Recap:

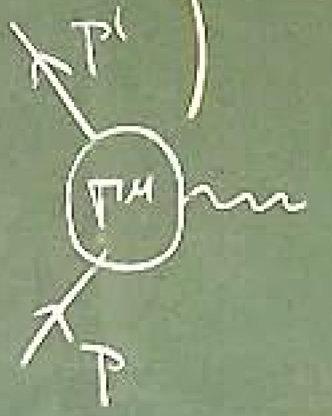
6 Radiative Corrections

6.3. The Electron Vertex Function

6.3.3. Evaluation

Form factors

$$\boxed{\Gamma^\mu(p',p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2)}$$



$$\sum_{n=0}^{\infty} \alpha^n F_1^{(n)}(q^2) = 1 + \alpha F_1^{(1)}(q^2) + O(\alpha^2)$$

$$\sum_{n=0}^{\infty} \alpha^n F_2^{(n)}(q^2) = \alpha F_2^{(1)}(q^2) + O(\alpha^2)$$

$F_1^{(1)}(q^2)$: • UV-divergence → Pauli-Villars regulator ($\Lambda \to \infty$) → Dropped out due to "ad hoc subtraction" $F_1^{(1)}(q^2) - F_1^{(1)}(0)$ (↑ LSZ reduction formula, → Field strength renormalization)

• IR-divergence → Photon mass ($\mu \to 0$) → still there ?!

$$\boxed{F_1^{(1)}(q^2) = \frac{1}{2\pi}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1)\left\{\log\left(\frac{\cdots}{\cdots}\right) + \frac{\cdots}{\cdots + z\mu^2} - \frac{\cdots}{\cdots + z\mu^2}\right\}}$$

$F_2^{(1)}(q^2)$: no divergencies

$$\boxed{F_2^{(1)}(q^2) = \frac{1}{2\pi}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(\ldots)\left\{\frac{\cdots}{\cdots}\right\}} \Rightarrow g = 2 + 2F_2(0)$$

Anomalous magnetic moment:

$$\Rightarrow \boxed{a_e \equiv \frac{g-2}{2} = F_2(0) = \alpha F_2^{(1)}(0) + O(\alpha^2) = \frac{\alpha}{2\pi} \approx 0.0011614}$$

"Schwinger term"

6.3.4. The Infrared Divergence

1) Goal: $|F_1(q^2)| \longrightarrow \infty$ for $\mu \to 0$

2) P-S ed 10

$$F_1(q^2) \overset{\mu\to 0}{\sim} 1 - \frac{\alpha}{2\pi} f_{IR}(q^2) \log\left(\frac{A}{\mu^2}\right) + O(\alpha^2)$$

where $A \in \{-q^2, m^2\}$

and $$f_{IR}(q^2) = \int_0^1 d\xi \, \frac{m^2 - q^2/2}{m^2 - q^2 \xi(1-\xi)} - 1 \geq 0$$

3) Cross section of electron scattering and a static potential.

$$\frac{d\sigma(p \to p')}{d\Omega} \overset{\mu\to 0}{\sim} \underbrace{\left(\frac{d\sigma}{d\Omega}\right)_0}_{\text{Tree level}} \cdot \left\{ 1 - \frac{\alpha}{\pi} f_{IR}(q^2) \log\left(\frac{A}{\mu^2}\right) \boxed{+ O(\alpha^2)} \right\}$$

4) Limit $-q^2 \to \infty$

$$f_{IR}(q^2) \sim \int_0^1 d\xi \, \frac{-q^2/2}{-q^2 \xi(1-\xi) + m^2} \overset{\circ}{\sim} \log\left(\frac{-q^2}{m^2}\right)$$

$$\to F_1(-q^2 \to \infty) \overset{\mu\to 0}{\sim} 1 - \frac{\alpha}{2\pi} \underbrace{\log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right)}_{\text{Sudakov double logarithm}} \boxed{+ O(\alpha^2)}$$

5) Compare to Bremsstrahlung:

- $$\frac{d\sigma(p \to p')}{d\Omega} \overset{\mu\to 0}{\sim} \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right) + O(\alpha^2) \right]$$
- $$\frac{d\sigma(p \to p' + \gamma)}{d\Omega} \overset{\mu\to 0}{\sim} \left[+ \frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right) + O(\alpha^2) \right] \left(\frac{d\sigma}{d\Omega}\right)_0$$

Both are divergent but sum is finite and independent μ!

