Recap

7. Systematics of Renormalization

7.2 Renormalized Perturbation Theory

ϕ^4-theory in $d = 3+1$

rescaled field: $(\phi_r = \frac{1}{\sqrt{Z}}\phi)$

$$\mathcal{L} = \boxed{\frac{1}{2}(\partial\phi_r)^2 - \frac{m^2}{2}\phi_r^2} - \frac{\lambda}{4!}\phi_r^4$$
$$+ \underbrace{\boxed{\frac{1}{2}\delta_Z(\partial\phi_r)^2 - \frac{\delta_m}{2!}\phi_r^2 - \frac{\delta_\lambda}{4!}\phi_r^4}}_{\text{counterterms}}$$

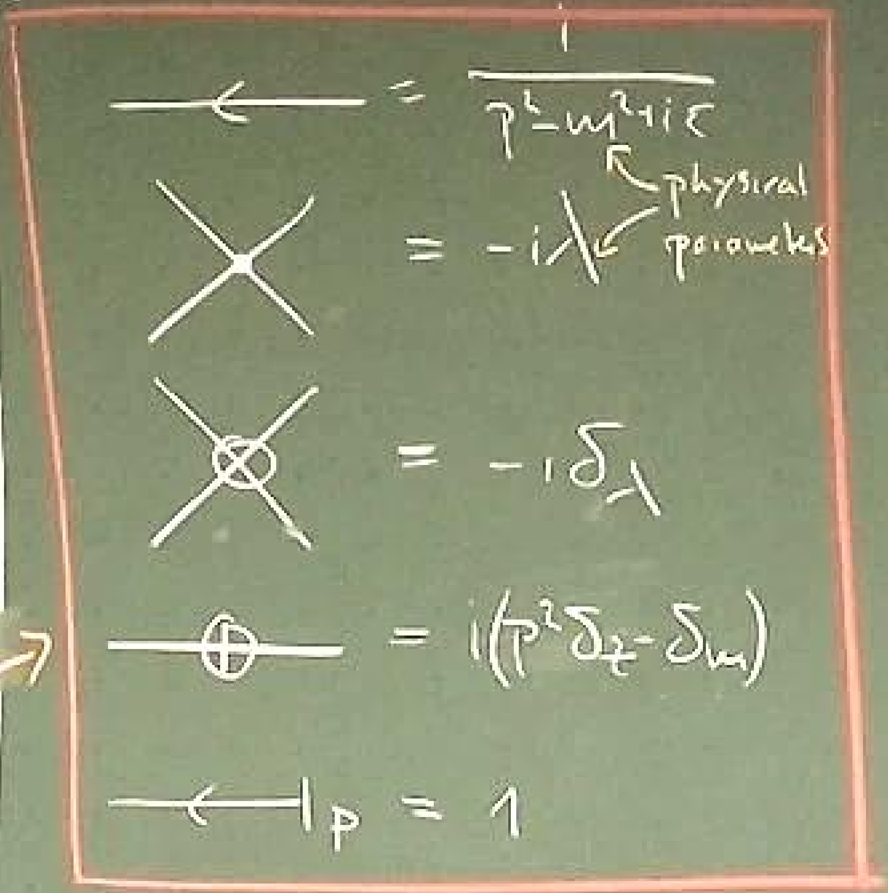
free theory

counterterms

(cutoff dependent & divergent)

perturbations

→ Feynman rules:



