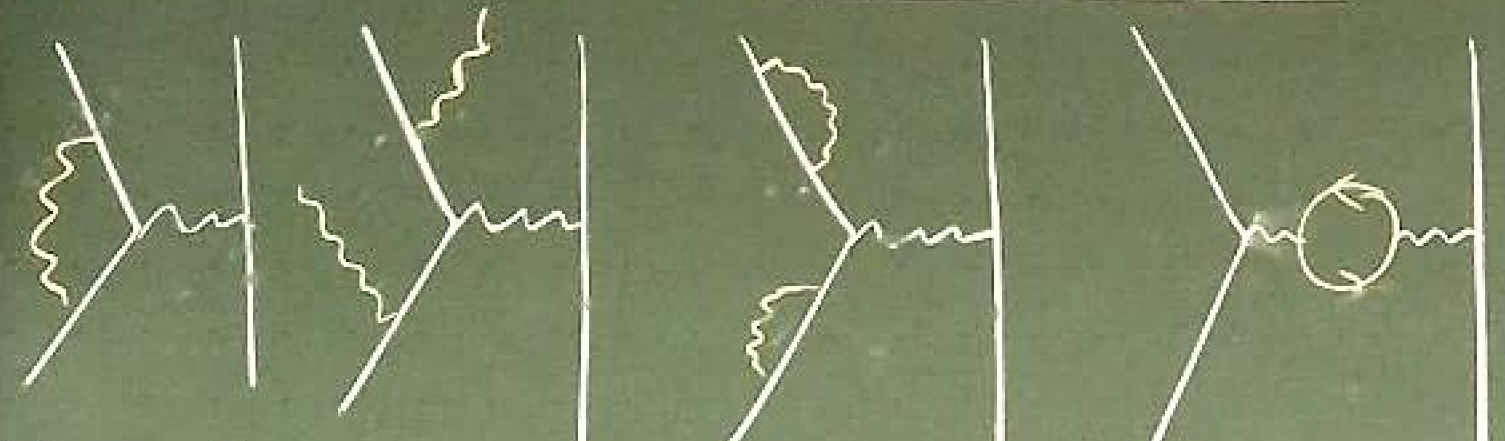
6 Radiative Corrections of QED

6.5 Electric Charge Renormalization



Vertex correction — Bremsstrahlung — Field strength & mass renormalization (Electron self-energy) — Vacuum polarization (Photon self-energy)

IR-div — IR-div

UV-div

cancel (Sudakov form factor)

UV-div

IR-div

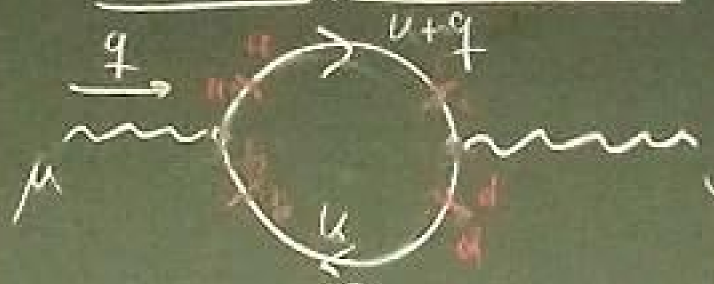
cancel

↓

Today

(Renormalization, LSZ reduction, $F_1(0)=1 \Leftrightarrow \delta Z_2 = -F_1^{(1)}(0)$)

