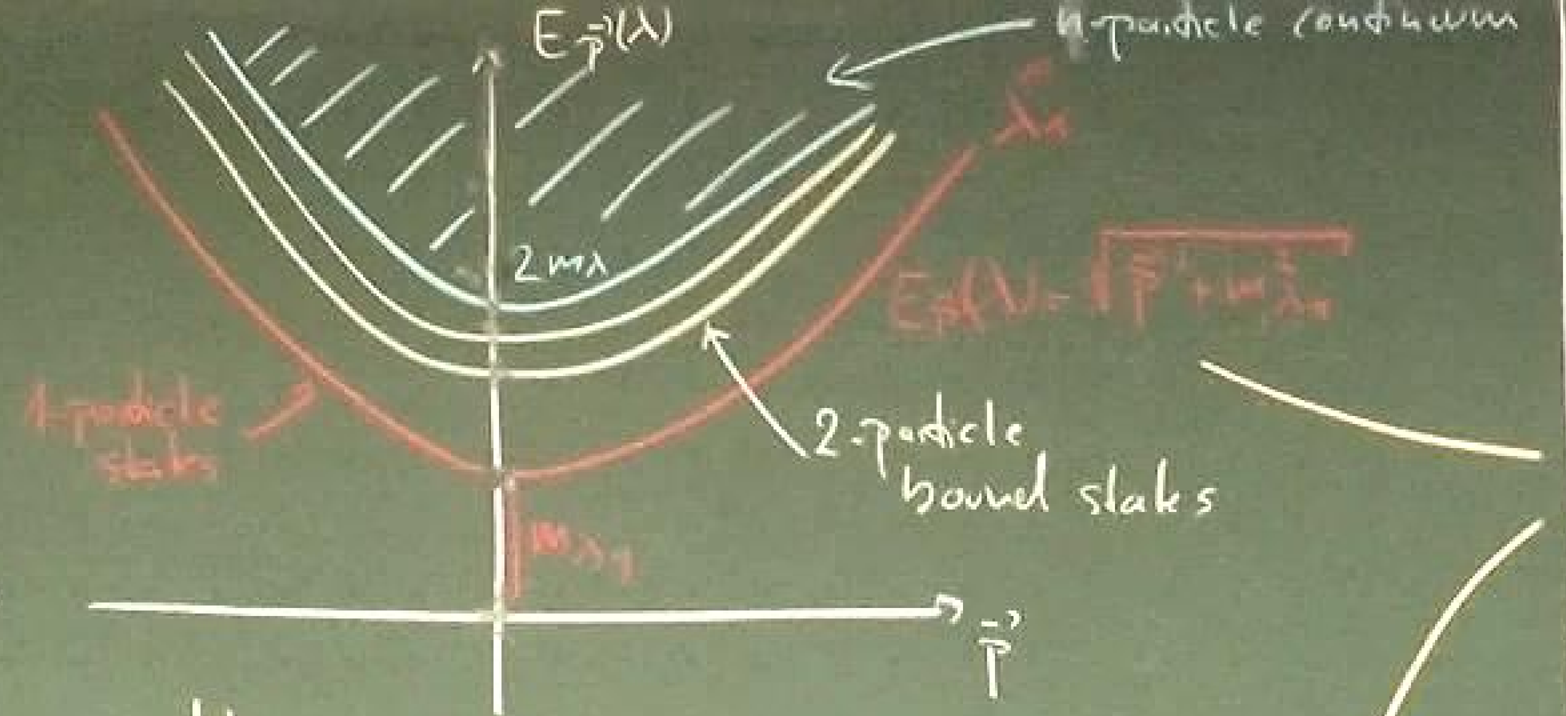
Recap

6.4. Field-Strength Renormalization

6.4.1 Structure of Two-Point Correlators in interacting Theories

- $P^\mu = \begin{pmatrix} H \\ \vec{P} \end{pmatrix}$, $P^\mu |\lambda_{\vec{p}}\rangle = \begin{pmatrix} E_{\vec{p}}(\lambda) \\ \vec{p} \end{pmatrix} |\lambda_{\vec{p}}\rangle$

eigenstate of interacting Hamiltonian H



7| Källén-Lehmann spectral representation

Feynman propagator (mass μ)

