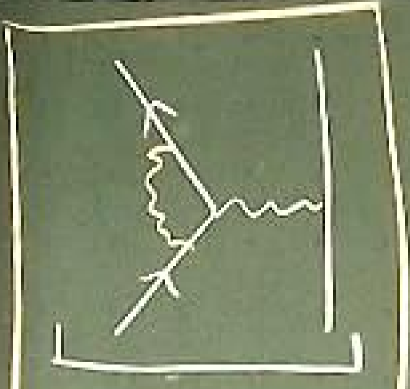
Recap

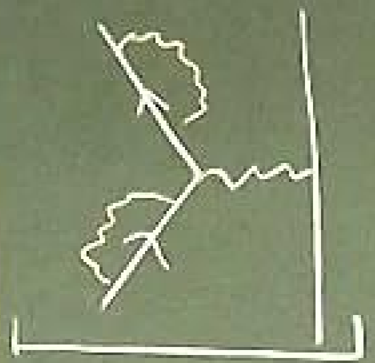
6 Radiative Corrections in QED

6.1 Overview

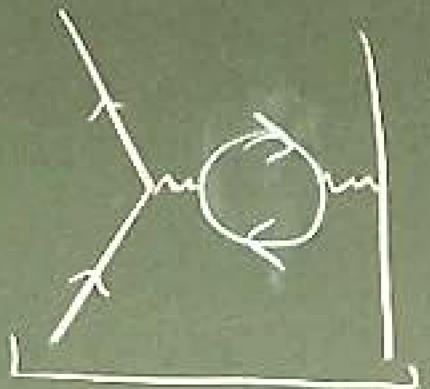


Vertex collection

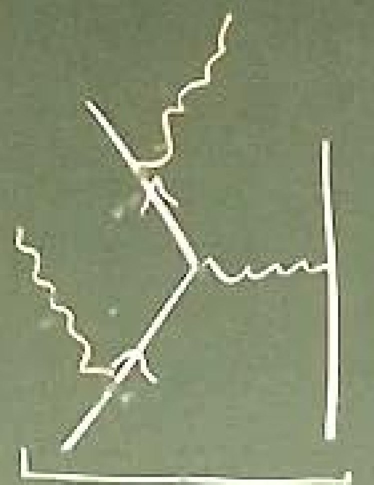
Today



External leg correction



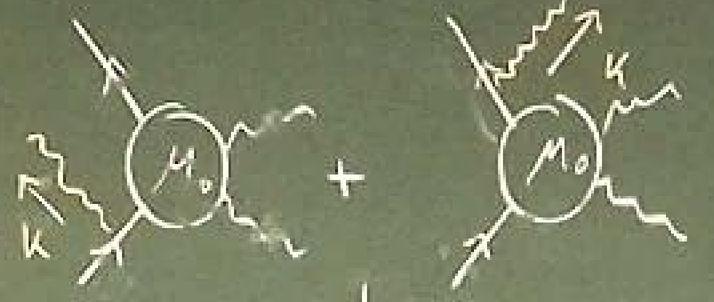
Vacuum polarization



Bremsstrahlung ✓

(→ IR divergence)

6.2 Soft Bremsstrahlung



- Soft photons: $|\vec{k}| \ll |\vec{p}\,' - \vec{p}|$
- Regularization: $\mu > 0$
- Relativistic limit

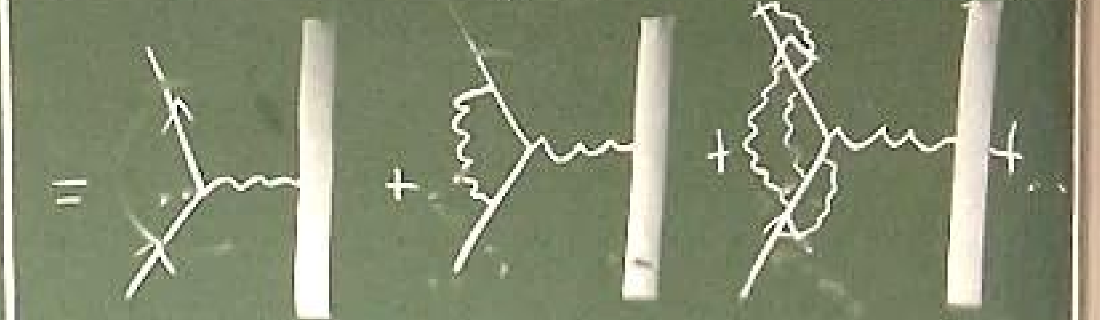
$\to 0$ $\quad q = p' - p \quad$ $\to \infty$

