Recap

7. Systematics of Renormalization

7.1 Counting UV-divergences

Superficial degree of divergence:

$$D_{QED} = dL - P_e - 2P_\gamma = d - \left(\frac{4-d}{2}\right)V - \left(\frac{d-2}{2}\right)N_\gamma - \left(\frac{d-1}{2}\right)N_e$$

\# vertices — # external photons — # external electrons

spacetime dimension, Jacobian + integration, electron propagator, photon propagator, mass dimension of coupling constant

Classification:

Super-renormalizable:

- Positive mass dimension (QED in $d<4$)
- Only finite number of Feynman diagrams diverge

Renormalizable:

- Zero mass dimension (QED in $d=4$)
- Only finite number of amplitudes diverge

Non-renormalizable:

- Negative mass dimension (QED in $d>4$, Einstein gravity)
- All amplitudes diverge

UV div can be "tamed" by renormalization procedure (today)

QED in $d=4$.

3 div amplitudes with 4 div quantities.

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$$\sim \overbrace{\text{const} \log\Lambda}^{a_1(\Lambda)} + \not{p}\, \overbrace{\text{const} \log\Lambda}^{a_2(\Lambda)}$$

- 

$$\sim \underbrace{-ie\gamma^\mu \log\Lambda}_{a_3(\Lambda)}$$

- 

$$\sim (g^{\mu\nu}q^2 - q^\mu q^\nu)\, \overbrace{\text{const} \log\Lambda}^{a_4(\Lambda)}$$

7.2 Renormalized Perturbation Theory

Goal: Compute finite predictions from given physical parameters m, e for $\Lambda \to \infty$

Recipe (Bare perturbation theory)

(i) Compute amplitude. ← UV divergent $\Lambda \to \infty$

$$\mathcal{M} = \mathcal{M}(m_0, e_0; \Lambda) + O(\alpha_0^n)$$

(ii) Compute physical mass, charge, field strength renormalization.

$$m = m(m_0, e_0, \Lambda) + O(\alpha_0^n)$$
$$e = e(m_0, e_0; \Lambda) + O$$
$$Z = Z(m_0, e_0; \Lambda) + O \quad \leftarrow \text{only needed scattering amplitudes}$$

(iii) Renormalization: Eliminate m_0, e_0 in favour of m and e ← given by experiments

$$e_0 = e_0(m, e, \Lambda)$$
$$m_0 = m_0(m, e, \Lambda)$$

(iv) Then

$$\boxed{\mathcal{M}(m, e) \equiv \lim_{\Lambda \to \infty} \mathcal{M}(m_0(m, e, \Lambda), e_0(m, e, \Lambda); \Lambda)}$$

is finite and independent of Λ in all orders of α

⇓

Works for all renormalizable QFTs

Alternative: Renormalized perturbation theory

1) $\lambda \phi^4$-theory in $d = 3+1$

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2} m_0^2 \phi^2 - \frac{\lambda_0}{4!}\phi^4$$

2) $D_{\phi^4} = 4 - N_\phi$ → divergent amplitudes

- $D_{\phi^4} = 4$  ignore
- $D_{\phi^4} = 2$  $\sim \underline{\Lambda^2} + \underline{p^2 \log \Lambda}$
- $D_{\phi^4} = 0$  $\sim \underline{\log \Lambda}$

→ 3 divergent quantities

→ Absorb in 3 unobservable parameters: m_0, λ_0, field ϕ

3| Recall:

$$\int d^4x\, e^{ipx}\langle\Omega|T\phi(x)\phi(0)|\Omega\rangle = \frac{iZ}{p^2-m^2} + \dots\dots$$

Absorb Z in rescaled fields:

$$\boxed{\phi_r \equiv \frac{1}{\sqrt{Z}}\,\phi}$$

$$\to \int d^4x\, e^{ipx}\langle\Omega|T\phi_r(x)\phi_r(0)|\Omega\rangle = \frac{i}{p^2-m^2} + \dots$$

4| Lagrangian:

$$\mathcal{L} = \frac{1}{2}Z(\partial_\mu\phi_r)^2 - \frac{1}{2}m_0^2 Z\phi_r^2 - \frac{\lambda_0}{4!}Z^2\phi_r^4$$

5| $$\mathcal{L} = \boxed{\frac{1}{2}(\partial\phi_r)^2 - \frac{1}{2}m^2\phi_r^2 - \frac{\lambda}{4!}\phi_r^4}$$ physical parameters (fixed)