Spectral density

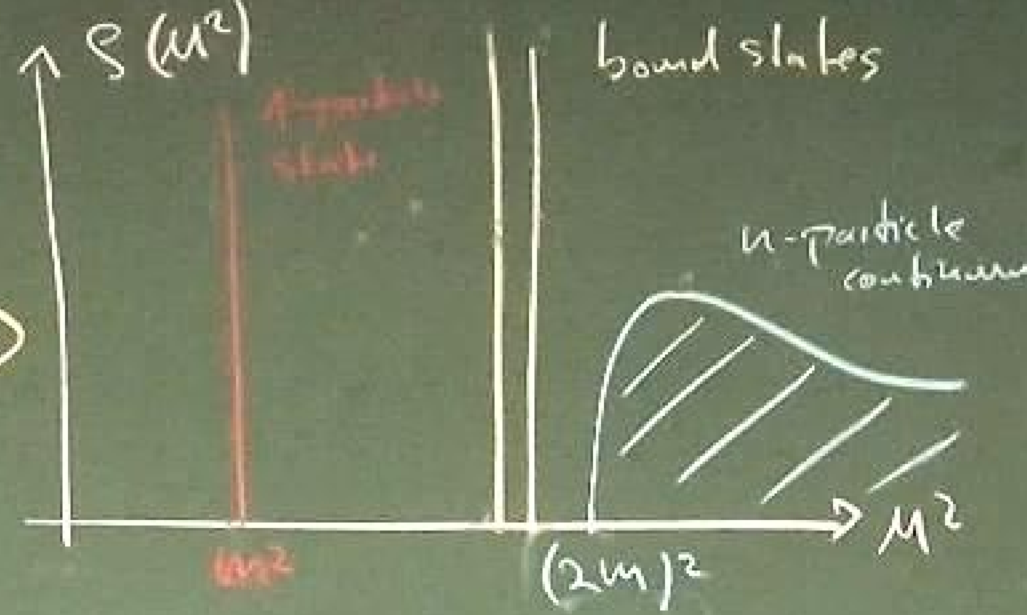
$$\Rightarrow \langle\Omega| T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \frac{d\mu^2}{2\pi}\, \rho(\mu^2)\, D_F(x-y,\mu^2)$$

with

$$\rho(\mu^2) = 2\pi \sum_\lambda \delta(\mu^2 - m_\lambda^2)\,|\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$

8| Typical spectral density:

$\Rightarrow$



$$\rightarrow \rho(M^2) = 2\pi\, \delta(M^2 - m_1^2)\, \underbrace{|\langle\Omega|\phi(0)|\lambda_0\rangle|^2}_{\lambda=1,\ \equiv Z}$$

$$+ \{\text{multi-particle states for } M^2 \gtrsim (2m_1)^2\}$$

- Field strength renormalization: $Z = |\langle\Omega|\phi(0)|\lambda_0 = 1_0\rangle|^2$
- Physical mass (observable): $m \equiv m_1 \quad (H|1_0\rangle = m_1|1_0\rangle)$
- Bare mass: $m_0 \quad (H = \ldots\ldots \frac{1}{2} m_0 \phi^2)$

