x} = dx)$ $O(A) = O(\tilde{A})$

Gauge invariance. $S[\tilde{A}] = S[A]$ $O(\tilde{A})$

$$= \det\left(\frac{\delta G}{\delta\alpha}\right)\underbrace{\int \mathcal{D}\alpha}_{\substack{\infty \\ \text{(Gauge orbit)}}}\underbrace{\int \mathcal{D}\tilde{A}\, e^{iS[\tilde{A}]}\, \delta(G(\tilde{A}))}_{\substack{\text{Only physically} \\ \text{distinct configurations}}}$$

v) $G(A) = \partial^\mu A_\mu - \omega(x)$

$$\rightarrow \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) = \det\left(\frac{1}{e}\partial^2\right)$$

[$G(A^\alpha) = \partial^\mu\left(A_\mu + \frac{1}{e}\partial_\mu\alpha\right) - \omega(x)$

$\frac{\delta G}{\delta\alpha} = \frac{1}{e}\partial^2$]

$$= \det\left(\frac{1}{e}\partial^2\right)\left(\int \mathcal{D}\alpha\right)\int \mathcal{D}A\, e^{iS[A]}\, \delta(\partial^\mu A_\mu - \omega(x)) \ominus$$

vi) Normalization

$$\ominus\, \underbrace{N(\xi)\int \mathcal{D}\omega\, e^{-i\int d^4x \frac{\omega^2}{2\xi}}}_{1}$$

$$\times \det\left(\frac{1}{e}\partial^2\right)\left(\int \mathcal{D}\alpha\right)\int \mathcal{D}A\, e^{iS[A]}\, \delta(\partial^\mu A_\mu - \omega(x))$$

$$= N(\xi)\det\left(\frac{1}{e}\partial^2\right)\left(\int \mathcal{D}\alpha\right)$$

$$\times \int \mathcal{D}A\, \underbrace{e^{iS[A]}}_{O(A)}\, \underbrace{\exp\left[-i\int d^4x\, \frac{(\partial^\mu A_\mu)^2}{2\xi}\right]}_{\substack{\text{New term} \\ \text{(breaks gauge inv.!)}}}$$

[See P-Set 13]

viii) & $O(\hat{A})$ gauge invariant operator:

$$O(\hat{A}^\alpha) = O(\hat{A})$$

$$\boxed{\langle\Omega|\mathcal{T}\,O(\hat{A})|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}A\, O(A)\, e^{i\int_{-T}^{T} d^4x\left[\mathcal{L}-\frac{1}{2\xi}(\partial^\mu A_\mu)^2\right]}}{\int \mathcal{D}A\, e^{i\int_{-T}^{T} d^4x\left[\mathcal{L}-\frac{1}{2\xi}(\partial^\mu A_\mu)^2\right]}}}$$

Note. Unknown/infinite prefactors cancelled!

5) New action:

$$S_\xi[A] = \int d^4x\left[-\frac{1}{4}F_{\mu\nu}^2 \boxed{-\frac{1}{2\xi}(\partial^\mu A_\mu)^2}\right]$$

$$\overset{\text{P.I., FT}}{=} \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,\tilde{A}_\mu(k)\left[-k^2 g^{\mu\nu} + \left(1 \boxed{-\frac{1}{\xi}}\right)k^\mu k^\nu\right]\hat{A}_\nu(-k)$$

New!

6) Propagator:

$$D_F^{\mu\nu}(x-y) = \langle\Omega|\mathcal{T}\,A^\mu(x)A^\nu(y)|\Omega\rangle$$

$$\to \langle\Omega|\tilde{A}^\mu(k)\tilde{A}^\nu(q)|\Omega\rangle = 0 \quad k \neq q$$

$$\tilde{D}_F^{\mu\nu}(q) = \langle\Omega|A^\mu(q)\tilde{A}^\nu(-q)|\Omega\rangle \;\Leftrightarrow$$

$$\stackrel{\circ}{=} \frac{\int \mathcal{D}\tilde{A}\, \tilde{A}^\mu(q)\, \tilde{A}^\nu(-q)\, \exp\left\{ \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(k) \underbrace{\left[-k^2 g^{\mu\nu} + (1-\xi^{-1}) k^\mu k^\nu \right]}_{M^{\mu\nu}(k)} \tilde{A}_\nu(-k) \right\}}{\text{— " —}} \underset{\substack{\uparrow \\ \text{P-Set 12}}}{\stackrel{\circ}{=}} i \left(M^{-1}(q) \right)^{\mu\nu}$$

