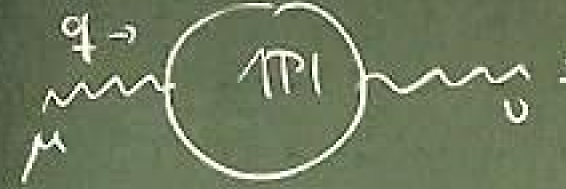
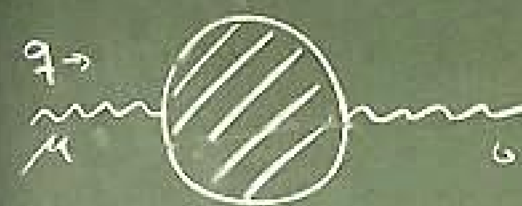
Recap:

6.5. Electric Charge Renormalization



$$= i\Pi^{\mu\nu}(q) = i\,(q^2 g^{\mu\nu} - q^\mu q^\nu)\,\Pi(q^2)$$

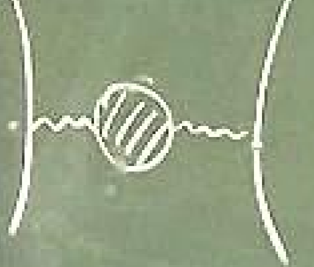
($\Pi(q^2)$: regular at $q^2 \to 0$)



$$\stackrel{!}{=} \frac{-i g_{\mu\nu}}{q^2[1-\Pi(q^2)]}$$

(in S-matrix elements)

$\measuredangle$ $q^2 \to 0$ (on-shell) $\to$ Charge renormalization:



$$= \frac{e_0^2\, g_{\mu\nu}}{q^2\,[1-\Pi(0)]}, \qquad \frac{e_0^2}{1-\Pi(0)} = e^2, \qquad 1-\Pi(0) = \frac{1}{Z_3}$$

Bare charge e_0 $\quad (\mathcal{H}_{int} = e_0 \bar\psi \gamma^\mu \psi A_\mu)$

Physical charge $e = \sqrt{Z_3}\, e_0$

Fine structure constant: $\dfrac{e^2}{4\pi} = \alpha \sim \dfrac{1}{137}$, $\quad \alpha \equiv Z_3 \alpha_0 = Z_3 \dfrac{e_0^2}{4\pi}$

q^2-dependence of coupling strength:

$$\alpha_{eff}(q^2) \equiv \frac{e_0^2/4\pi}{1-\Pi(q^2)} = \frac{\alpha}{1-[\Pi_2(q^2)-\Pi_2(0)]} + O(\alpha^2)$$

$$[\Pi_2(q^2)-\Pi_2(0)] \equiv \hat{\Pi}_2(q^2)$$

---

6) Computation of $\Pi_2(q^2)$

$$i\Pi_2^{\mu\nu}(q) = \text{(loop diagram)}$$

$$= -4 i e^2 \int_0^1 dx \int \frac{d^d l_E}{(4\pi)^d} \times \frac{\left(1-\frac{2}{d}\right) l_E^2 + \ldots}{(l_E^2+\Delta)^2} \qquad (*)$$

- $\Delta = m^2 - x(1-x)\, q^2$
- $d = 4 \to$ Integral diverges
  $\to$ Regularization

iii) Dimensional regularization

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1\, l_E^2}{(l_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \qquad (**)$$

Gamma function:

$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$ for $z \in \mathbb{C}$, simple poles at $z \in \{0,-1,-2,\dots\}$

iv) $n=2$

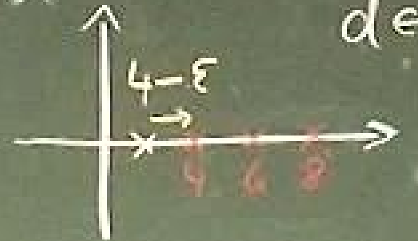
$$\int \frac{d^d l_E}{(2\pi)^d} \frac{(1-\frac{2}{d})\, l_E^2}{(l_E^2+\Delta)^2} \propto \left(1-\frac{2}{d}\right)\Gamma\left(2-\frac{d}{2}-1\right) = \Gamma\left(2-\frac{d}{2}\right)$$

$z\Gamma(z) = \Gamma(1+z)$

→ Divergence for $d=2$ cancelled!