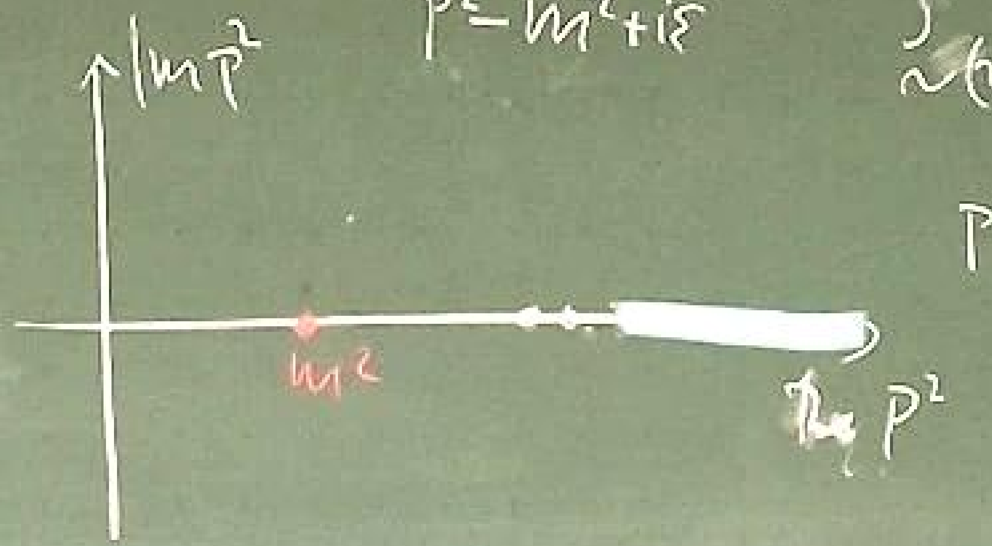
- Free theory: $m = m_0,\ Z = 1$
- Interacting ...: $m \neq m_0,\ Z \neq 1$

g) $$\frac{i\cdot 1}{p^2 - m_0^2 + i\varepsilon} \underset{\text{free}}{=} \int d^4x\, e^{ipx} \langle\Omega|T\,\phi(x)\phi(0)|\Omega\rangle$$

$$\underset{\text{general}}{=} \int_0^\infty \frac{dM^2}{2\pi} \frac{i\rho(M^2)}{p^2 - M^2 + i\varepsilon}$$

$$= \frac{i\cdot Z}{p^2 - m^2 + i\varepsilon} + \int_{\sim(2m)^2}^\infty \frac{i\rho M^2}{p^2 - M^2 + i\varepsilon}$$

$p^2 \in \mathbb{C}$



6.4.2. Application to QED: The Electron Self-Energy

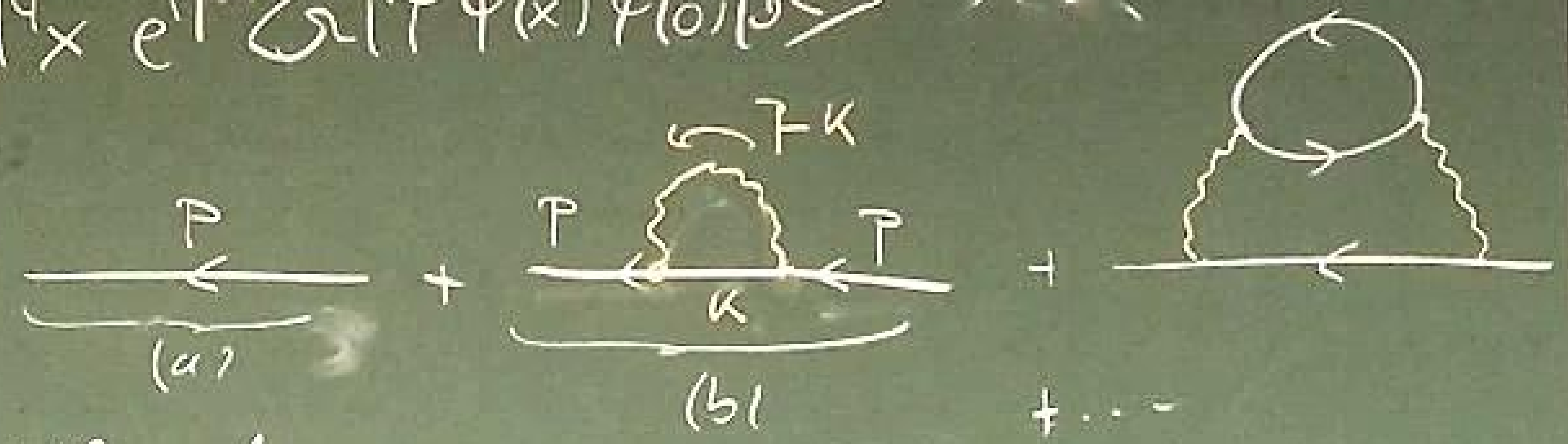
1) $\phi^4 \mapsto$ QED

$$\int d^4x\, e^{ipx} \langle \Omega | T\, \psi(x) \bar{\psi}(0) | \Omega \rangle$$