1) One-loop correction:



$$\bar\psi\psi\bar\psi\psi = \bar\psi\psi\,\bar\psi\psi = -\psi\bar\psi\,\psi\bar\psi$$

$$\psi\bar\psi = S_F$$

$$= (-1)(-ie)^2 \int \frac{d^4k}{(2\pi)^4}\, \gamma^\mu_{ab}\, \frac{i(\not k + m)_{bd}}{k^2-m^2}\, \gamma^\nu_{dc}\, \frac{(\not k + \not q + m)_{ca}}{(k+q)^2-m^2}$$

$$= (-1)(-ie)^2 \int \frac{d^4k}{(2\pi)^4}\, \mathrm{Tr}\left[\gamma^\mu \frac{i(\not k + m)}{k^2-m^2}\, \gamma^\nu \frac{i(\not k + \not q + m)}{(k+q)^2-m^2}\right]$$

$$\equiv i\Pi_2^{\mu\nu}(q)$$

2) Sum of all 1PI diagrams:

$$\equiv i\Pi^{\mu\nu}(q) = i\left[\Pi_2^{\mu\nu}(q) + O(e^4)\right]$$

$q^2 \quad q$

i) Only tensors. $q^\nu q^\mu$, $g^{\mu\nu}$

$$\Rightarrow \Pi^{\mu\nu}(q) = A(q^2)\, g^{\mu\nu} + B(q^2)\, q^\mu q^\nu$$

ii) Ward identity:

$$q_\mu \Pi^{\mu\nu}(q) \overset{*}{=} 0$$

$$\rightarrow B = -\frac{A}{q^2} \Rightarrow \Pi^{\mu\nu}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\frac{A}{q^2}$$

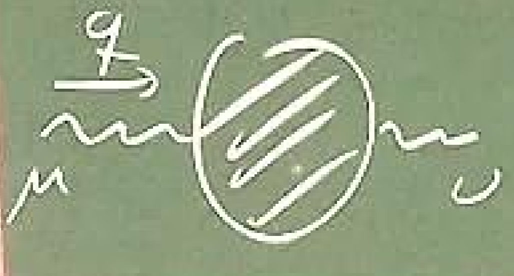
iii) $\xrightarrow{*}$ $\Pi^{\mu\nu}(q)$ has no pole at $q^2=0$

$$\rightarrow \Pi(q^2) \equiv \frac{A(q^2)}{q^2}$$

is regular at $q^2=0$

$$\boxed{\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\,\Pi(q^2)}$$

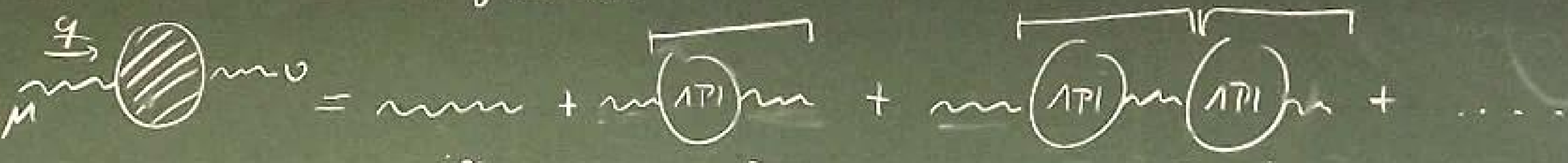
4) Ward identity $\xrightarrow{*}$



$$\hat{=}\ \frac{-i\, g_{\mu\nu}}{q^2[1-\Pi(q^2)]}$$

valid in S-matrix elements

3) Sum of all diagrams:



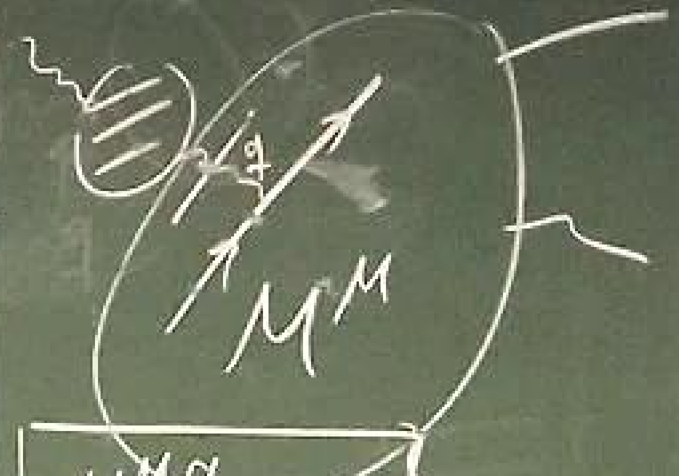
$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\left[i(q^2 g^{\rho\sigma} - q^\rho q^\sigma)\,\Pi(q^2)\right]\frac{-ig_{\sigma\nu}}{q^2} + \dots$$