$= \frac{i}{p^2 - m^2 + i\varepsilon}$ ← physical parameters

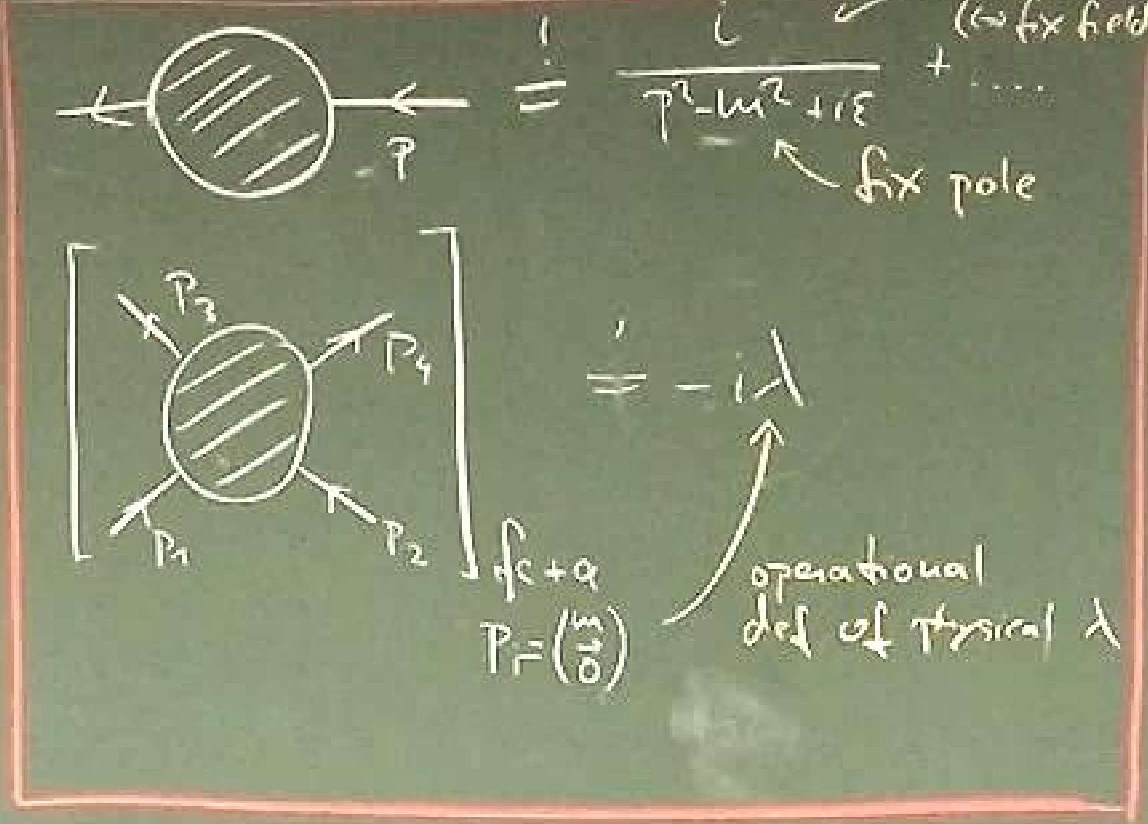
$= -i\lambda$

$= -i\delta_\lambda$

$= i(p^2\delta_Z - \delta_m)$

$p = 1$

Renormalization conditions:



$\overset{!}{=} \frac{i}{p^2 - m^2 + i\varepsilon} + \dots\dots$

fix residue ($\Leftrightarrow$ fix field ϕ_r)

fix pole

$\overset{!}{=} -i\lambda$

$p_i = \begin{pmatrix} m \\ \vec{0} \end{pmatrix}$

operational def of physical λ

Procedure:

(i) Sum relevant amplitudes.

(ii) Loopintegrals → Regulator Λ, ε

(iii) Amplitude.

$$\mathcal{M} = \mathcal{M}(p, m, \lambda, \delta_m, \delta_Z, \delta_\lambda, \Lambda)$$

(iv) Use renormalization conditions to solve for $\delta_m, \delta_Z, \delta_\lambda$:

$$\delta_\cdot = \delta_\cdot(m, \lambda, \Lambda)$$

divergent!

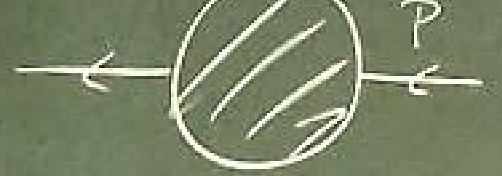
(v) Finite amplitude.

$$\mathcal{M} = \lim_{\Lambda\to\infty} \mathcal{M}(p, m, \lambda, \delta_\cdot(m, \lambda, \underline{\Lambda}), \underline{\Lambda})$$

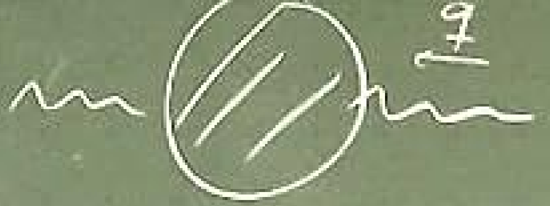
cancel!