6) Solution

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} = \frac{d\sigma(p \to p')}{d\Omega} + \frac{d\sigma(p \to p' + \gamma)}{d\Omega}\bigg|_{\omega < E_{min}}$$

7| For general q:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} \overset{\mu\to 0}{\sim} \left(\frac{d\sigma}{d\Omega}\right)_0 \left\{ \underbrace{1-\frac{\alpha}{\pi} f_{IR}(q^2)\log\left(\frac{\Lambda}{\mu^2}\right)}_{\text{Elastic scattering}} + \underbrace{\frac{\alpha}{\pi} I(p,p')\log\left(\frac{E_{min}^2}{\mu^2}\right)}_{\text{Bremsstrahlung}} + O(\alpha^2)\right\}$$

with $I(p,p') = \int \frac{d\Omega_k}{4\pi} \sum_r \left| \frac{p'\epsilon^r}{p'\hat{k}} - \frac{p\epsilon^r}{p\hat{k}} \right|$

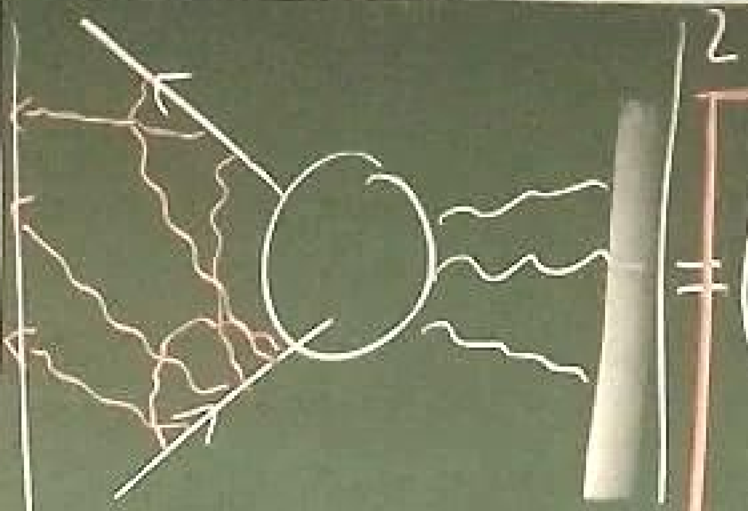
8| Show: $I(p,p') \overset{*}{=} 2 f_{IR}(q^2)$

9| Then:

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} \overset{\mu\to 0}{\sim} \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1-\frac{\alpha}{\pi} f_{IR}(q^2)\log\left(\frac{\Lambda}{E_{min}^2}\right) + O(\alpha^2)\right] \overset{-q^2\gg m^2}{\approx} -\text{''}- \left[1-\frac{\alpha}{\pi}\overbrace{\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{E_{min}^2}\right)}^{\text{Sudakov double log.}} + O(\alpha^2)\right]}$$

$\nwarrow$ μ dropped out, dependent on E_{min}

6.3.5. Summation and Interpretation of IR divergences



$$= \boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} = \left(\frac{d\sigma}{d\Omega}\right)_0 \exp\left[-\frac{\alpha}{\pi} f_{IR}(q^2)\log\left(\frac{-q^2}{E_{min}^2}\right)\right] \overset{-q^2\gg m^2}{\sim} -\text{''}- \exp\left[-\frac{\alpha}{\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{E_{min}^2}\right)\right]}$$

6.4 Field-Strength Renormalization

6.4.1 Structure of Two-Point Correlators in Interacting Theories

∢ ϕ^4-theory

1| Goal: Structure of $\langle\Omega| T\phi(x)\phi(y) |\Omega\rangle$ for interacting theories.

2| Interpretation for free theories.

$\langle 0| T\phi(x)\phi(y) |0\rangle$ = Propagation amplitude for particle from $y \to x$ $(x^0 > y^0)$

3| Mathematical preliminaries

i| ∢ Hilbert space $\mathcal{H}_{int}$

ii| $O(1,3) \ltimes \mathbb{R}^{1,3}$ = Poincaré group

↳ Unitary rep. $U(\Lambda, a_\mu)$ on $\mathcal{H}_{int}$

$$= e^{i a_\mu P^\mu} e^{i \Lambda_{\mu\nu} M^{\mu\nu}}$$

↳ $[P^\mu, P^\nu] = 0$

$$P^\mu = \begin{pmatrix} P^0 = H \\ P^i = \vec{P} \end{pmatrix} \Rightarrow [H, \vec{P}] = 0$$

→ Basis $|\lambda_{\vec{p}}\rangle$.