$$\overset{*}{=} \frac{i Z_2 (\not{p} + m)}{p^2 - m^2 + i\varepsilon} + \ldots\ldots$$

2) α Perturbation theory:

$$\int d^4x\, e^{ipx} \langle \Omega | T\, \psi(x) \bar{\psi}(0) | \Omega \rangle$$



3) α^0-order:

$$(a) = \frac{i(\not{p} + m_0)}{p^2 - m_0^2 + i\varepsilon}$$

4) α^1-order:

$$(b) = \frac{i(\not{p}+m_0)}{p^2 - m_0^2 + i\varepsilon} \underbrace{\left[(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i(\not{k} + m_0)}{k^2 - m_0^2 + i\varepsilon} \gamma_\mu \frac{-i}{(p-k)^2 + i\varepsilon} \right]}_{\equiv -i\Sigma_2(p)} \frac{i(\not{p}+m_0)}{p^2 - m_0^2 + i\varepsilon}$$

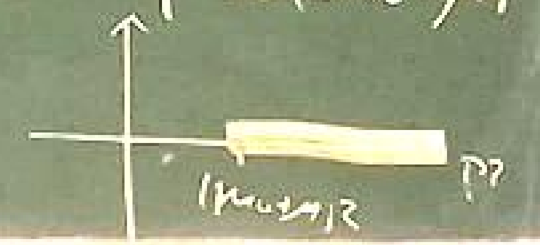
→ (P-Set 10)

→ IR- and UV-divergence

→ Solve like vertex function

→ $$\Sigma_2(p) \overset{\Lambda \to \infty}{\sim} \frac{\alpha}{2\pi} \int_0^1 dx\, (2m_0 - x\not{p}) \times \log\left[\frac{x\Lambda^2}{(1-x)m_0^2 + x\mu^2 - x(1-x)p^2} \right]$$

↑ branch cut emanating from $p^2 = (m_0 + \mu)^2$

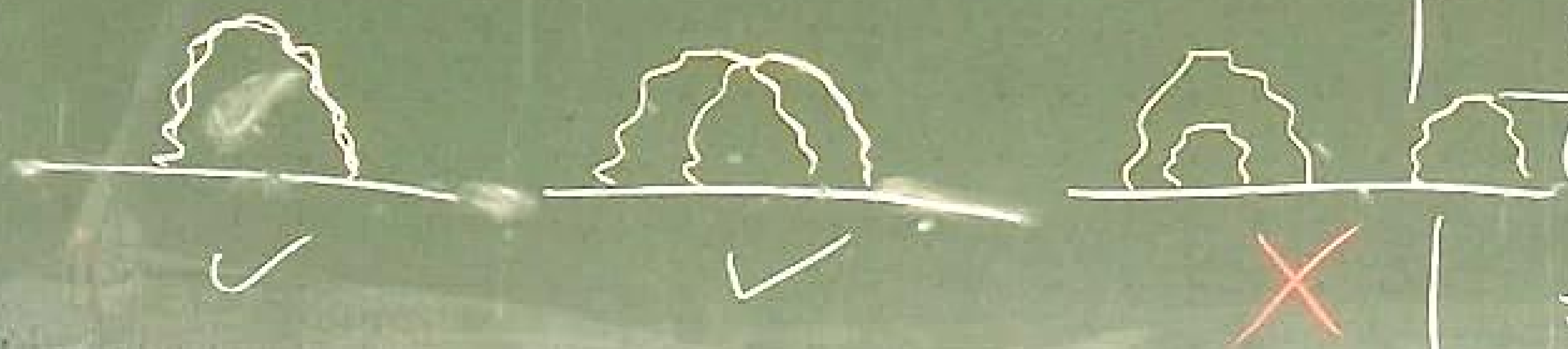


5] Summation to all orders in α.