Application to QED:

1| $\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \overline{\psi}(i\not\partial - m_0)\psi - e_0\overline{\psi}\gamma^\mu\psi A_\mu$

2| 

$$= \frac{i Z_2}{\not p - m} + \dots$$



$$= \frac{-i Z_3 g_{\mu\nu}}{q^2} + \dots$$

3| Renormalize fields:

$$\psi_r := \frac{1}{\sqrt{Z_2}}\psi, \quad A_r^\mu := \frac{1}{\sqrt{Z_3}}A^\mu$$

4| $\mathcal{L} = -\frac{1}{4}Z_3(F_r^{\mu\nu})^2 + Z_2\overline{\psi}_r(i\not\partial - m_0)\psi_r$
$- \boxed{\frac{e_0 Z_2 Z_3^{\frac{1}{2}}}{e}} e\overline{\psi}_r\gamma^\mu\psi_r(A_r)_\mu$

5' $\equiv Z_1$

free theory

6| $\mathcal{L} = \boxed{-\frac{1}{4}(F_r^{\mu\nu})^2 + \overline{\psi}_r(i\not\partial - m)\psi_r} - e\overline{\psi}_r\gamma^\mu\psi_r(A_r)_\mu$

$\boxed{-\frac{1}{4}\delta_3(F_r^{\mu\nu})^2 + \overline{\psi}(i\delta_2\not\partial - \delta_m)\psi_r - e\delta_1\overline{\psi}_r\gamma^\mu\psi_r(A_r)_\mu}$

with $\delta_i = Z_i - 1$, $\delta_m = Z_2 m_0 - m$

↓ 4 counter terms

7| Feynman rules:

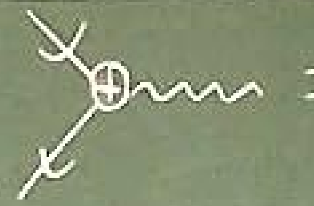
Edges: 

$$= \frac{i}{\not p - m + i\varepsilon}$$

$$= \frac{-i g_{\mu\nu}}{q^2 + i\varepsilon}$$

Vertices: 

$$= -ie\gamma^\mu$$

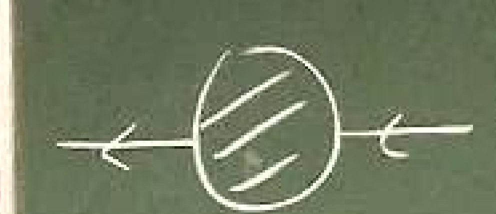


$$= -ie\delta_1\gamma^\mu$$

$$= -i(\not p\delta_2 - \delta_m)$$

$$= -i(g^{\mu\nu}q^2 - q^\mu q^\nu)\delta_3$$

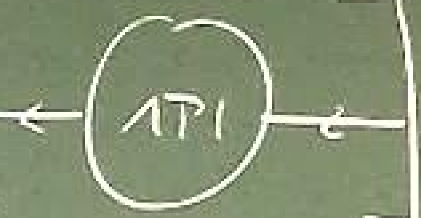
8) 4 counterterms → 4 renormalization conditions.