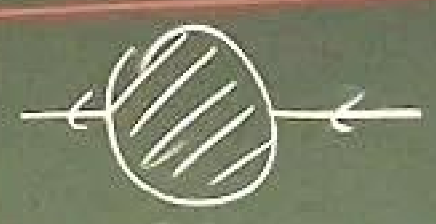
(*) $$+\boxed{\underbrace{\frac{1}{2}(Z-1)}_{\delta_Z}(\partial\phi_r)^2 - \frac{1}{2}\underbrace{(m_0^2 Z - m^2)}_{\delta_m}\phi_r^2 - \frac{1}{4!}\underbrace{(\lambda_0 Z^2 - \lambda)}_{\delta_\lambda}\phi_r^4}$$

Counter terms (cutoff dependent)

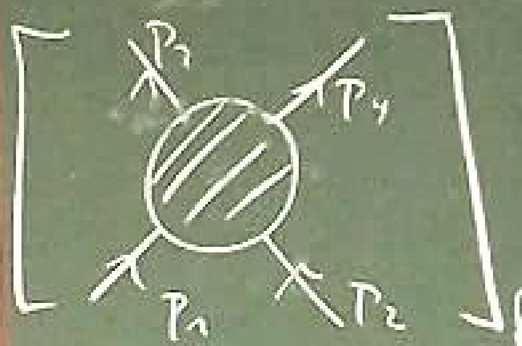
$\to \delta_Z, \delta_m, \delta_\lambda$ absorb diverging shifts of bare and physical quantities

6| Renormalization conditions:

(Experimental input) fix residue + pole



$$\doteq \frac{i}{p^2-m^2} + \dots$$



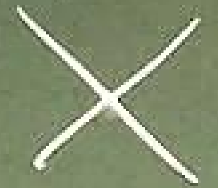
$$\Big[\;\cdots\;\Big]_{fc+a} \doteq -i\lambda$$

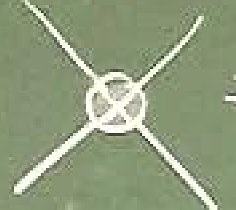
$p_i = (m, \vec{0})$

7| Perturbation theory of (*)

→ Feynman rules for renormalized perturbation theory.

1. Edges:  $= \dfrac{i}{p^2 - m^2 + i\varepsilon}$

2. Vertices:  $= -i\lambda$

 $= -i\delta_\lambda$

 $= i(p^2\delta_Z - \delta_m)$

3. External lines:  $= 1$
4. Impose momentum conservation
5. Integrate momenta
6. Divide by sym. factor.

8| Procedure for computing amplitudes.

(i) Sum all relevant diagrams

(ii) If loop integrals diverge → introduce regulator

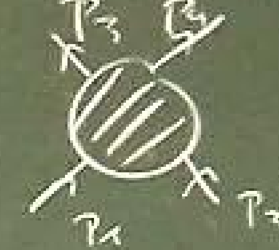
(iii) Results depend on $\{\delta_{\lambda, m, Z}\}$, physical parameters $\{m, e\}$ and regulator $\{\Lambda, \varepsilon\}$

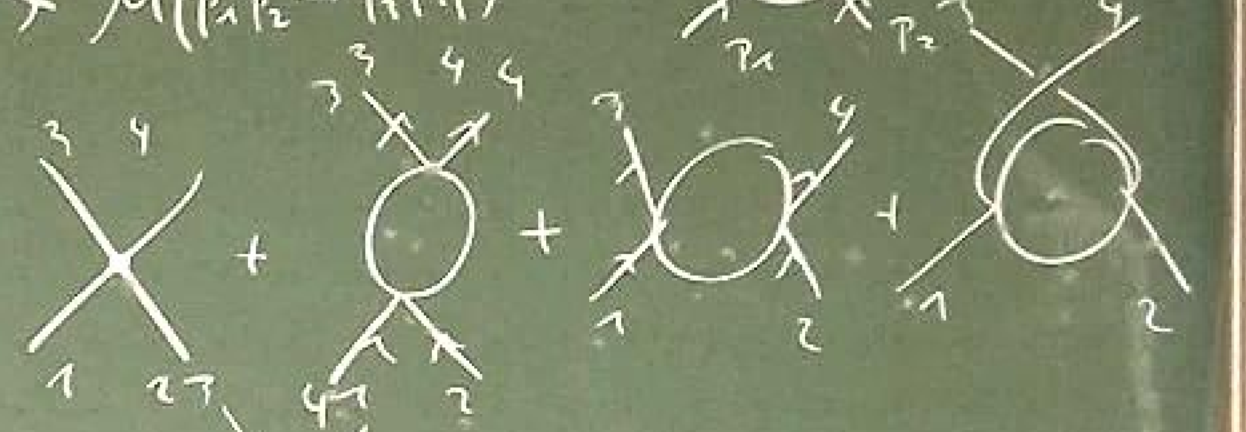
(iv) Choose $\{\delta_\cdot\}$ such that renormalization conditions are satisfied

