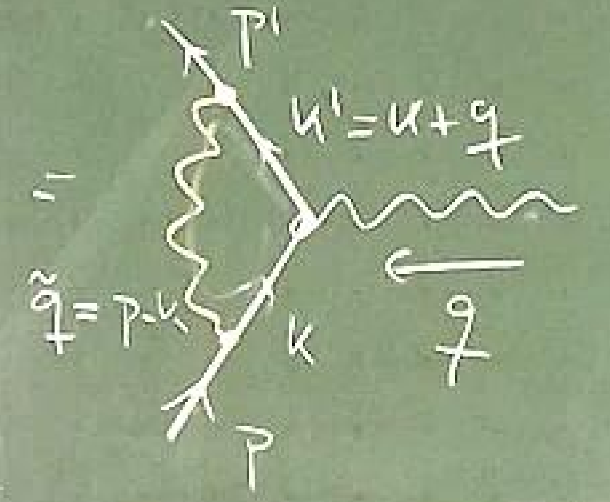
Recap.

6. Radiative Corrections

6.3. The Electron Vertex function

6.3.3. Evaluation

1) $\bar{u}(p')\left[\alpha \Gamma^{(1)}(p',p)\right]^{\mu} u(p)$

$=$ 

$\stackrel{\circ}{=}$ $$2ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\boxed{\bar{u}(p')\left[\not{k}\,\gamma^\mu \not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu\right] u(p)}\ (**)}{\boxed{(\tilde{q}^2+i\varepsilon)(k'^2-m^2+i\varepsilon)(k^2-m^2+i\varepsilon)}\ (*)}$$

2) Feynman parameters

3) $$\frac{1}{(*)} = \int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1)\, \frac{2}{D^3}$$

with $D \equiv l^2 - \Delta + i\varepsilon$

$l \equiv k + yq - zp \qquad \Delta \equiv -xyq^2 + (1-z)^2 m^2 > 0$

4) $k^\mu = l^\mu - y q^\mu + z p^\mu$

$$(**) \stackrel{\circ}{\simeq} \bar{u}(p')\Big\{ -\frac{1}{2}\gamma^\mu l^2 + \left[-y\not{q} + z\not{p}\right]\gamma^\mu \times \left[(1-y)\not{q} + z\not{p}\right] + m^2\gamma^\mu - 2m\left[(1-2)y q^\mu + 2z p^\mu\right] \Big\} u(p)$$

- $\displaystyle\int \frac{d^4l}{(2\pi)^4} \frac{l^\mu}{D(l)} = 0$

Script

- $\displaystyle\int \frac{d^4l}{(2\pi)^4} \frac{l^\mu l^\nu}{D(l)} = \int \frac{d^4l}{(2\pi)^4} \frac{g^{\mu\nu}}{4} \frac{l^2}{D(l^2)}$

$\stackrel{\circ}{=}$

- $\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu \not{p}$
- $\not{p}u(p) = m\,u(p)$

  $\bar{u}(p')\not{p}' = m\,\bar{u}(p')$
- $x+y+z=1$

$\underline{5]}$ C integral / $\int dx \int dy$) odd

$\hookrightarrow C \cong 0$

$$\ominus\ \bar{u}(p')\Big\{\gamma^\mu \underbrace{\Big[-\tfrac{1}{2}l^2 + (1-x)(1-y)q^2 + (1-\underline{2z-z^2})m^2\Big]}_{A} + (p+p')^\mu \underbrace{[mz(z-1)]}_{B} + \underset{(p'-p)^\mu}{q^\mu} \underbrace{[m(z-2)(x-y)]}_{C}\Big\}u(p)$$

$\underline{6]}$ Use Gordon identity on $(p+p')^\mu$.

$$\bar{u}(p')[\delta\Gamma^{(1)}(p',p)]^\mu u(p) = 2ie^2 \int \frac{d^4l}{(2\pi)^4} \int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1)\, \frac{2}{D^3}$$