→ $\propto \Gamma(2-\frac{d}{2})$ has poles at $d = 4, 6, \dots$

$d \in \mathbb{C}$



$$\Rightarrow \Gamma\left(2-\frac{d}{2}\right) = \Gamma\left(\frac{\varepsilon}{2}\right) \doteq \frac{2}{\varepsilon} - \gamma + O(\varepsilon) \qquad (***)$$

(γ: Euler-Mascheroni constant)

vi) Evaluate (*) using (**):

$$i\Pi_2^{\mu\nu}(q) \overset{\circ}{=} (q^2 g^{\mu\nu} - q^\mu q^\nu)\; i\Pi_2(q^2)$$

$$\Pi_2(q^2) = \frac{-8e^2}{(4\pi)^{d/2}} \int_0^1 dx\, \frac{x(1-x)\,\Gamma(2-\frac{d}{2})}{[m^2 - x(1-x)q^2]^{2-\frac{d}{2}}}$$

Use (***) and $(4\pi)^{\frac{d}{2}} = e^{\frac{d}{2}\log 4\pi}$ to expand in ε.

$$\Pi_2(q^2) \overset{\circ}{=} \frac{-2\alpha}{\pi}\int_0^1 dx\, x(1-x)\left[\frac{2}{\varepsilon} - \log(\Delta) - \gamma + \log(4\pi)\right] + O(\varepsilon)$$

7) $O(\alpha)$ charge renormalization:

$$\frac{e^2-e_0^2}{e_0^2} = Z_3 - 1 = \frac{\Pi(0)}{1-\Pi(0)}$$

$$= \Pi_2(0) + O(\alpha^2)$$

$$\overset{\varepsilon\to 0}{\sim} -\frac{2\alpha}{3\pi\varepsilon} \xrightarrow{\varepsilon\to 0} -\infty$$

→ e finite → bare charge diverges: $|e_0| \to \infty$

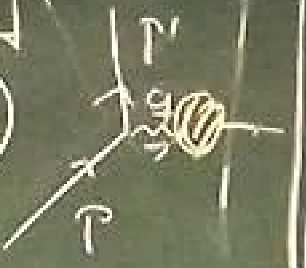
8) $O(\alpha)$ q^2-dependence of $\alpha_{eff}(q^2)$

$$\hat{\Pi}_2 = \Pi_2(q^2) - \Pi_2(0) \overset{\varepsilon\to 0}{=} \frac{-2\alpha}{\pi}\int_0^1 dx\, x(1-x) \log\left(\frac{m^2}{m^2 - x(1-x)q^2}\right)$$

9) Analysis & Interpretation of $\Pi_2(q^2)$

i) ⊗ Effective potential in non-rel. limit

($q^2 = (p-p')^2 \approx -|\vec{p}-\vec{p}'|^2$, $|q^2| \ll m^2$)



$$V(\vec{x}) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\vec{x}} \frac{-e^2}{|\vec{q}|^2[1-\hat{\Pi}_2(-|\vec{q}|^2)]}$$

$$= -''- \left(\frac{-e^2}{|\vec{q}|^2}\right)\left[1 + \hat{\Pi}_2(-|\vec{q}|^2) + O(\alpha^2)\right] \quad (⊗)$$

$$\overset{|q^2| \ll m^2}{=} -''- \; -''- \left[1 + \frac{\alpha}{15\pi m^2}|\vec{q}|^2\right] + O(\alpha^3)$$