Energy

· $H |\lambda_{\vec{p}}\rangle = E_{\vec{p}}(\lambda) |\lambda_{\vec{p}}\rangle$

· $\vec{P} |\lambda_{\vec{p}}\rangle = \vec{p} |\lambda_{\vec{p}}\rangle$

Momentum

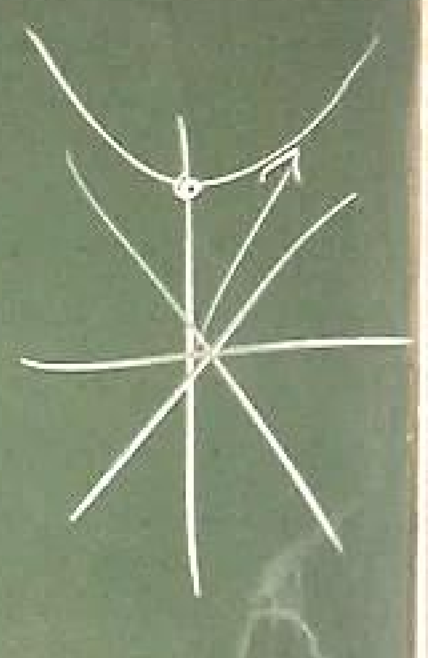
iii| $P^\mu |\lambda_{\vec{p}}\rangle = p^\mu |\lambda_{\vec{p}}\rangle$

$m_\lambda^2 = p^2 > 0$

∢ Boost $\Lambda_{\vec{p}} \in SO^+(1,3)$

$$\Lambda_{\vec{p}}^{-1} \begin{pmatrix} E_{\vec{p}} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} m_\lambda \\ \vec{0} \end{pmatrix}$$

$E_{\vec{p}}(\lambda) = \sqrt{\vec{p}^2 + m_\lambda^2}$



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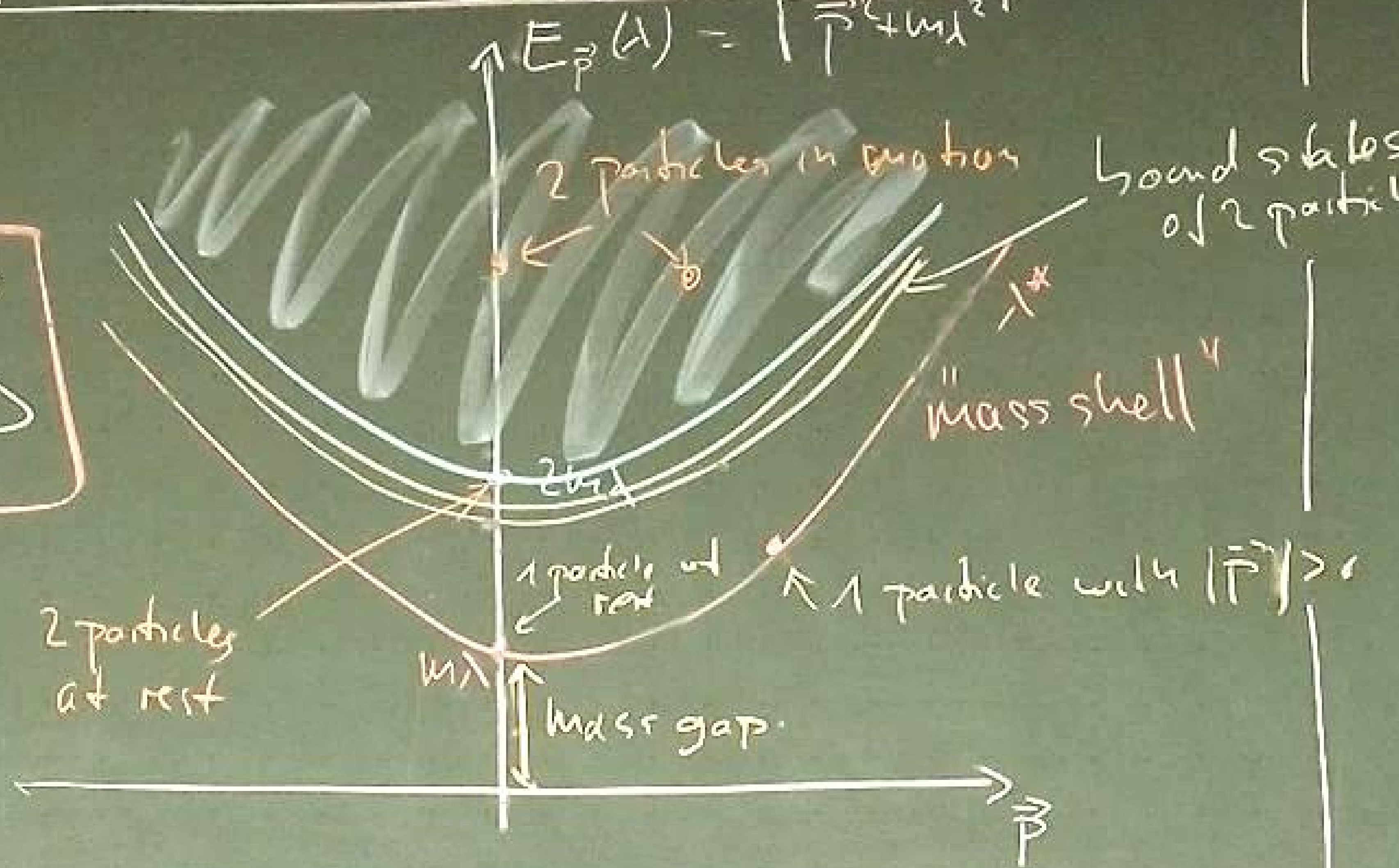
iii| $\Lambda_{\vec p}^{-1}\begin{pmatrix}E_p\\ \vec p\end{pmatrix} = \begin{pmatrix}m_\lambda\\ 0\end{pmatrix}$)