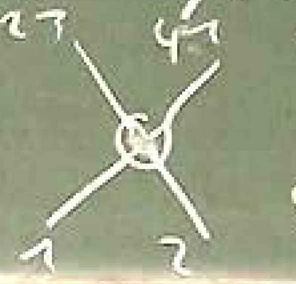
(v) With these $\{\delta_\cdot\}$ the amplitude is finite (independent of regulator)

9| Bare perturbation theory and renormalized —"— are equivalent

10| Example:

1| $i\mathcal{M}(p_1 p_2 \to p_3 p_4) =$ 

$\approx$ 

$+$ 

← will see. $\delta_\lambda = O(\lambda^2)$

$$\doteq -i\lambda + (-i\lambda)^2\left[iV(s) + iV(t) + iV(u)\right] \boxed{-i\delta_\lambda} = \frac{(-i\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}\frac{i}{(k+p)^2-m^2}$$

Mandelstam variables:

$s = (p_1+p_2)^2$

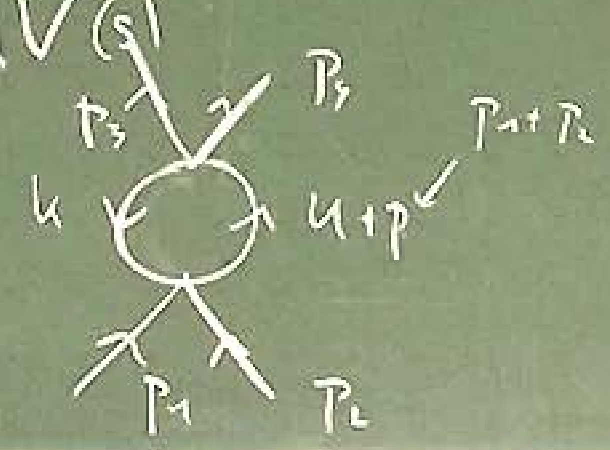
$t = (p_3-p_1)^2$

$u = (p_4-p_1)^2$

ii] Evaluate loop integral.

$(-i\lambda)^2\, iV(s)$

$=$



Feynman paramtr, substitution, Wick rotation, Dimensional regularisation

$$\underset{\varepsilon\to 0}{\sim} -(-i\lambda)^2\frac{i}{32\pi^2}\int_0^1 dx\left\{\frac{2}{\varepsilon} - \gamma + \log(4\pi) - \log\left[m^2 - x(1-x)p^2\right]\right\}$$

iii] Enforce renormalization condition $\left(\quad \Big|_{p=0} \doteq -i\lambda \right)$ to compute δ_λ

$$\boxed{i\mathcal{M}\Big|_{s=4m^2,\, t=0,\, u=0} \doteq -i\lambda}$$

$$\delta_\lambda = -\lambda^2\left[V(4m^2) + 2V(0)\right]$$

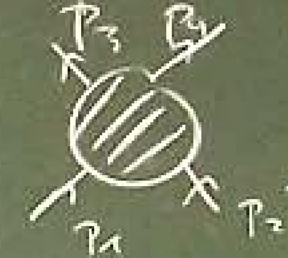
$$\sim \boxed{\lambda^2}\frac{1}{32\pi^2}\int_0^1 dx\left\{\frac{6}{\varepsilon} - 3\gamma + 3\log(4\pi) - \log\left[m^2 - x(1-x)4m^2\right] - 2\log\left[m^2\right]\right\}$$

(v) With these $\{\delta_i\}$ the amplitude is finite (independent of regulator)