i] Definition:

One-particle irreducible (1PI) diagram = Bridgeless one-particle diagram

Example



Furthermore:

$$-i\Sigma(p) := \{\text{Sum of all 1PI diagrams}\}$$

$$= -i\Sigma_2(p) + O(\alpha^2)$$

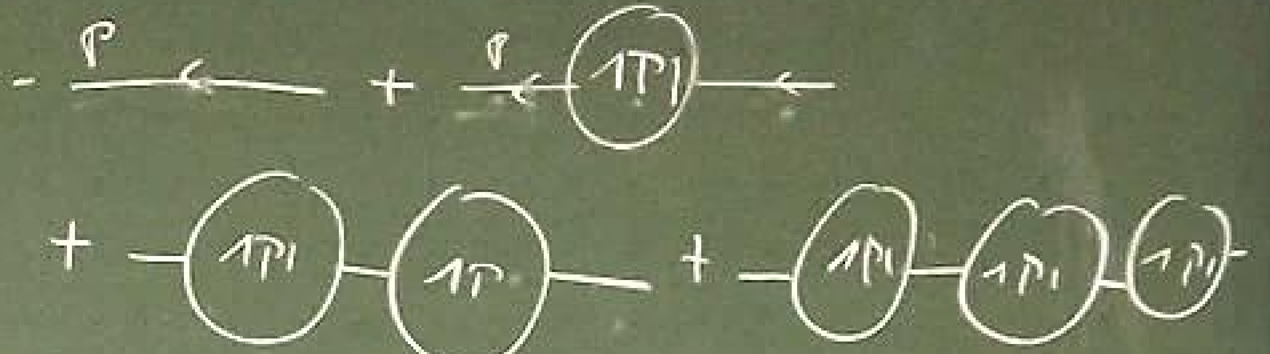
$$\frac{i(\not p + m_0)}{(\not p - m_0)(\not p + m_0)} = \frac{i}{\not p - m_0}$$

$\not p^2 = p^2$, $\not p$; $p^2 = \not p^2$

$$\Sigma(p) = f(\gamma^\mu p_\mu) + g(p^\mu p_\mu) + c(\gamma^\mu p_\mu)$$

ii] $\int d^4x\, e^{ipx} \langle\Omega| T \psi(x) \bar\psi(0) |\Omega\rangle$

$= \{\text{Sum of all one-particle diagrams}\}$



$$= \frac{i(\not p + m_0)}{p^2 - m_0^2} + \frac{i(\not p + m_0)}{p^2 - m_0^2}\left(-i\Sigma(p)\right)\frac{i(\not p + m_0)}{p^2 - m_0^2} + \dots$$

$$= \frac{i}{\not p - m_0} \sum_{n=0}^{\infty} \left(\frac{\Sigma(p)}{\not p - m_0} \right)^n$$

$$= \frac{i}{\not p - m_0} \cdot \frac{1}{1 - \frac{\Sigma(\not p)}{\not p - m_0}}$$

$$= \frac{i}{\not p - m_0 - \Sigma(\not p)}$$

6) Laurent series:

$$\frac{i}{\not p - m_0 - \Sigma(\not p)} = \frac{i Z_2}{\not p - m} + \dots$$

→ Expect for $\not p = m\,\mathbb{1} = m$ a simple pole.