Show: $U^\dagger(\Lambda,0)\, P^\mu\, U(\Lambda,0) \hat{=} \Lambda^\mu{}_\nu P^\nu$

$$\boxed{\begin{array}{c}\forall |\lambda_{\vec p}\rangle\ \exists \Lambda_{\vec p}\ \exists |\lambda_0\rangle : |\lambda_0\rangle \equiv U(\Lambda_{\vec p}^{-1})|\lambda_{\vec p}\rangle \\ H|\lambda_0\rangle = m_\lambda |\lambda_0\rangle \text{ and } \vec P|\lambda_{\vec 0}\rangle = \vec 0\, |\lambda_{\vec 0}\rangle\end{array}}$$

$$P^\mu|\lambda_0\rangle = \begin{pmatrix}m_\lambda\\ 0\end{pmatrix}|\lambda_0\rangle$$

iv| Typical spectrum of $P^\mu = (H, \vec P)$



v| Identity on $\mathcal{H}_{int}$

$$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p(\lambda)}\, |\lambda_{\vec p}\rangle\langle\lambda_{\vec p}|$$

4| $x^0 > y^0$

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle$$

$$= \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec p}(\lambda)} \langle\Omega|\phi(x)|\lambda_{\vec p}\rangle\langle\lambda_{\vec p}|\phi(y)|\Omega\rangle \quad + \text{const}$$

5| $\langle\Omega|\phi(x)|\lambda_{\vec p}\rangle = \langle\Omega| e^{iPx}\phi(0)e^{-iPx}|\lambda_{\vec p}\rangle$

$$= \langle\Omega|\phi(0)|\lambda_{\vec p}\rangle e^{-ipx}\Big|_{p^0=E_p(\lambda)}$$

$$\underline{U(\Lambda_p)}\, U^{-1}(\Lambda_p)\, U(\Lambda_p)|\lambda_0\rangle$$

$$\phi(\Lambda^{-1}_p \vec 0)$$

$$= \langle\Omega|\phi(0)|\lambda_0\rangle e^{-ipx}\Big|_{p^0=E_p(\lambda)}$$

6| $\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda |\langle\Omega|\phi(0)|\lambda_0\rangle|^2$

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p(\lambda)} e^{-ip(x-y)}\Big|_{p^0=E_p(\lambda)}$$

$$\overset{x^0>y^0}{=} \sum_\lambda |\dots|^2 \underbrace{\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m_\lambda^2+i\epsilon}e^{-ip(x-y)}}_{D_F(x-y;\,m_\lambda)}$$

$$\overset{x^0<y^0}{=} \langle\Omega|\phi(y)\phi(x)|\Omega\rangle$$

7| → Källén-Lehman spectral representation

$$\boxed{\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \frac{dM^2}{2\pi}\rho(M^2)\, D_F(x-y;M^2)}$$

with spectral density:

$$\boxed{\rho(M^2) = 2\pi\sum_\lambda \delta(M^2-m_\lambda^2)\,|\langle\Omega|\phi(0)|\lambda_0\rangle|^2}$$