7 Systematics of Renormalization

IR-Divergences: Cancel in complete amplitudes (Bremsstrahlung)

↳ not a fundamental problem

UV-Divergences: Diverging physical resp. bare quantities (mass, charge, field strength)

↳ fundamental problem

$\Downarrow$

$\boxed{\text{Study UV-divergences systematically}}$

(Ultimate Solution: Renormalization)

7.1 Counting UV-Divergences

1| Goal: Classify UV-div. in QED

2| Def

N_e = # ext. el. lines

N_γ = # ext. phot. lines

P_e = # electr. prop. $\rightarrow \prod^{P_e} \frac{1}{\not{k}_i - m}$

P_γ = # phot prop $\rightarrow \prod_i^{P_\gamma} \frac{1}{k_i^2}$

V = # vertices

L = # independent loops $\rightarrow \prod_i^{L} \int \frac{d^4k_i}{(2\pi)^4}$

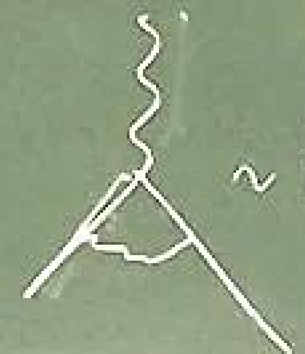
3| Superficial degree of divergence

$$\boxed{D_{QED} = L(3+1) - (P_e + 2P_\gamma)}$$

Intuition:

$$D_{QED} \begin{cases} >0: & \text{Divergence with } \Lambda^{D_{QED}} \\ =0: & \text{Divergence with } \log\Lambda \\ <0: & \text{No divergence} \end{cases}$$

Example:



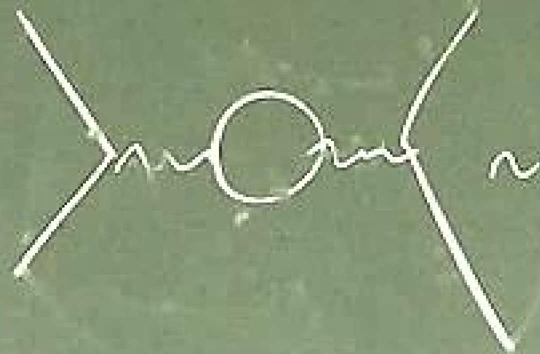
$\sim \log\Lambda$ and $D_{QED} = 4-(2+2\cdot 1) = 0$

However: Not always correct

•  $\sim \log\Lambda$

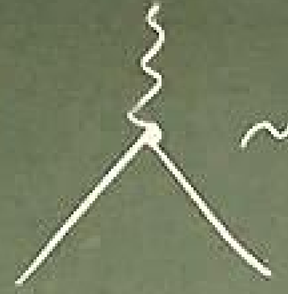
$D_{QED} = 4-(2+2\cdot 0) = 2$

→ Div. is weaker than expected due to symmetry

•  $\sim \log\Lambda$ but $D_{QED} = 4-(2+2\cdot 2)$

$= -2 < 0$

→ Diverging subdiagrams can make diagrams diverge despite $D_{QED} < 0$.