$\log\frac{1}{1-x} = x + O(x^2)$

$$\overset{\circ}{\approx} -\frac{\alpha}{|\vec{x}|} - \frac{4\alpha^2}{15m^2}\delta^{(3)}(\vec{x})$$

→ EM force becomes stronger at small distances
(Coulomb potential is an approximation)

iii) Exp. verification:

⊗ s-orbitals of hydrogen atom

$$\Delta E \approx \int d^3x\, |\psi(\vec{x})|^2 \left(-\frac{4\alpha^2}{15m^2}\delta^3(\vec{x})\right)$$

$$= -\frac{4\alpha^2}{15m^2}|\psi(0)|^2 \overset{l=0}{<} 0$$

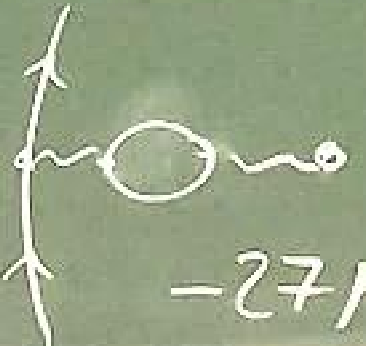
cf. Darwin term:

$$H_{Darwin} = \frac{\pi\alpha}{2m^2}\delta^{(3)}(\vec{x})$$

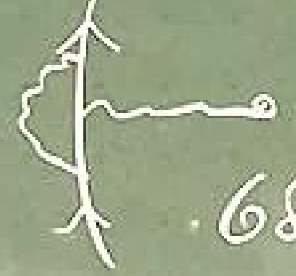
Lamb shift:

Schrödinger	Dirac	Lamb shift (Lamb-Retherford experiment)
	$2P_{3/2}$ –	
2S – 2P ---	$2S_{1/2}$ – $2P_{1/2}$ –	$2S_{1/2}$, $2P_{1/2}$] 1000 MHz
1S	$1S_{1/2}$	$1S_{1/2}$ –

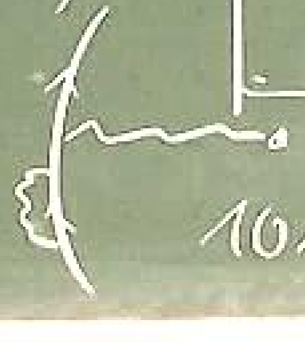
3 Effects.



−27 MHz



.68 MHz



1011 MHz

IV) Skip approx. (*), complex analysis.

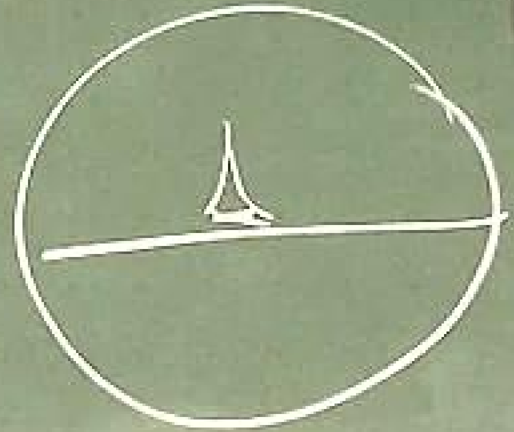
$$V(\vec{x}) = -\frac{\alpha}{r}\left(1 + \frac{\alpha}{4\sqrt{\pi}} \frac{e^{-2mr}}{(mr)^{3/2}} + \dots\right)$$