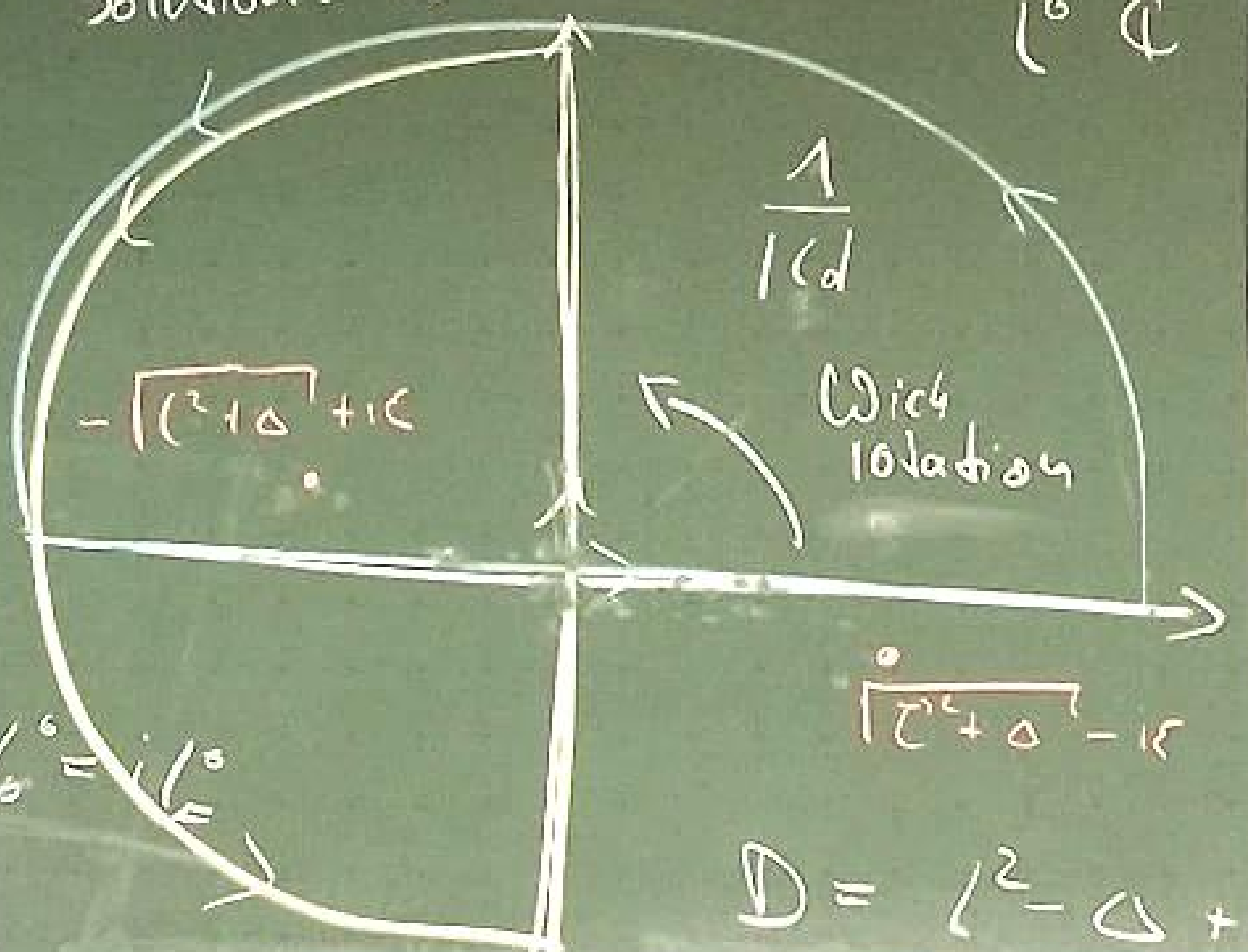
$$\times\ \bar{u}(p')\Big\{\gamma^\mu\Big[-\tfrac{1}{2}l^2 + (1-x)(1-y)q^2 + (1-\underline{4z+z^2})m^2\Big] + \frac{i\sigma^{\mu\nu}q_\nu}{2m}[2m^2 z(1-z)]\Big\}u(p)$$

$$l^2 \to l^2 - \Delta^2$$

7) Momentum integral

1) Problem: $l^2 = l^{0\,2} - \vec{l}^{\,2}$

Solution: Wick rotation



$$D = l^2-\Delta+i\epsilon \to l^{0\,2} - (\vec l^{\,2}+\Delta)+i\epsilon$$

New contour:

$$\boxed{\begin{aligned} & l^0 \equiv i l^0_E, \quad \vec l = \vec l_E \qquad l_E \in \mathbb{R}^4 \\ \Rightarrow\ & l^2 = -(l^0_E)^2 - \vec l_E^{\,2} = -l_E^2 \end{aligned}} \quad (*)$$

ii) Then

$$\lim_{\epsilon\to 0}\int \frac{d^4 l}{(2\pi)^4}\frac{1}{(l^2-\Delta+i\epsilon)^m}$$

$$\overset{(*)}{=} \frac{i}{(-1)^m}\frac{1}{(2\pi)^4}\int d^4 l_E \frac{1}{(l_E^2+\Delta)^m}$$

$$= \frac{i(-1)^m}{(2\pi)^4}\underbrace{\int d\Omega_4}_{2\pi^2}\int dl_E \frac{l_E^3}{(l_E^2+\Delta)^m}$$

$$\overset{\boxed{m>2}}{=} \frac{i(-1)^m}{(4\pi)^2}\frac{1}{(m-1)(m-2)\,\Delta^{m-2}} \quad (*)$$

and

$$\lim_{\epsilon\to 0}\int \frac{d^4 l}{(2\pi)^4}\frac{l^2}{(l^2-\Delta+i\epsilon)^m}$$

$$\overset{\boxed{m>3}}{=} \frac{i(-1)^{m-1}}{(4\pi)^2}\frac{2}{(m-1)(m-2)(m-3)\,\Delta^{m-3}} \quad (**)$$

Problem: for $m=3$ integral diverges!