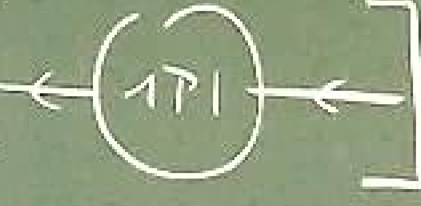


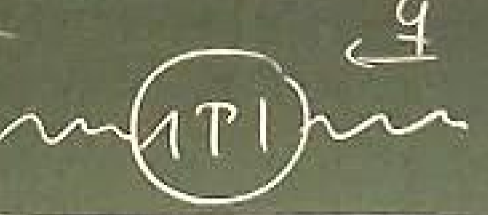
$$\overset{!}{=} \frac{1}{\not{p}-m} + \ldots$$

fixes pole

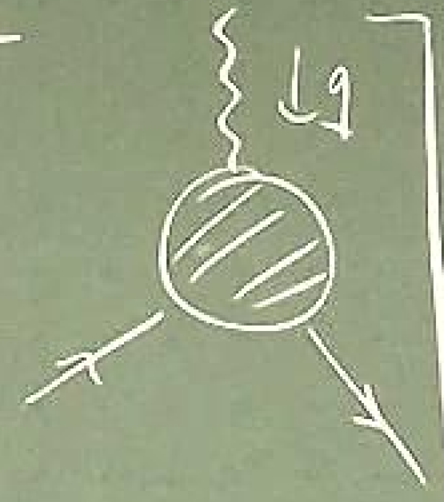
fixes residue (chooses ψ_r)

① $\Big[$  $\Big]_{\not{p}=m} = -i\Sigma(\not{p}=m) \overset{!}{=} 0$

② $\frac{d}{d\not{p}}\Big[$  $\Big]_{\not{p}=m} = -i \left.\frac{d\Sigma}{d\not{p}}\right|_{\not{p}=m} \overset{!}{=} 0$

③ $\Bigg[\frac{\text{diagram}}{(g^{\mu\nu}q^2 - q^\mu q^\nu)}\Bigg]_{q^2=0}$  $= i\,\Pi(q^2=0) \overset{!}{=} 0$

fixes residue of photon propagator (chooses A_r)

④ $\Big[$  $\Big]_{fc+a,\ q=0} = -ie\Gamma^\mu(q=0) \overset{!}{=} -ie\gamma^\mu$

$$\left.\frac{\delta F}{\delta x}\right|_{x=x_{cl}} \overset{!}{=} 0$$

$$F = \frac{S}{\hbar} = \frac{1}{\hbar}\int dt\, L(x(t))$$

((correspondence principle))

8 Functional Methods

- So far:

Hamiltonian → Canonical quantization

→ Feynman rules

- Alternative:

Lagrangian → Path integral

→ Feynman rules!

8.1 Path Integrals in Quantum Mechanics

1] Particle in 1D: $H = \frac{P^2}{2m} + V(x)$

2] Time evolution operator:

$$U(x_a, x_b, T) = \langle x_b | e^{-\frac{1}{\hbar} HT} | x_a \rangle$$

3] PI: alternative expression for U:

$$U(x_a, x_b, T) = \sum_{\substack{\text{All paths } x(t)\\ x(0) = x_a,\ x(T) = x_b}}' e^{iF[x(t)]} = \int \mathcal{D}x(t)\, e^{iF[x(t)]}$$

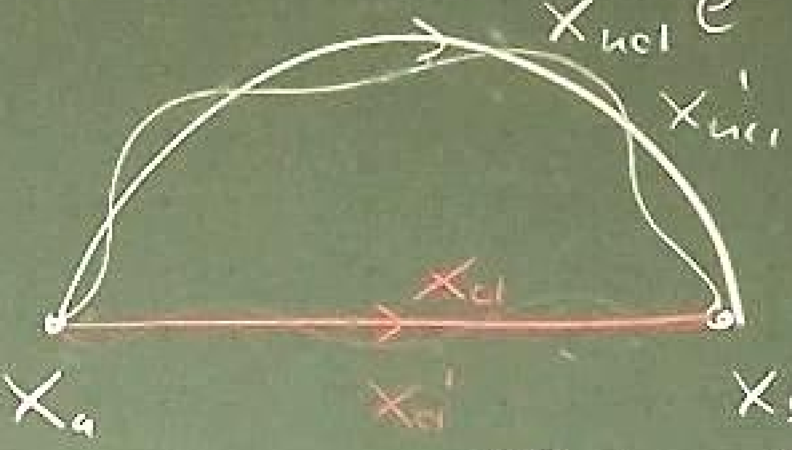
4] Conditions on F.