$$m - m_0 - \Sigma(\not p = m) \overset{!}{=} 0$$

$$\Leftrightarrow \boxed{m - m_0 = \Sigma(\not p = m)}$$ ← implicit eq. for m

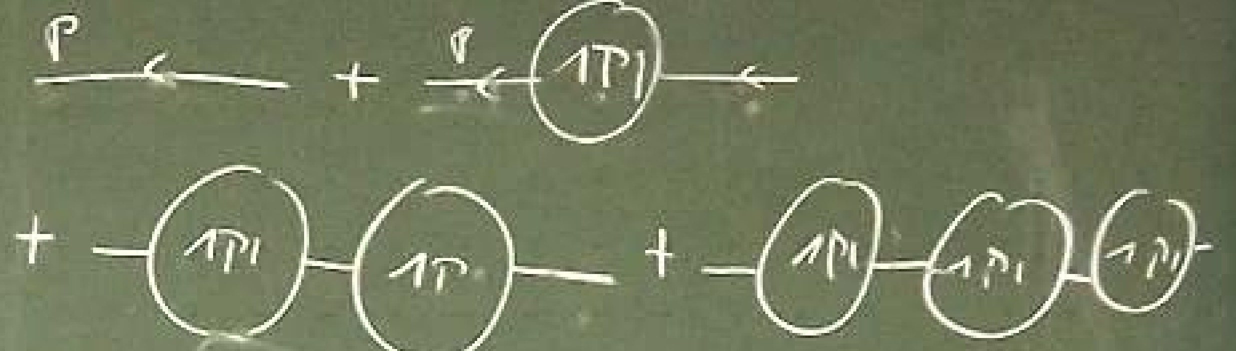
• $$\not p - m_0 - \Sigma(\not p) = (\not p - m)\left(1 - \frac{d\Sigma(\not p)}{d\not p}\right)\Big|_{\not p = m} + O((\not p - m)^2)$$

$$\Rightarrow \boxed{Z_2 = \left(1 - \frac{d\Sigma}{d\not p}\Big|_{\not p = m}\right)^{-1}}$$

$$\frac{1}{1-x} = 1 + x + O(x^2)$$

$$\int d^4x\, e^{ipx} \langle \Omega | T \psi(x) \bar\psi(0) | \Omega \rangle$$

$= \{$ Sum of all one-particle diagrams $\}$



$$\frac{i(\not p + m_0)}{p^2 - m_0^2} + \frac{i(\not p + m_0)}{p^2 - m_0^2}(-i\Sigma_p)\frac{i(\not p + m_0)}{p^2 - m_0^2} + \dots$$

$$\frac{i}{\not p - m_0}\sum_{n=0}^{\infty} \left(\frac{\Sigma(\not p)}{\not p - m_0}\right)^n$$

7) Results in leading order: $O(\alpha)$

i) Physical mass

$$\delta m = m - m_0 = \Sigma(\not p = m) = \Sigma_2(\not p = m) + O(\alpha^2)$$

(where $m = m_0 + O(\alpha)$)

$$= \Sigma_2'(m_0) + O(\alpha^2)$$

$m = m_0 + O(\alpha)$ (P-Set 10)

$$\overset{\Lambda \to \infty}{\sim} \frac{3\alpha}{4\pi} m_0 \log\left(\frac{\Lambda^2}{m^2}\right) \xrightarrow{\Lambda \to \infty} \infty$$

→ Mass shift is UV-divergent

(→ later: Renormalization)

ii) Field strength renormalization

$$\delta Z_2 = Z_2 - 1 = \left.\frac{d\Sigma}{d\not p}\right|_{\not p = m} + O(\alpha^2)$$

$$= \left.\frac{d\Sigma_2}{d\not p}\right|_{\not p = m} + O(\alpha^2)$$

$$\stackrel{\text{?}}{=} \frac{\alpha}{2\pi}\int_0^1 dx \left\{ -x \log\left[\frac{x\Lambda^2}{(1-x)^2 m^2 + x\mu^2}\right] + 2(2-x)\,\frac{x(1-x)m^2}{(1-x)^2 m^2 + x\mu^2} \right\}$$

→ Field-strength renormalisation is also UV-divergent

- $\delta Z_2 = -F_1^{(1)}(0)$

$$F_1(q^2) = 1 + F_1^{(1)}(q^2) + \delta Z_2$$

(↑ LSZ reduction formula)

$$= 1 + \left(F_1^{(1)}(q^2) - F_1^{(1)}(0)\right)$$

6.5. Electric Charge Renormalization