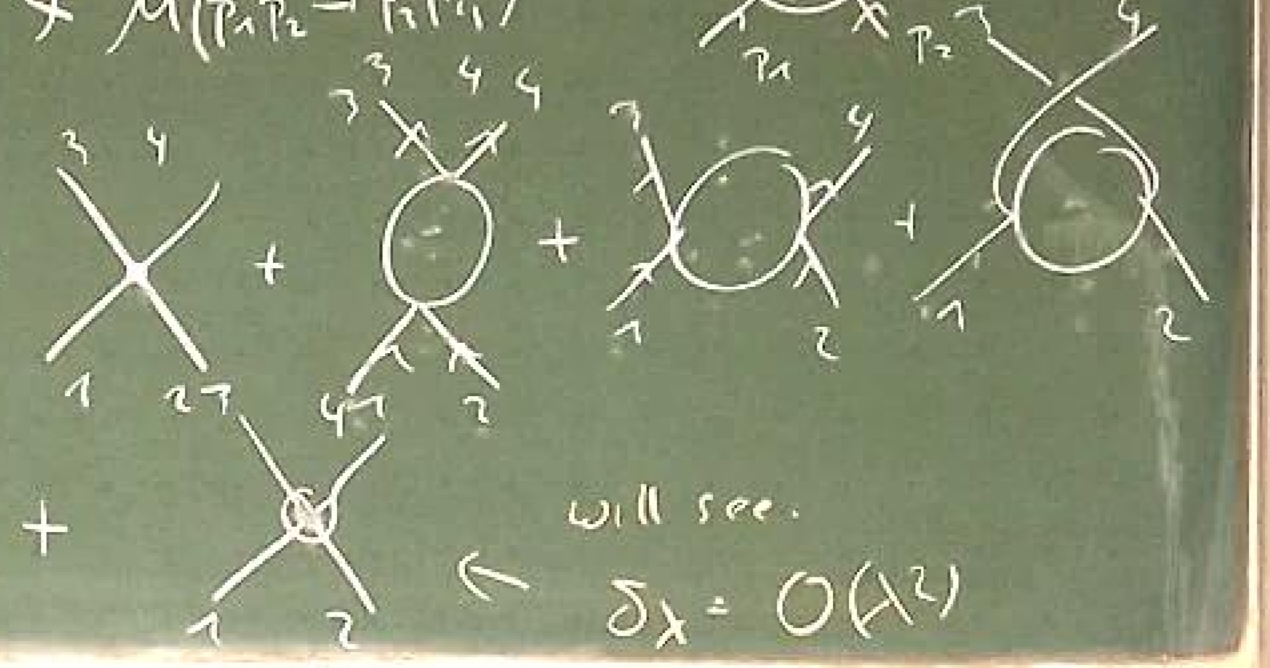
9] Bare perturbation theory and renormalized —"— are equivalent

10] Example.

1] $i\mathcal{M}(p_1p_2\to p_3p_4) =$



$\approx$



will see. $\delta_\lambda = O(\lambda^2)$

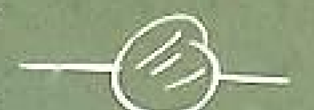
IV) Amplitude

$$i\mathcal{M}(p_1 p_2 \to p_3 p_4) = -i\lambda - i\lambda^2 \mathcal{F}(\{p_i\}, m)$$

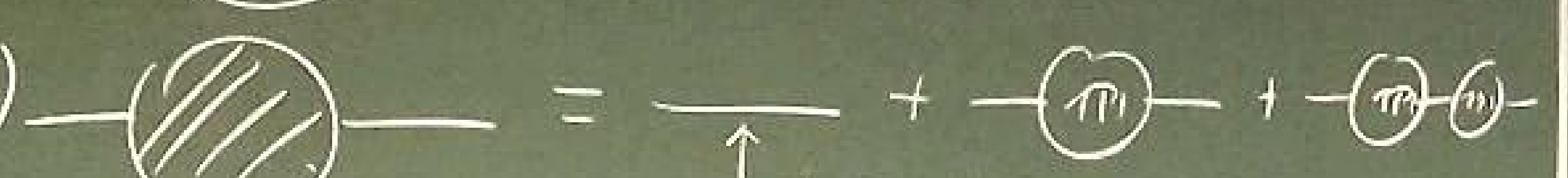
finite function of momenta (and m)

Important: regulator ε drops out.

V) Enforce renormalization condition:

 $\doteq \frac{i}{p^2-m^2}$

a)  $:= -i\mathcal{M}^2(p^2)$

b) 

$$= \frac{i}{p^2-m^2-\mathcal{M}^2(p^2)} \doteq \frac{i}{p^2-m^2} + \dots$$

$\uparrow \frac{i}{p^2-m^2}$

c) • $\left.\mathcal{M}^2(p^2)\right|_{p^2=m^2} \doteq 0$ fixes pole

• $\left.\frac{d\mathcal{M}^2(p^2)}{dp^2}\right|_{p^2=m^2} \doteq 0$ fixes residue (*)

$$\mathrm{Res}_a \frac{1}{f} = \frac{1}{f'(a)} = 1 \Leftrightarrow f'(a) = 1$$

d) 

$$-i\mathcal{M}^2(p^2) \simeq \dots = (-i\lambda)\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2} + i(p^2\delta_Z - \delta_m)$$

$\Downarrow$ (*)

$$\boxed{\delta_Z = 0 \text{ and } \delta_m = -\frac{\lambda}{2}\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(1-\frac{d}{2})}{(m^2)^{1-d/2}}}$$