$$\Delta^\rho_\nu \equiv \delta^\rho_\nu - q^\rho q_\nu/q^2 \quad (\text{use } g^{\rho\sigma}g_{\sigma\nu} = \delta^\rho_\nu)$$

$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\underbrace{\Delta^\rho_\nu\, \Pi(q^2)} + \frac{-ig_{\mu\rho}}{q^2}\Delta^\rho_\sigma\Delta^\sigma_\nu\, \Pi^2(q^2) + \dots$$

$$\Delta^\rho_\sigma\Delta^\sigma_\nu = \Delta^\rho_\nu$$

$$= -\frac{ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\Delta^\rho_\nu \underbrace{\sum_{n=1}^{\infty} \Pi^n(q^2)}_{\frac{1}{1-\Pi(q^2)}-1} = \frac{-i}{q^2[1-\Pi(q^2)]}\left(g_{\mu\nu} - \cancel{\frac{q_\mu q_\nu}{q^2}}\right) + \frac{-i}{q^2}\cancel{\left(\frac{q_\mu q_\nu}{q^2}\right)}$$

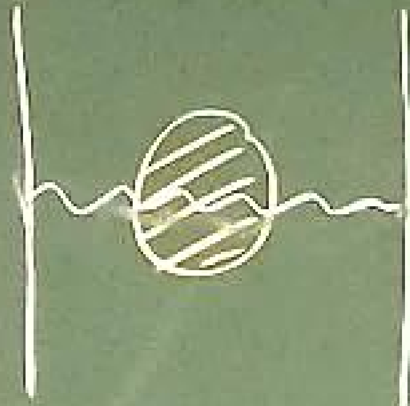


$$\boxed{M^\mu q_\mu = 0}$$

5]  $\doteq \dfrac{-i g_{\mu\nu}}{q^2[1-\Pi(q^2)]}$

Charge renormalization:

i] Define $\boxed{Z_3 \equiv \dfrac{1}{1-\Pi(0)}}$



$\xrightarrow{q^2\to 0} = \dots \dfrac{\overbrace{e_0^2 g_{\mu\nu} Z_3}^{e^2}}{q^2} \dots$

ii] → Charge renormalization:

$$\boxed{\begin{array}{l} \bullet \text{ Bare charge } e_0 \quad (\text{given by } \mathcal{L}_{int} = e_0 \bar\psi \gamma^\mu \psi A_\mu) \\ \bullet \text{ Physical charge } \quad e \equiv \sqrt{Z_3}\, e_0 \\ \bullet \text{ Fine-structure constant } \quad \dfrac{e^2}{4\pi} = \alpha \equiv Z_3 \alpha_0 = Z_3 \dfrac{e_0^2}{4\pi} \end{array}}$$

$\sqrt{\cdot}\ \alpha = \alpha_0 + O(\alpha_0^2) \rightarrow \alpha^2 = \alpha_0^2 + O(\alpha_0^3)$

$\alpha_0 = \alpha + O(\alpha^2) \rightarrow O(\alpha^2) = O(\alpha_0^2)$

$\bullet\ Z_3 = 1 + O(\alpha_0)$

iii] $\dfrac{-i g_{\mu\nu}}{q^2} \boxed{\dfrac{e_0^2 / 4\pi}{1-\Pi(q^2)}}$

$= \dfrac{-i g_{\mu\nu}}{q^2} \dfrac{e^2 [1-\Pi(0)]}{1-\Pi(q^2)}$

$\dfrac{1}{1-x} = 1 + x + O(x^2)$

$= \dfrac{-i g_{\mu\nu}}{q^2} \dfrac{e^2[1-\Pi_2(0)]}{1-\Pi_2(q^2)} + O(\alpha^2)$