→ UV-divergence

iii) Pauli-Villars regularization:

$$\frac{-ig_{\mu\nu}}{q^2+i\varepsilon} \longmapsto \frac{-ig_{\mu\nu}}{q^2+i\varepsilon} - \frac{-ig_{\mu\nu}}{q^2-\Lambda^2+i\varepsilon}$$

for $\Lambda \to \infty$

Hope: Λ drops out in physical quantities

$\overset{\circ}{\Rightarrow}$ $\Delta_\Lambda = -xyq^2+(1-z)^2m^2+z\Lambda^2$

iv) Then:

- Additional term in (*) vanishes with $\frac{\Lambda}{\Lambda^2} \to 0$
- (**)

$$\lim_{\varepsilon\to 0}\int\frac{d^4l}{(2\pi)^4}\left[\frac{l^2}{(l^2-\Delta+i\varepsilon)^3}-\frac{l^2}{(l^2-\Delta_\Lambda+i\varepsilon)^3}\right]$$

$$\doteq \frac{i}{(4\pi)^2}\log\left(\frac{\Delta_\Lambda}{\Delta}\right)$$

$$\overset{\Lambda\to\infty}{\sim} \frac{i}{(4\pi)^2}\log\left(\frac{z\Lambda^2}{\Delta}\right)$$

8) Result.

$$\bar{u}(p')\left[\alpha\Gamma^1(p',p)\right]^\mu u(p) \doteq \frac{\alpha}{2\pi}\int_0^1 dx\int_0^1 dy\int_0^1 dz\,\delta(x+y+z-1)\,\bar{u}(p')\Bigg\{\gamma^\mu\underbrace{\left[\log\left(\frac{z\Lambda^2}{\Delta}\right)+\frac{(1-x)(1-y)q^2}{\Delta}+\frac{(1-4z+z^2)m^2}{\Delta}\right]}_{\sim F_1(q^2)} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\underbrace{\frac{2m^2z(1-z)}{\Delta}}_{\sim F_2(q^2)}\Bigg\}u(p)$$

9) αF_1 $\quad 1+O(\alpha)$

i) Problem 1: $F_1(0)=1 \Rightarrow F^{(n)}(0)=0$ $\quad n=1,2,\dots$

But. $F_1^{(n)}(0)\neq 0$

Fix. $F_1^{(n)}(q^2)\mapsto F^{(n)}(q^2)-F^{(n)}(0)$

ii) Problem 2.

F_2 has a IR-divergence for $q^2\to 0$.

α $q^2=0$

$$\int_0^1 dx\int_0^1 dy\int_0^1 dz\,\delta(x+y+z-1)\frac{1-4z+z^2}{(1-z)^2} = \int_0^1 dz\int_0^{1-z} dy\,\frac{-2+(1-z)(3-z)}{(1-z)^2}$$

$$= \int_0^1 dz\,\frac{-2}{(1-z)} + \text{finite} = \infty$$

ii) Fix 2: Add small photon mass: $\mu > 0$.

$$\boxed{\Delta \mapsto \Delta_\mu = -xyq^2 + (1-z)^2 m^2 + z\mu^2}$$

iii) Fix 1 + Fix 2: $F_1(q^2) = 1 + \alpha F_1^{(1)}(q^2) + O(\alpha^2)$

with

$$\boxed{F_1^{(1)}(q^2) = \frac{1}{2\pi}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1) \left\{ \underbrace{\log\left(\frac{m^2(1-z)^2}{m^2(1-z)^2 - q^2xy}\right)}_{\Lambda \text{ vanished!}} + \frac{m^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy + \boxed{z\mu^2}} - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + \boxed{z\mu^2}} \right\}}$$

iv) $\not\!\!\; F_2$: $F_2(q^2) = \alpha F_2^{(1)}(q^2) + O(\alpha^2)$

with

$$\boxed{F_2^{(1)}(q^2) = \frac{1}{2\pi}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(\dots) \left[\frac{2m^2 z(1-z)}{m^2(1-z)^2 - q^2xy}\right]}$$

11/ g-factor:

$$F_2(q^2=0) = \frac{\alpha}{2\pi}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1)\, \frac{2z}{1-z} + O(\alpha^2)$$

$$= \frac{\alpha}{2\pi}\int_0^1 dz \int_0^{1-z} dy\, \frac{2z}{1-z} + O(\alpha^2)$$

$$= \frac{\alpha}{2\pi} + O(\alpha^2)$$

Anomalous magnetic moment:

$$\Rightarrow \boxed{\begin{aligned} a_e = \frac{g-2}{2} &\approx \frac{\alpha}{2\pi} \approx 0.0011614 \\ a_e^{exp} &\approx 0.0011597 \end{aligned}}$$

$\alpha^2 \sim 0.5 \cdot 10^{-4}$

- $a_e^{SM} = 0.001\ 159\ 652\ 18\ 2031\ldots$

  $a_e^{exp} = 0.001\ 159\ 652\ 18\ 073$

- $a_\mu^{(1)} = \frac{\alpha}{2\pi}$

  $a_\mu^{exp} - a_\mu^{SM} = 261(63)(48) \times 10^{-11}$

  g-2 experiment

$$a_e^{SM} = a_e^{QED} + a_e^{W} + a_e^{H}$$