•  ~ 1 but. $D_{QED} = 4\;0-(0+2\cdot 0) = 0$

→ tree-level diagrams do not diverge

4] • $L = P_e + P_\gamma - V + 1$

• $V = 2P_\gamma + N_\gamma = \frac{1}{2}(2P_e + N_e)$

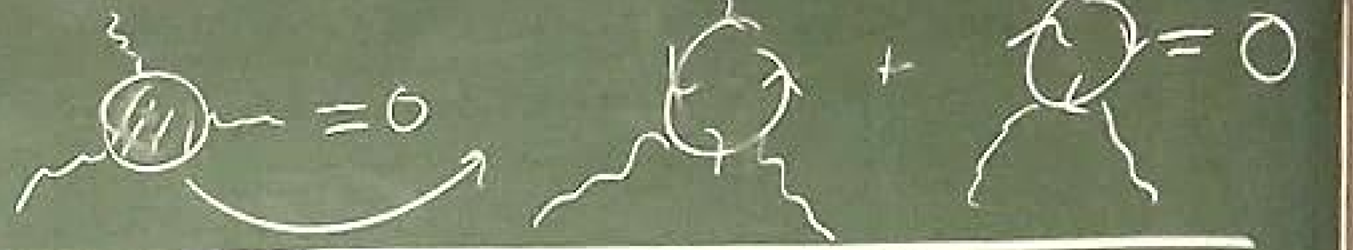
$\Rightarrow \boxed{D_{QED} \doteq 4 - N_\gamma - \frac{3}{2}N_e}$

→ independent of number of vertices!

5] Aside. Furry's theorem.

$\boxed{\text{Sum of all Feynman diagrams with and odd number of photons as only external lines vanishes.}}$

Example.



6) Enumerate all amplitudes with D_{QED}:

$$\boxed{D_{QED} = 4 - N_\gamma - \frac{3}{2} N_e}$$

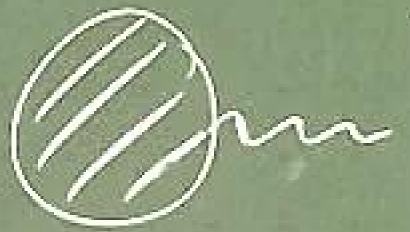
i) $\boxed{N_e = 0}$

a) $N_\gamma = 0 \to D = 4$

$\sim$ badly divergent

↳ Unobservable vacuum energy shift → Ignore

b) $N_\gamma = 1 \to D = 3$



$\overset{\text{Furry}}{=} 0$

c) $N_\gamma = 2 \to D = 2$

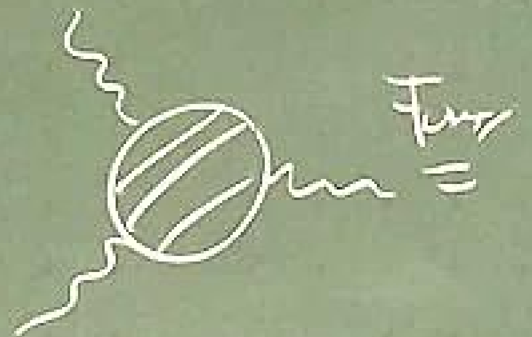
$= (g^{\mu\nu} q^2 - q^\mu q^\nu)\,\Pi(q^2)$

(1)

$\sim (g^{\mu\nu} q^2 - q^\mu q^\nu)\left(\frac{\text{const.}}{\varepsilon}\right)$

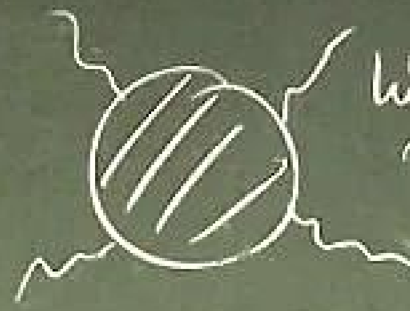
$\underbrace{\text{const.}\log\Lambda}_{a_0(\Lambda)}$

