Recap:

8 Functional Methods

8.1 Path Integrals in QM

$$\boxed{U(x_a, x_b; T) = \langle x_b | e^{-\frac{i}{\hbar} HT} | x_a \rangle = \underbrace{\int_{x(0)=x_a}^{x(T)=x_b} \mathcal{D}x(t)}_{\text{Path integral}} \; e^{\frac{i}{\hbar} \overbrace{S[x(t)]}^{\text{classical action}}}}$$

Example 8.1: $L = \frac{m}{2}\dot{x}^2 - V(x)$

$\downarrow$ Time slicing

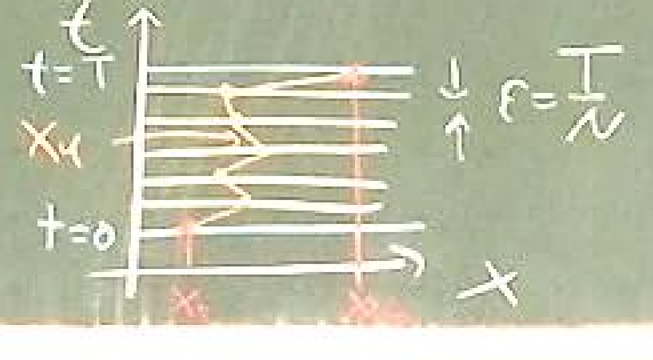
$$\boxed{i\hbar \partial_T U = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_b^2} + V(x_b) \right] U}$$

Schrödinger Equation

Path Integral Measure:

$$\int \mathcal{D}x(t) \equiv \lim_{N\to\infty} \frac{1}{C_\varepsilon} \int \frac{dx_1}{C_\varepsilon} \cdots \int \frac{dx_{N-1}}{C_\varepsilon}$$

C_ε: depends on theory



Generalization: [P-Set 12]

(Canonical quantization $\to$ Path Integral)

1] ∢ Coordinates q_i, conj. mom. p_i, Hamiltonian $H(\vec{q}, \vec{p})$

2] Canonical quantization.

$$[q_i, p_j] = i\hbar\delta_{ij} \;\to\; U(\vec{q}_a, \vec{q}_b; T) = \langle \vec{q}_b | e^{-\frac{i}{\hbar} HT} | \vec{q}_a \rangle$$

3] Time slicing: $e^{-\frac{i}{\hbar}\hat{H}T} \underset{\hbar=1}{=} \underbrace{e^{-iH\varepsilon} e^{-iH\varepsilon} \cdots e^{-iH\varepsilon}}_{\times N}$

4] Insert $N-1$ identities. $k = 1, \ldots, N-1$

$$\mathbb{1}_k = \int d\vec{q}_k \, |\vec{q}_k \rangle\langle \vec{q}_k|$$

$$\to \langle \vec{q}_{k+1} | e^{-i\hat{H}\varepsilon} | \vec{q}_k \rangle = \langle \vec{q}_{k+1} | \left[\mathbb{1} - i\hat{H}\varepsilon + O(\varepsilon^2) \right] | \vec{q}_k \rangle$$

5] $\hat{H} = \hat{H}_1(\vec{q}) + \hat{H}_2(\vec{p})$

$\langle \vec{q}_{n+1} | \hat{H}_2(\vec{p}) | \vec{q}_n \rangle \; e^{i \vec{q}_n \vec{p}_n}$

$\mathbb{1}_n = \int \frac{d\vec{p}_n}{(2\pi)^d} |\vec{p}_n\rangle\langle\vec{p}_n|$

$$\boxed{\langle \vec{q}_{n+1} | \hat{H} | \vec{q}_n \rangle \doteq \int \frac{d\vec{p}_n}{2\pi} H\left(\frac{\vec{q}_{n+1}+\vec{q}_n}{2}, \vec{p}_n \right) e^{i \vec{p}_n (\vec{q}_{n+1}-\vec{q}_n)}}$$

Weyl transform

6] Hamiltonian phase-space path integral

$$\boxed{U(\vec{q}_a, \vec{q}_b, T) = \langle \vec{q}_b | e^{-i\frac{T}{\hbar}H} | \vec{q}_a \rangle \doteq \underbrace{\int_{\vec{q}_a}^{\vec{q}_b} \mathcal{D}\vec{q}(t) \int \mathcal{D}p(t)}_{\lim_{n\to\infty} \prod_n \int \frac{dq_n \, dp_n}{2\pi\hbar} \text{ : Canonical measure}} \exp\left[\frac{i}{\hbar} \int_0^T dt \, \underbrace{\left(\vec{p}\dot{\vec{q}} - H(\vec{q},\vec{p}) \right)}_{\hat{=}\, L(\dot{\vec{q}},\vec{q}) \text{ with } \vec{p} = \frac{\partial L}{\partial \dot{\vec{q}}}} \right]}$$

