## 1.2 Symmetries and Conservation laws

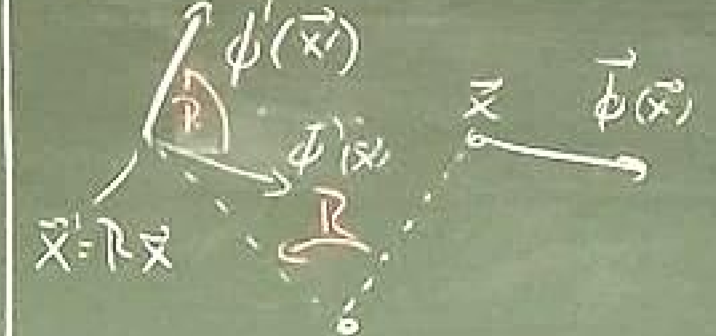
1] Transformations of **fields** decompose into two transformations:

$$x \mapsto x'(x) \quad , \quad \phi(x) \mapsto \phi'(x') = \mathcal{F}(\phi(x))$$

These are *active* transformations!

**Example 1.?** Rotation of vector field $\vec{\phi}$

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad R \in SO(3)$$




$$\vec{x}' = R\vec{x}$$

$$\vec{\phi}'(\vec{x}') = R\vec{\phi}(\vec{x}) = R\vec{\phi}(R^{-1}\vec{x}')$$

$$\Downarrow$$

$$\boxed{\vec{\phi}'(\vec{x}) = R\vec{\phi}(R^{-1}\vec{x})}$$

Vector field

$$\phi \mapsto \phi'$$

2] Change of the action under $\phi \to \phi'$ ?

$$S' = S[\phi'] = \int d^d x \, \mathcal{L}(\phi'(x), \partial_\mu \phi'(x))$$

$$= \int d^d x' \, \mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x'))$$

$$= \int d^d x' \, \mathcal{L}(\mathcal{F}(\phi(x)), \partial'_\mu \mathcal{F}(\phi(x)))$$

$$\underset{x' = x'(x)}{\overset{\text{Substitution}}{=}} \int d^d x \left| \frac{\partial x'}{\partial x} \right| \mathcal{L}\left(\mathcal{F}(\phi(x)), \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \mathcal{F}(\phi(x))\right)$$

Example 1.3 · Translation

1) $x' = x + a$

$\phi'(x') := \phi(x) = \phi(x'-a)$

2) $\mathcal{F} = \mathbb{1}$, · $\phi'(x') = \mathcal{F}(\phi(x))$

$= \phi(x(x'))$

· $\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu$

3)

$S[\phi'] = \int d^dx\, \mathcal{L}(\phi'(x), \partial'_\mu \phi'(x)) =$

$= \int d^dx\, \mathcal{L}(\phi(x), \partial_\mu \phi(x)) = S[\phi]$

$\to$ Action is translation invariant!

Example 1.4. Scale transformations

$\Rightarrow$ see script.

Example 1.5. Phase rotation

1) $x' := x$, $\phi'(x') := e^{i\theta}\phi(x)$

2) $\mathcal{F}(\phi) = e^{i\theta}\phi$, $\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu$, $\left|\frac{\partial x'}{\partial x}\right| = 1$

$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi^*) - \frac{1}{2} m \phi \phi^*$

Infinitesimal transformations

1) $\nexists$ Infinitesimal transformations (IT).

$x'^\mu = x^\mu + \boxed{\omega_a} \boxed{\frac{\delta x^\mu}{\delta \omega_a}(x)}$

$\phi'(x') = \phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) = \mathcal{F}(\phi(x))$

For now: $\omega_a = \omega_a(x)$

2] Generator of IT:

$$\delta_\omega \phi(x) = \phi'(x) - \phi(x) \equiv -i\omega_a G_a \phi(x)$$

With

$$\phi'(x) = \phi(x) - \omega_a \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) + O(\omega^2)$$

$$\Rightarrow \boxed{iG_a \phi = \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \phi - \frac{\delta \mathcal{F}}{\delta \omega_a}}$$

Example 1.6. Translation

1] $x'^\mu = x^\mu + \omega^\mu = x^\mu + \omega^\nu \underbrace{\frac{\delta x^\mu}{\delta \omega^\nu}}_{\delta^\mu_\nu}$

2] $\frac{\delta \mathcal{F}}{\delta \omega^\nu} = 0$ (Scalar field)

3] $iG_\mu \phi = \delta^\nu_\mu \partial_\nu \phi - 0$

$\rightarrow G_\mu = -i\partial_\mu = P_\mu$

$-i\partial_t \psi = H\psi$

$S[\phi'] = S[\phi] + \int d^4x\, \partial_\mu K^\mu$

Ex. 1.7

Ex. 1.8. Spatial Rotations

$$G_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + S_{\mu\nu}$$

Noether's theorem

1] Transformations (*) for which

$\boxed{S[\phi] = S[\phi']}$ $\Leftarrow$ (strict) symmetry of the action

for $\omega_a$ independent of $x$ (rigid transform)

2] Assume in (*) $\omega_a = \omega_a(x)$ is not rigid

$\ulcorner\; O(\omega_a) = O(\partial_\mu \omega_a)$

3]. $\frac{\partial x'^\nu}{x^\mu} = \delta^\nu_\mu + \partial_\mu\left(\omega_a \frac{\delta x^\nu}{\delta \omega_a}\right)$

$\left|\frac{\partial x'}{\partial x}\right| = 1 + \partial_\mu\left(\omega_a \frac{\delta x^\mu}{\delta \omega_a}\right)$

$\ulcorner\; \det(\underline{1}+\underline{A}) = 1 + \mathrm{Tr}[A] + O(A^2)$

4] Inverse Jacobian matrix.