$1 - x = \dfrac{1}{1+x} + O(x^2)$

$= \dfrac{-i g_{\mu\nu}}{q^2} \dfrac{e^2}{[1-\Pi_2(q^2)][1+\Pi_2(0)]} + O(\alpha^2)$

$= \dfrac{-i g_{\mu\nu}}{q^2} \boxed{\dfrac{e^2 / 4\pi}{1-[\Pi_2(q^2)-\Pi_2(0)]}} + O(\alpha^2)$

$$\boxed{\alpha_{\text{eff}}(q^2) \equiv \frac{e_0^2/4\pi}{1-\Pi(q^2)} = \frac{\alpha}{1-[\Pi_2(q^2)-\Pi_2(0)]} + O(\alpha^2)}$$

6| Computation of Π_2:

i| $$i\Pi_2^{\mu\nu}(q) = -(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^\mu \frac{i(\not{k}+m)}{k^2-m^2} \gamma^\nu \frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2}\right]$$

$$\left[\frac{i\alpha}{\not{k}-m} = \frac{i\alpha_0}{\not{k}-m_0} + O(\alpha_0^2)\right]$$

Trace identities

$$= -4e^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu (k+q)^\nu + k^\nu (k+q)^\mu - g^{\mu\nu}(k\cdot(k+q)-m^2)}{(k^2-m^2)((k+q)^2-m^2)}$$

Feynman parameters, $\ell \equiv k + xq$, Wick rotation $\ell^0 \equiv i\ell_E^0$ $\qquad \Delta \equiv m^2 - x(1-x)q^2$

$$\overset{d=4}{=} -4ie^2 \int_0^1 dx \int \frac{d^4\ell_E}{(2\pi)^4} \frac{-\frac{2}{d} g^{\mu\nu}\ell_E^2 + g^{\mu\nu}\ell_E^2 - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}(m^2 + x(1-x)q^2)}{(\ell_E^2+\Delta)^2}$$

(Eq. (6.59 1))

$$\boxed{\ell^\mu \ell^\nu \triangleq \frac{1}{d}\ell^2 g^{\mu\nu}}$$ spacetime dimension

$$g_{\mu\nu} g^{\nu\mu} = \delta^\mu_\mu = d$$

ii| Strong UV-divergence: UV cutoff: $|\ell_E| < \Lambda$

$$i\Pi_2^{\mu\nu}(q) \sim e^2 \Lambda^2 g^{\mu\nu} \xrightarrow{\Lambda\to\infty} \infty$$

→ Regularization needed

iii) Dimensional regularization. (PSet 11)

1. Lower spacetime dimension $d \in \mathbb{N}$ until UV div. vanishes
2. Generalize expressions to $d \in \mathbb{R}$
3. Take limit $d \nearrow 4$ in observable quantities.

$$\boxed{\begin{aligned} \int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2+\Delta)^n} &= \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \\ -''- \quad \frac{l_E^2}{(l_E^2+\Delta)^n} &= \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \end{aligned}}$$

· $\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t}$

$\Gamma(n) = (n-1)!$

& $n=2$

$\Gamma(z) = \overbrace{0, -1, -2, -3,}^{\text{poles}}$

$\Gamma(2-\frac{d}{2})$ poles $d = 4, 6, \ldots\ldots$

$\longrightarrow$ $d = 4-\varepsilon$ and use

$$\boxed{\Gamma\left(2-\frac{d}{2}\right) = \Gamma\left(\frac{\varepsilon}{2}\right) = \frac{2}{\varepsilon} - \gamma + O(\varepsilon)}$$

$\uparrow$ Euler-Mascheroni constant