$\equiv S[\vec{q},\vec{p}] \;\hat{=}\; S[\vec{q}]$

- $\frac{\delta S}{\delta \vec{p}} = \dot{\vec{q}} - \frac{\partial H}{\partial \vec{p}} \overset{!}{=} 0$

  $\to \dot{\vec{q}} = \frac{\partial H}{\partial \vec{p}}$

- $\frac{\delta S}{\delta \vec{q}} \overset{\text{P.I.}}{=} -\dot{\vec{p}} - \frac{\partial H}{\partial \vec{q}} \overset{!}{=} 0$

  $\to \dot{\vec{p}} = -\frac{\partial H}{\partial \vec{q}}$

8.2 Path Integral for Fields

Identification: $q_i \leftrightarrow \phi(x)$

Example 8.2: Real scalar field

$$\langle \phi_b | e^{-\frac{i}{\hbar} H T} | \phi_a \rangle$$

$$= \int_{\phi_a}^{\phi_b} \mathcal{D}\phi \int \mathcal{D}\pi \, \exp\left[\frac{i}{\hbar}\int_0^T d^4x \left(\pi\dot{\phi} - \frac{1}{2}\pi^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right)\right]$$

$$\stackrel{\circ}{=} \int_{\phi_a}^{\phi_b} \mathcal{D}\phi \, \exp\left[\frac{i}{\hbar}\int_0^T d^4x \, \mathcal{L}(\phi, \partial_\mu\phi)\right]$$

Evaluate π integral

Lagrangian

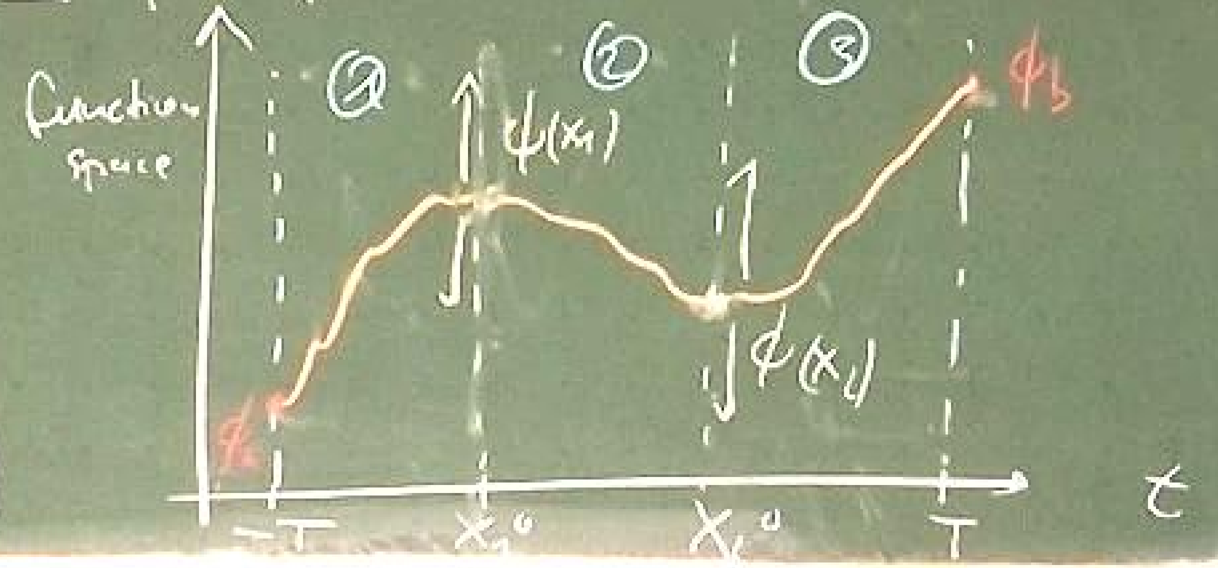
$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$$

Goal: Derive correlation functions:

1) $\langle \Omega | T \phi_H(x_1) \phi_H(x_2) | \Omega \rangle$

$$\overset{?}{\longleftrightarrow} \boxed{\int_{\phi(-T)=\phi_a}^{\phi(T)=\phi_b} \mathcal{D}\phi \, \phi(x_1)\phi(x_2) \, e^{i\int_{-T}^{T} d^4x \, \mathcal{L}} \qquad T > x_2^0 > x_1^0 > -T} \quad (*)$$

2) Split functional integral:



$$\int_{\phi_a}^{\phi_b} \mathcal{D}\phi = \int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \int_{\phi(x_1^0,\vec{x}) = \phi_1(\vec{x})}^{\phi(x_2^0,\vec{x}) = \phi_2(\vec{x})} \mathcal{D}\phi(x)$$

3) $(*) = \int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x})\, \phi_1(\vec{x}_1)\, \phi_2(\vec{x}_2)$

$$\times \underbrace{\langle \phi_b | e^{-iH(T-x_2^0)} | \phi_2 \rangle}_{3} \underbrace{\langle \phi_2 | e^{-iH(x_2^0-x_1^0)} | \phi_1 \rangle}_{2} \underbrace{\langle \phi_1 | e^{-iH(x_1^0+T)} | \phi_a \rangle}_{1}$$

Use. • $\hat{\phi}_S(\vec{x}_1)|\phi_1\rangle = \phi_1(\vec{x})|\phi_1\rangle$ • $\int \mathcal{D}\phi_1 |\phi_1\rangle\langle\phi_1| = \mathbb{1}$

$(\text{cf. } \hat{q}_i|\vec{q}\rangle = q_i|\vec{q}\rangle)$ $\left(\int d\vec{q}\, |\vec{q}\rangle\langle\vec{q}| = 1\right)$

$$\overset{x_2^0 > x_1^0}{=} \langle \phi_b | e^{-iH(T-x_2^0)} \underbrace{\phi_S(\vec{x}_2)\, e^{-iH(x_2^0-x_1^0)}}_{\phi_H(x_2)} \underbrace{\phi_S(\vec{x}_1)\, e^{-iH(x_1^0+T)}}_{\phi_H(x_1)} |\phi_a\rangle$$

$$= \langle \phi_b | \underbrace{e^{-iHT}}_{\to \propto |\Omega\rangle\langle\Omega|} T\{\phi_H(x_1)\phi_H(x_2)\} \underbrace{e^{-iHT}}_{\to \propto |\Omega\rangle\langle\Omega|} |\phi_a\rangle$$

$$\xrightarrow{T \to \infty(1-i\epsilon)} C \langle \Omega | T\{\phi_H(x_1)\phi_H(x_2)\} | \Omega \rangle$$

$$\boxed{\langle \Omega | T\{\hat{\phi}_H(x_1)\hat{\phi}_H(x_2)\} | \Omega \rangle = \lim_{T \to \infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)\, e^{i\int_{-T}^{T} d^4x\, \mathcal{L}(\phi)}}{\int \mathcal{D}\phi\, e^{i\int_{-T}^{T} d^4x\, \mathcal{L}(\phi)}}}$$

8.3 Application: Quantization of the EM field

Goal: Compute $\frac{-i g_{\mu\nu}}{k^2+i\varepsilon}$ using PIs

1| Action:

$$S[A] = \int d^4x \left[-\frac{1}{4} (F_{\mu\nu})^2 \right]$$

Partial integration, $A \xrightarrow{|x^\mu| \to \infty} 0$

$$\overset{\circ}{=} \frac{1}{2} \int d^4x \, A_\mu(x) \left(\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu \right) A_\nu(x)$$

Fourier transform

$$= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \underbrace{\tilde{A}_\mu(k) \left(-k^2 g^{\mu\nu} + k^\mu k^\nu \right) \tilde{A}_\nu(-k)}_{(*)}$$

2| Set $\tilde{A}_\mu(k) = k_\mu \, \alpha(k)$ ← arbitrary scalar function

$$\downarrow$$

$$(*) \equiv 0$$

$$\downarrow$$

$$S[A] = 0 \longrightarrow \int \mathcal{D}A \, \underbrace{e^0}_{1} = \infty$$

3| Problem: Gauge invariance.

$$A_\mu \longmapsto A_\mu + \frac{1}{e} \partial_\mu \alpha$$

→ Overcounting of gauge-equivalent configurations