d) $N_\gamma = 3 \to D = 1$



$\overset{\text{Furry}}{=} 0$

e) $N_\gamma = 4 \to D = 0$



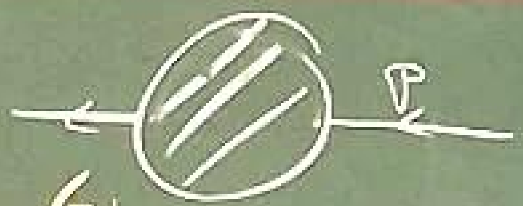
Wait. ~ 1

$\sim \frac{\alpha^2}{m_e^4}$

80 eTV

ii) $\boxed{N_e = 2}$

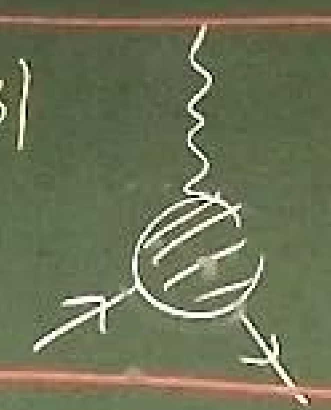
a) $N_\gamma = 0 \to D = 1$



$\sim \underbrace{\text{const}\log\Lambda}_{a_1(\Lambda)} + \cancel{p}\,\underbrace{\overset{\text{const}}{\log\Lambda}}_{a_2(\Lambda)}$

(2)

b) $N_\gamma = 1 \to D = 0$



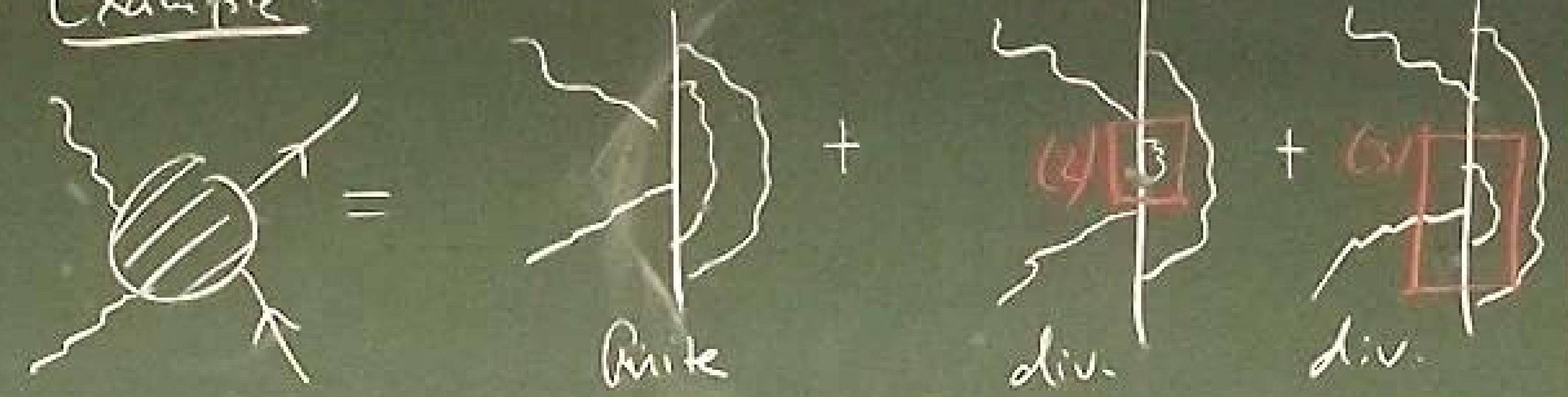
$$\sim \frac{-ie\gamma^\mu \log\Lambda}{a(\Lambda)}$$

- Weinberg theorem:

Diagrams in QED diverge if and only if they contain (1) (2) and/or (3) as subdiagrams.

→ QED has only 4 diverging quantities: a_0, a_1, a_2, a_3.

Example:



7] Idea: Absorb finite number of div. quantities in a finite number of div. but unobservable parameters of Lagrangian. → Renormalization

8] Generalization of QED in d spacetime dimensions

$$\boxed{D_{QED} = dL - P_e - 2P_\gamma \overset{!}{=} d + \left(\frac{d-4}{2}\right)V - \left(\frac{d-2}{2}\right)N_\gamma - \left(\frac{d-1}{2}\right)N_e}$$

→ Observations:

- $d<4$: Only a finite number of Feynman diagrams are divergent

  ⇒ Super-renormalizable theory

- $d=4$: Only a finite number of amplitudes are divergent.