i] Describe system

ii] Function of path x(t)

iii] Classical path $x_{cl}(t)$ dominates for $\hbar \to 0$

$$U(x_a, x_b, T) \approx \sum_{x(t) \text{ close to } x_{cl}(t)}' e^{iF[x(t)]}$$



$$e^{iF[x_{ncl}]} + e^{iF[x_{ncl}']} \approx 0$$

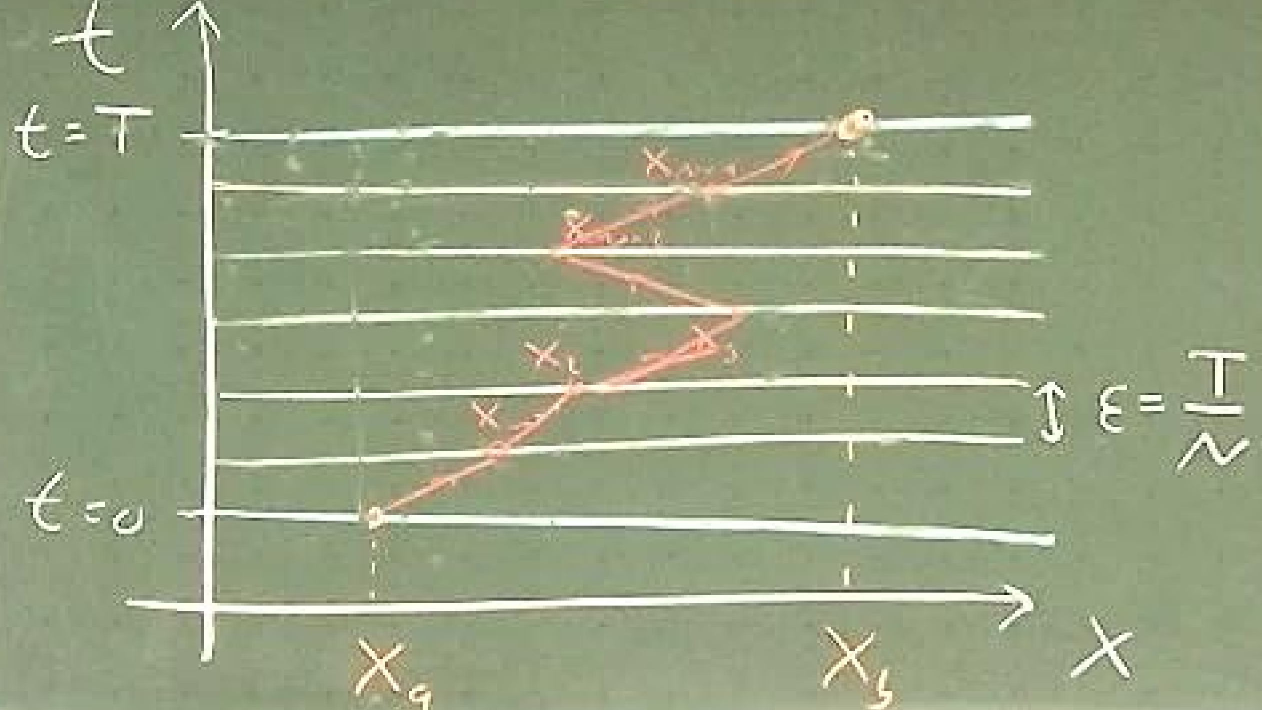
$$e^{iF[x_{cl}]} + e^{iF[x_{cl}']} \neq 0$$

5) Propagador.

$$\boxed{\mathcal{U}(x_a, x_b; T) = \int_{x(0)=x_a}^{x(T)=x_b} \mathcal{D}x(t)\; e^{\frac{i}{\hbar}S[x(t)]} \overset{?}{=} \langle x_b | e^{-\frac{i}{\hbar}HT} | x_a \rangle}$$

define?