Uehling potential

Length scale of modification: $\frac{2\pi}{m} = \frac{h}{mc} = \lambda_c$

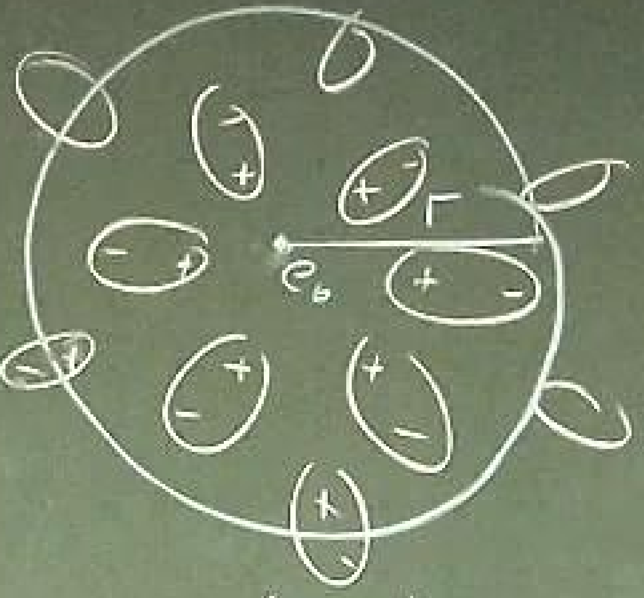
Atomic physics: $a_0 \sim 22\lambda_c$

↑ Bohr radius



V) Interpretation: Vacuum polarization:

$\frac{1}{q} \sim r$



→ Screening due to vacuum fluctuations lowers bare charge e_0 to physical charge e

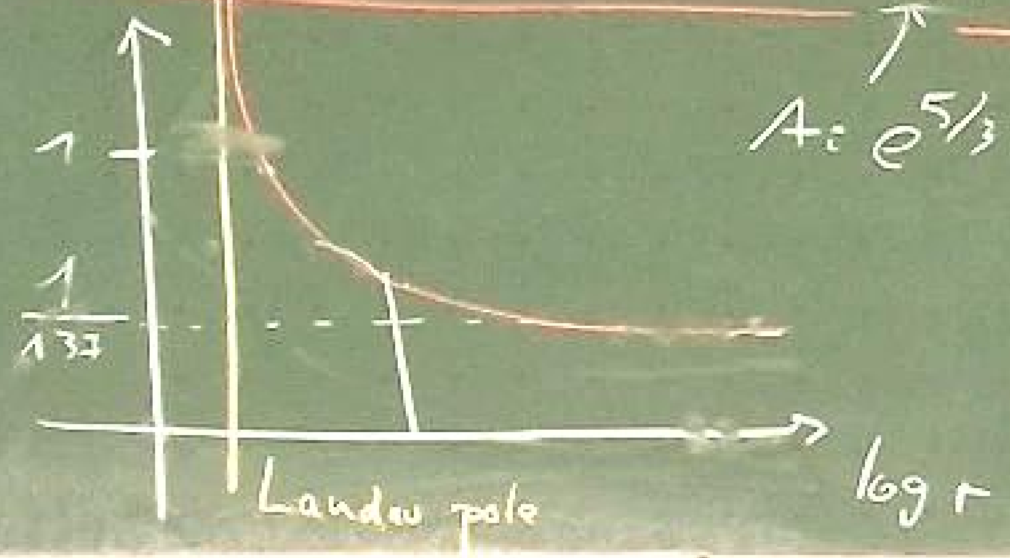
vi) ∗ Relativistic limit: $-q^2 \gg m^2$.

$$\Pi_2(q^2) \doteq \frac{\alpha}{3\pi}\left[\log\left(\frac{-q^2}{m^2}\right) - \frac{5}{3} + O\left(\frac{m^2}{q^2}\right)\right]$$

→ "Running" of α_{eff} with length scale $r = \frac{1}{q}$..

$$\alpha_{eff}(q^2) \overset{-q^2\gg m}{\approx} \frac{\alpha}{1-\frac{\alpha}{3\pi}\log\left(\frac{-q^2}{Am^2}\right)}$$

$A: e^{5/3}$



∗ Landau pole:

$$1-\frac{\alpha}{3\pi}\log\left(\frac{-q^2}{Am^2}\right) \overset{!}{=} 0 \Rightarrow \Lambda_L \sim m e^{\frac{3\pi}{2\alpha}} \sim 10^{286}\ \text{eV}$$

$$E_{LCH} \sim 10^{13}\ \text{eV} \ll E_{Planck} \sim 10^{28}\ \text{eV}$$

→ Practically irrelevant.

