7. Systematics of Renormalization

7.1 Counting UV-divergences

Superficial degree of divergence:
\[ D_{\text{QED}} = \frac{1}{n-1} = \frac{1}{2} N = \frac{1}{2} m \]

Superficial dimension: \( \frac{1}{2} N = \frac{1}{2} m \)

\( \text{QED}; m = \frac{1}{2} \)

Classification:

\( \text{Non-renormalizable} \)

- Negative mass dimension (QED in d > 4)
  - All amplitudes diverge

\( \text{Renormalizable} \)

- Zero mass dimension (QED in d = 4)
  - Only finite number of amplitudes diverge

\( \text{Super-renormalizable} \)

- Positive mass dimension (QED in d < 4)
  - Only finite number of Feynman diagrams diverge
  - UV divergences can be "tamed" by renormalization procedure (today)
Renormalized Perturbation Theory

Goal. Compute finite predictions from given physical parameters, i.e., for $\Lambda \to \infty$

Recite (Basic perturbation theory)

(i) Compute amplitude. $M = M\left(w_0, e_0, \lambda \right) + O\left(w_0^3\right)$

(ii) Compute physical mass, charge, field strength renormalization.

$M = m_0 + O\left(w_0^3\right)$

$e = e_0 + O\left(w_0^3\right)$

$\lambda = \lambda_0 + O\left(w_0^3\right)$

(iii) Renormalization

Eliminate $w_0, e_0$ in favor of $M$ and $e$ by giving $e_0$ new components

$e = e_0(M, \omega, \lambda)$

$(\omega) = \frac{M}{M_0}(M, \omega, \lambda)$

Then

$M(M, \omega, \lambda)$ is finite and independent of $\Lambda$ in all order of $\lambda$.

Works for all renormalizable QFTs.

Alternative. Renormalized perturbation theory

$1. e^2\phi^2$-theory in $d = 3 + 1$

$\phi^4$ theory

$2. \phi^4 = 4 - N_d \rightarrow$ divergent amplitudes

$\Phi_0 = 4 \rightarrow$ square

$\Phi_0 = 2 \rightarrow \sim \frac{\Lambda^2}{\Phi_0^2} \sim \log \Lambda$

$\Phi_0 = 0 \rightarrow 3$ divergent quantities

$\rightarrow$ Absorb in $3$ unobservable parameters $\omega_0, \lambda_0$, field $\lambda$. 
2. Recall
\[ \int d^4x e^{ix\cdot\xi} \langle 0\left\vert \phi_1\phi_2\phi_3\phi_4\right\rangle = \frac{i}{p^2-m^2 + \ldots} \]
Absorb Z by rescaled fields:
\[ \phi_t \equiv \frac{1}{Z} \phi \]

4. Lagrangian
\[ L = \frac{1}{2} \partial^\mu \phi^\dagger \partial_\mu \phi + \frac{1}{2} m^2 \phi^\dagger \phi - \frac{1}{4} \phi^\dagger \phi \]

\[ \Delta \]

\[ \text{physical parameters} \]
\[ (\text{fixed}) \]

\[ \Delta \]

\[ \phi^\dagger \partial_\mu \partial^\mu \phi - \frac{1}{8} \phi^\dagger \phi^3 \]
\[ \text{Counter terms} \]
\[ \Delta \]
\[ \text{(cut-off dependent)} \]
\[ \Delta \]

\[ \Delta , \delta m, \delta A \text{ absorbs diverging shifts of bare and physical quantities} \]

6. Renormalisation conditions:
\[ \text{(Experimental result)} \]
\[ \gamma^\text{residue} + \text{loop pole} \]
1. Perturbation theory of \( x \)

- Feynman rules for renormalized perturbation theory.

1. Edge: \( x = -i \lambda \)

2. Value: \( x = -i \delta \)

   i. Sum all relevant diagrams → introduce regulator
   ii. If loop integrals change → introduce \( \delta \) regulator
   iii. Results depend on \( \delta \) for physical processes when \( \delta \to 0 \)
   iv. Choose \( \delta \) such that renormalization conditions are satisfied

5. External lines: \( P = 1 \)

6. Impose normal counterterms

6. Integrate momenta

6. Divide by sym. factor

1. Basic perturbation theory and renormalized theory are equivalent

1. Example:

   \[ M(p_p, -p_h) = \frac{1}{p_p^2 - m^2} \]

   \[ \approx \]

   \[ \approx X + \ldots \]

   \[ \approx \]

   \[ \frac{d^2 x}{dx} = 0(x^2) \]
Mandula variables:

\[ s = (p_1 + p_2)^2 \]
\[ t = (p_1 - p_3)^2 \]
\[ u = (p_2 - p_3)^2 \]

Evaluate loop integral:

\[ \frac{1}{M^2} V(4\omega) + 2V(0) \]

\[ \Delta = -\lambda^2 \left[ V(4\omega) + 2 V(0) \right] \]

\[ \frac{\lambda^2}{32\pi^2} \int_{0}^{\infty} dx \left\{ \frac{e^{-x}}{x} - 3x + 3\log(4\pi^2) - \log (m^2 - x^2 - r^2) \right\} \]

[Diagram]

(0) With these delta \( \delta \) the amplitude is finite (independent of regulator).

(1) Bare perturbation theory and renormalized \( \Delta \) are equivalent.

(1a) Example:

\[ \Delta'M(p_1, p_2, r_1, r_2) \]

\[ \approx \frac{1}{16\pi^2} \]

\[ \Delta X + \Delta X \]

\[ \Delta \approx 0(\lambda^2) \]
Amplitude:

\[ M(\pi^+\pi^- \to \rho^-) = -i G \int \frac{d^3 p \, f_{\pi\rho\pi}}{2 \pi^2 \nu} \]

Important: Regularize $\mu$ drops out. finite function of momenta (and $\omega$)

Enforce renormalization condition:

\[ \left. M^2(p) \right|_{p^2=m^2} = 0 \]

\[ \frac{dM^2}{dp^2} = \frac{d^4 \Gamma}{d^4 p} \]

\[ \Gamma_{\pi^+\pi^-} = \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2-m^2} \]

\[ S_\text{A} = \frac{1}{2} \left( \frac{p^4}{m^2} \right) \]

\[ S_{\text{A}} = 0 \quad \text{and} \quad S_{\text{B}} = \frac{1}{2} \left( \frac{p_1 \cdot p_2}{m^2} \right) \]

\[ S_{\text{C}} = \frac{1}{2} \left( \frac{p_1 \cdot p_2}{m^2} \right) \]

\[ S_{\text{D}} = \frac{1}{2} \left( \frac{p_1 \cdot p_2}{m^2} \right) \]

\[ S_{\text{E}} = \frac{1}{2} \left( \frac{p_1 \cdot p_2}{m^2} \right) \]