  ⇒ Renormalizable theory

- $d>4$: All amplitudes diverge at sufficiently high order in perturbation theory.

  ⇒ Non-renormalizable theory

Alternative approach:

1] $\forall\ \phi^n$-theory

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{n!}\phi^n \qquad n \in \mathbb{N}$$

2] Def:

N_ϕ = # external lines

P_ϕ = # propagators

V = # vertices

L = # independent loops.

3] Superficial degree of divergence

$$\boxed{\begin{aligned} D_{\phi^n} &= dL - 2P_\phi \\ &\hat{=}\ d + \left[n\left(\frac{d-2}{2}\right) - d\right]V - \left(\frac{d-2}{2}\right)N_\phi \end{aligned}}$$

graph identities

$n=4, d=4 \rightarrow D$ not dep on V

$\rightarrow \phi^4$ renormalizable

4] Alternative: Dimensional analysis

i] $\hbar = c = 1, \quad \lambda_c = \frac{h}{mc} = \frac{2\pi}{m}$

$\rightarrow L = [\lambda_c] = M^{-1}$

dimension of length

dimension of mass

ii] $[S] = 1 \qquad [d^dx] = L^d = M^{-d}$

iii] $S = \int d^dx\, \mathcal{L} \longrightarrow [\mathcal{L}] = M^d$

iv] $[\partial] = \frac{1}{L} = M$

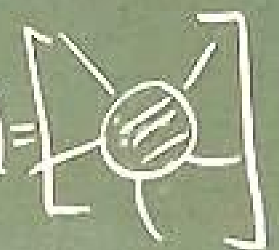
$\rightarrow [\phi] = M^{\frac{d-2}{2}}$

, $[m] = M$

, $[\lambda] = M^{\boxed{d - n\frac{d-2}{2}}}$ mass dimension of coupling constant.

v] $\forall$ Amplitude $\mathcal{M}$ with N_ϕ external lines

$\eta\phi^{N_\phi}$

$[\mathcal{M}] =$  $=$  $\hat{=}\ [\eta] = M^{d - N_\phi\frac{d-2}{2}}$

$\mathcal{M} = -\lambda + \dots$

vi) γ Diagram with Vertices

$$\rightarrow [\mathcal{M}] \overset{\Lambda\to\infty}{\sim} [\lambda^V \Lambda^{D_{\phi^n}}]$$

$$\hookrightarrow [\Lambda] = \frac{1}{L} = \mathcal{M}$$

$$\log_{\mathcal{M}} [\lambda]^V [\Lambda]^{D_{\phi^n}} = [\mathcal{M}] = \mathcal{M}^{d - N_\phi \frac{d-2}{2}}$$

$$V \log_{\mathcal{M}} [\lambda] + D_{\phi^n} \underbrace{\log_{\mathcal{M}} [\Lambda]}_{1} = d - N_\phi \frac{d-2}{2}$$

$$\boxed{\rightarrow D_{\phi^n} = d - \underbrace{\log_{\mathcal{M}} [\lambda]}_{d - n\frac{d-2}{2}} V - \frac{d-2}{2} N_\phi}$$

5)

Super-renormalizable:	$\log_{\mathcal{M}} [\lambda] > 0$
Renormalizable:	$\log_{\mathcal{M}} [\lambda] = 1$
Non-renormalizable:	$\log_{\mathcal{M}} [\lambda] < 0$

Aside: Quantum gravity

$$S_{EH} = \frac{\hat{c}^4}{16\pi \underset{\text{coupling constant}}{G}} \int d^4x \sqrt{|\det g(x)|} \left(R(g(x)) - \Lambda\right)$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}(x)$$

$$G = \frac{\hbar c}{m_P^2} = \frac{1}{m_P^2}$$

$$\rightarrow [G] = \mathcal{M}^{-2}$$

$$\rightarrow \boxed{\log_{\mathcal{M}} [G] = -2 < 0}$$

Gravity is (superficially) non-renormalizable