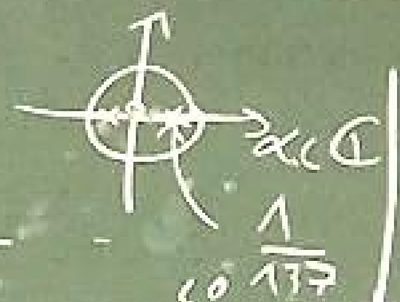
• Dyson argument:

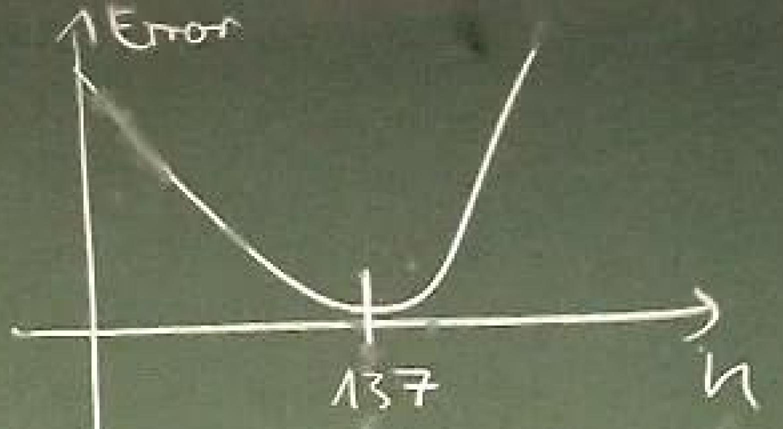
$$f(\alpha) = f_0 + f_1\alpha + f_2\alpha^2 \dots$$

$\alpha \in \mathbb{C}$ $+\, e^{-\frac{c}{\alpha}}$

→ $\alpha < 0$ convergence

$$V(x) = -\frac{\alpha}{r}$$

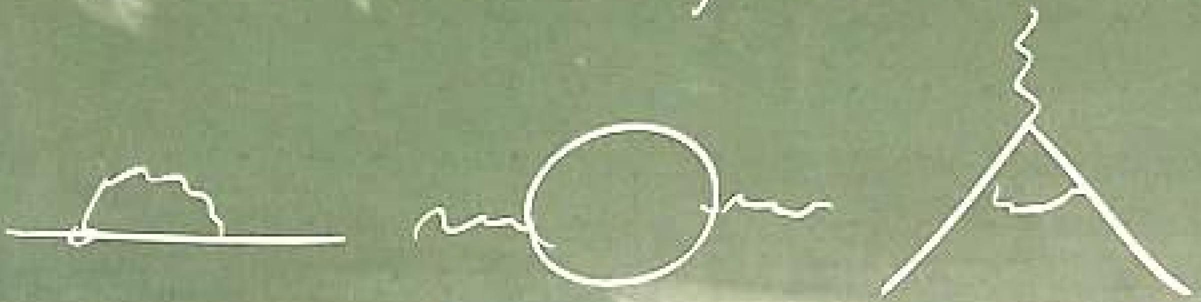




$$\cdot\ e^{-\frac{c}{\alpha}} \sim \alpha^N \leadsto N \sim \frac{1}{\alpha} \sim 137$$

7. Systematics of Renormalization

- **IR-divergencies:**
  - due massless particles
  - regulate with small photon mass μ
  - Cancel with bremsstrahlung.
- **UV-divergencies.**
  - due to unbounded high momenta:



- regulation with Pauli-Villars (Λ) or dimensional regularization (ε)
- Cancelled in obs. quantities
- Diverging difference between physical and bare parameters.

$\Downarrow$

Fundamental problem

$\Downarrow$

Study UV-divergencies systematically.