$\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu - \partial_\mu\left(\omega_a \frac{\delta x^\nu}{\delta \omega_a}\right)$

5]

$S' = S[\phi'] = \int d^dx \left[1 + \partial_\mu\left(\omega_a \frac{\delta x^\mu}{\delta \omega_a}\right)\right]$

$\times \mathcal{L}\Big(\phi + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}, \left[\delta^\nu_\mu - \partial_\mu\left(\omega_a \frac{\delta x^\nu}{\delta \omega_a}\right)\right]$

$\times \left[\partial_\nu\phi + \partial_\nu\left(\omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}\right)\right]\Big)$

6] Expand in 1st order of $\omega_a, \partial_\mu \omega_a$ *

7] ∡ $\delta S = S' - S = \int d^dx \left[\boxed{K}\omega_a - j^\mu_a \partial_\mu \omega_a\right]$

Symmetry $\Downarrow$

($\Rightarrow \omega_a$ rigid) $0$

$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi^*) - \frac{1}{2} m \phi\phi^*$

Infinitesimal transformations

1] ∡ Infinitesimal transformations (IT).

$\begin{cases} x'^\mu = x^\mu + \boxed{\omega_a} \boxed{\frac{\delta x^\mu}{\delta \omega_a}(x)} \\ \phi'(x') = \phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) = \mathcal{F}(\phi(x)) \end{cases}$

For now: $\omega_a = \omega_a(x)$

8) For non-rigid transformation

$$\delta S = -\int d^d x \; j^\mu_a \partial_\mu \omega_a$$

with the current:

$$\boxed{j^\mu_a(x) \hat{=} \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \right\} \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta \mathcal{F}}{\delta \omega_a}}$$

9) Integration by parts:

$$\delta S = \int d^d x \; \omega_a(x) \partial_\mu j^\mu_a = 0 \quad , \forall \omega_a(x)$$

10) $\forall\ \phi$ that obey the EOM

$\Rightarrow \delta S = 0$ for arbitrary variations $\phi' = \phi + \delta\phi$

$\Rightarrow$ In particular for non-rigid trafos

$\Downarrow$

$$\boxed{\partial_\mu j^\mu_a = 0 \quad \forall_{x,a}}$$

Noether's (first) theorem

11) Conserved charge:

$$\boxed{Q_a = \int_{\text{Space}} d^{d-1} x \; j^0_a}$$

Note 1.3:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c} T_{\mu\nu}$$

$$T^{\mu\nu} \neq T^{\nu\mu}$$

But: $\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho K^{\rho\mu\nu}$

Homework:

$$\frac{dQ_a}{dt} = \ldots \overset{\text{Gauss}}{=} -\int_{\text{Surface}} d\sigma_i \, j^i_a = 0$$

Note 1.1 $j^\mu_a$: canonical current

$$\tilde{j}^\mu_a = j^\mu_a + \partial_\nu B^{\mu\nu}_a \quad , \quad B^{\mu\nu}_a = -B^{\nu\mu}_a \text{ arbitrary}$$

$$\Rightarrow \partial_\mu \tilde{j}^\mu_a = 0$$

Note 1.2:

$$\underbrace{\text{Symmetric } \mathcal{L} \Rightarrow \text{Symmetric } S}_{\to \text{Conserved charges}} \Rightarrow \text{Symmetric EOM}$$

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\phi^2 \quad \Rightarrow (\partial^2+m^2)\phi = 0$$

$$\phi' = \lambda\phi \qquad SO(1,3) \ltimes \mathbb{R}^4$$

**Application**: Energy-Momentum Tensor (EMT)

1) IT spacetime. $x'^\mu = x^\mu + \varepsilon^\mu$

$$\rightarrow \frac{\delta x^\mu}{\delta \varepsilon^\nu} = \delta^\mu_\nu$$

$$\frac{\delta F}{\delta \varepsilon^\nu} = 0$$

2) Translation invariant action $S$

3) Conserved currents

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu \mathcal{L}$$

$$\boxed{T^{\mu\nu} = g^{\nu\rho}T^\mu{}_\rho = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - g^{\mu\nu}\mathcal{L}}$$

Energy momentum tensor.

with $\partial_\mu T^{\mu\nu} = 0$

$$\boxed{P^\nu = \int d^3x\, T^{0\nu}}$$

Conserved charges

4) Energy ($\nu = 0$).

$$P^0 = \int d^3x\, T^{00} = \underbrace{\int d^3x \Big\{ \underbrace{\frac{\partial\mathcal{L}}{\partial\dot\phi}}_{\pi}\dot\phi - \mathcal{L}\Big\}}_{\mathcal{H}}$$

$= H$ Hamiltonian

5) $\nu = i \Rightarrow$ Kinetic momentum

$$P^i = \int d^3x\, T^{0i} = \int d^3x \frac{\partial\mathcal{L}}{\partial\dot\phi}(-\partial_i\phi)$$

$$= -\int d^3x\, \pi \cdot (\partial_i\phi)$$