$$\mathcal{D}\tilde{A} = \prod_{\mu,\, k,\, k^0 > 0} d\left(\mathrm{Re}\,\tilde{A}^\mu(k)\right) d\left(\mathrm{Im}\,\tilde{A}^\mu(k)\right)$$

$$\tilde{A}^\nu(-q) = \left(\tilde{A}^\nu(q)\right)^*$$

$$\Rightarrow \boxed{\tilde{D}_F^{\mu\nu}(q) \stackrel{\circ}{=} \frac{-i}{q^2 + i\varepsilon} \left[g^{\mu\nu} - (1-\xi) \frac{q^\mu q^\nu}{q^2} \right]}$$

Z) Gauges:

- $\xi = 1$: $\boxed{\tilde{D}_F^{\mu\nu}(q) = \frac{-i g^{\mu\nu}}{q^2 + i\varepsilon}}$ (Feynman gauge)

- $\xi = 0$: $\tilde{D}_F^{\mu\nu}(q) = \frac{-i}{q^2 + i\varepsilon} \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right)$ (Landau gauge)

9 Non-Abelian Gauge Theories

Motivation:

i) Massless vector bosons
→ not spin ($SO(3)$) but helicity ($ISO(2)$)

ii)
$$\boxed{U(\Lambda) A_\mu(x) U^{-1}(\Lambda) = (\Lambda^{-1})_{\mu\nu} A^\mu(\Lambda x) + \partial_\mu \Omega(x,\Lambda)}$$
$$U(\Lambda)\psi(x)U^{-1}(\Lambda) = \Lambda^{-1}_{\frac{1}{2}}\psi(\Lambda x)$$

iii) Unitarity + Lorentz invariance
→ Gauge theory!

9.1. The Geometry of Gauge Invariance

1) Local $U(1)$ symmetry of Dirac field.
$$\tilde{\psi}(x) = e^{i\alpha(x)}\psi(x)$$
for arbitrary $\alpha(x): \mathbb{R}^{1,3} \to \mathbb{R}$

2) Goal: Construct invariant Lagrangian

3) No problem without derivatives.
Inv under global $U(1)$
→ Inv. local $U(1)$

Example: $\bar{\tilde{\psi}}\tilde{\psi} = \bar{\psi}\psi$

4) Directional derivative along $n \in \mathbb{R}^{1,3}$:
$$n^\mu \partial_\mu \psi := \lim_{\varepsilon \to 0} \frac{\psi(x+\varepsilon n) - \psi(x)}{\varepsilon}$$
→ $n^\mu \partial_\mu \psi$ has no simple transformation law under $U(1)$
$$(n^\mu \partial_\mu \tilde{\psi} \neq e^{i\alpha(x)} n^\mu \partial_\mu \psi)$$

5| ∃ "Comparator" $U: \mathbb{R}^{1,3} \times \mathbb{R}^{1,3} \to \mathbb{C}$

with transformation:

- $\tilde{U}(y,x) = e^{i\alpha(y)} U(y,x)\, e^{-i\alpha(x)}$
- $U(y,y) = 1$

($\cdot$ $U(y,x) = e^{i\phi(y,x)}$)

$\to$ $\psi(y)$ and $U(y,x)\,\psi(x)$ transform in the same way:

$$\boxed{\tilde{\psi}(y) = \underline{e^{i\alpha(y)}}\,\psi(y), \quad \tilde{U}(y,x)\,\tilde{\psi}(x) = e^{i\alpha(y)} U(y,x) \overbrace{e^{-i\alpha(x)} e^{i\alpha(x)}}^{1} \psi(x) = \underline{e^{i\alpha(y)}}\, U(y,x)\,\psi(x)}$$

6| Covariant derivative:

$$\boxed{n^\mu D_\mu \psi := \lim_{\varepsilon \to 0} \frac{\psi(x+\varepsilon n) - U(x+\varepsilon n, x)\,\psi(x)}{\varepsilon}}$$

$$n^\mu \tilde{D}_\mu \tilde{\psi} = \dots \underset{e^{i\alpha(x+\varepsilon n)}}{\tilde{\psi}} - \underset{e^{i\alpha(x+\varepsilon n)}}{\tilde{U}\,\tilde{\psi}}$$

$$= e^{i\alpha(x)}\, n^\mu D_\mu \psi$$