6) Define $\mathcal{D}$ via time slices:



$H = P^2 + x^2$

$\frac{1}{i} + (px + xp)$

$$\int \mathcal{D}x(t) := \lim_{N\to\infty} \frac{1}{C_\varepsilon} \underbrace{\int \frac{dx_1}{C_\varepsilon} \cdots \int \frac{dx_{N-1}}{C_\varepsilon}}^{N-1}$$

$$= \lim_{N\to\infty} \frac{1}{C_\varepsilon} \prod_{k=1}^{N-1} \int \frac{dx_k}{C_\varepsilon}$$

$\varepsilon = \frac{T}{N}$, C_ε constant

Example 8.1:

1) Lagrangian. $L = \frac{m}{2}\dot{x}^2 - V(x)$

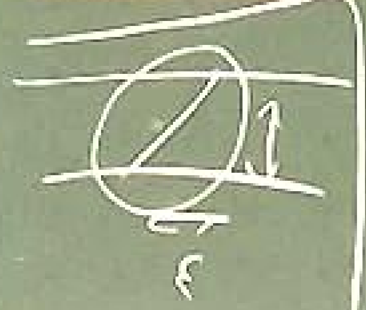
2) Action.

$$S = \int dt L \approx \sum_{k=0}^{N-1} \left[\frac{m}{2} \left(\frac{x_{k+1} - x_k}{\varepsilon} \right)^2 - \varepsilon V\left(\frac{x_{k+1} + x_k}{2} \right) \right]$$

3) Recursion.

$$\boxed{\mathcal{U}(x_a, x_b; T)} = \int_{-\infty}^{\infty} \frac{dx'}{C_\varepsilon} \exp\left[\frac{i}{\hbar} \frac{\overset{m + i\eta}{m}(x_b - x')^2}{2\varepsilon} - \frac{i}{\hbar} \varepsilon V\left(\frac{x_b + x'}{2} \right) \right]$$

$$\times\, \mathcal{U}(x_a, x', T - \varepsilon)$$



$x_b + (x' - x_b)$

Taylor at x_b

$V(x_b) + O(\varepsilon)$

$$= \int_{-\infty}^{\infty} \frac{dx'}{C_\varepsilon} \exp\left[\frac{i}{\hbar}\frac{m(x_b - x')^2}{2\varepsilon}\right]\left[1 - \frac{i}{\hbar}\varepsilon V(x_b) + \dots\right]$$

$$\times \left[1 + (x'-x_b)\frac{\partial}{\partial x_b} + \frac{(x'-x_b)^2}{2}\frac{\partial^2}{\partial x_b^2} + \dots\right] U(x_a, x_b, T-\varepsilon)$$

$$\overset{\circ}{=} \boxed{\frac{1}{C_\varepsilon}\sqrt{\frac{2\pi\hbar\varepsilon}{-im}}} \overset{!}{=} 1 \quad \left[1 - \frac{i}{\hbar}\varepsilon V(x_b) + \frac{i\hbar}{2m}\varepsilon\frac{\partial}{\partial x_b^2} + O(\varepsilon^2)\right] U(x_a, x_b, T-\varepsilon)$$

$\varepsilon \to 0$ on both sides

4] $\boxed{C_\varepsilon = \sqrt{\frac{2\pi\hbar\varepsilon}{-im}}}$ (depends on Lagrangian)

5] $U(x_a, x_b, T-\varepsilon) = \underbrace{U(x_a, x_b, T) - \varepsilon\,\partial_T U + O(\varepsilon^2)}$

$$\boxed{i\hbar\,\partial_T U = \underbrace{\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2} + V(x_b)\right]}_{H} U = HU}$$

Schrödinger equation

6] Initial condition:

$$U(x_a, x_b, \varepsilon) = \frac{1}{C_\varepsilon}\exp\left[\frac{i}{\hbar}\frac{m}{2\varepsilon}(x_b - x_a)^2 + O(\varepsilon)\right]$$

$$\overset{\varepsilon\to0}{\approx} \sqrt{\frac{-im}{2\pi\hbar\varepsilon}}\, e^{\frac{i}{\hbar}\frac{m}{2\varepsilon}(x_b - x_a)^2}$$

$$\xrightarrow{\varepsilon\to0} \delta(x_a - x_b) = U(x_a, x_b; 0) = \langle x_b | x_a \rangle$$

7] ▨