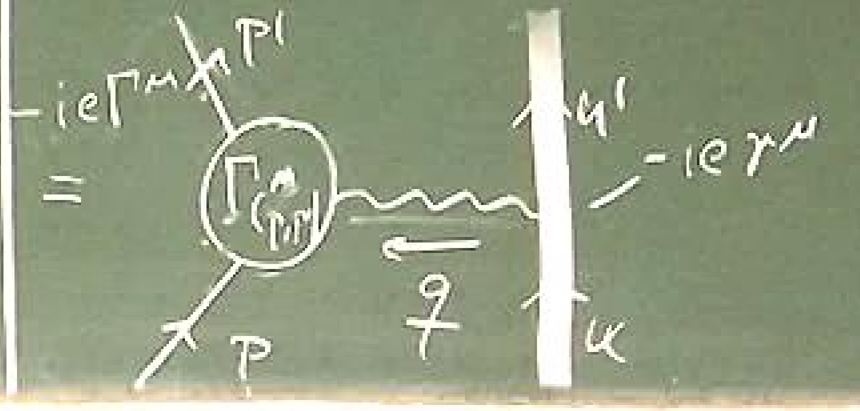
$$\boxed{d\sigma(p \to p' + \gamma) \approx d\sigma(p \to p') \cdot \frac{\alpha}{\pi} \underbrace{\log\left(\frac{-q^2}{\mu^2}\right) \log\left(\frac{-q^2}{m^2}\right)}_{\text{Sudakov double logarithm}}}$$

6.3 The Electron Vertex Function

6.3.1 Formal Structure

1] $i\mathcal{M}(e^-(p)\mu^-(k) \to e^-(p')\mu^-(k'))$





$$= ie^2 \left[\bar u_e(p') \boxed{\Gamma^\mu(p',p)} u_e(p)\right] \frac{1}{q^2} \left[\bar u_\mu(k') \gamma_\mu u_\mu(k)\right]$$

2| General form.

$$\Gamma^\mu(p,p') = f(p^\mu, p'^\mu, \gamma^\mu, m, e, \mathbb{C})$$

3| Restrictions.

i| Lorentz covariance. Γ^μ transform like γ^μ

$$\Gamma^\mu = A\gamma^\mu + \tilde B p^\mu + \tilde C p'^\mu$$

$$= A\gamma^\mu + B(p^\mu + p'^\mu) + C\cdot(\overbrace{p'^\mu - p^\mu}^{q^\mu})$$

ii| Recall: $\not p\, u(p) = m\cdot u(p)$

$$\bar u(p')\not p' = m\, u(p')$$

$$\to \underset{\substack{\uparrow \\ A,B,C}}{X} = X(p^\mu, p'^\mu, m, e, \mathbb{C})\cdot \mathbb{1}$$

iii| $p^2 = p'^2 = m^2 \to \underline{p p'}$

$$\underline{q^2} = (p'-p)^2 = 2(m^2 - \underline{pp'})$$

$$\to X = X(q^2, m, e, \mathbb{C})$$

iv| Ward identity for U(1) gauge symmetry of QED.

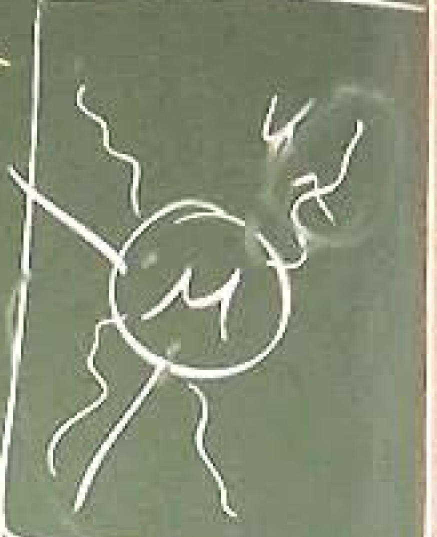
$$q_\mu \Gamma^\mu(p',p) \overset{*}{=} 0$$

$$\to 0 = q_\mu\Gamma^\mu \qquad p^2 = p'^2 = m^2$$

$$= A \underbrace{q_\mu\gamma^\mu}_{=0} + B\overbrace{q_\mu(p'^\mu + p^\mu)}^{=0}$$

$$+ C q^2$$

$$\to \boxed{C=0}$$



$$\mathcal{M}^\mu \varepsilon^*_\mu(k)$$
$$\Downarrow$$
$$\mathcal{M}^\mu k_\mu = 0$$

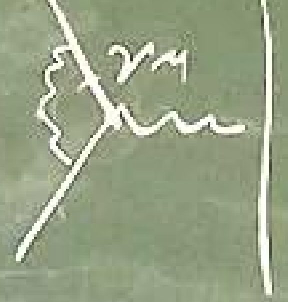
4| Gordon identity: $\frac{1}{2}[\gamma^\mu,\gamma^\nu]$

$$\bar u(p')\frac{p'^\mu+p^\mu}{2m}u(p) \overset{!}{=} \bar u(p')\gamma^\mu u(p) - \bar u(p')\frac{i\sigma^{\mu\nu}q_\nu}{2m}u(p)$$

5| $\Gamma^\mu = A\,\gamma^\mu + B\,(p^\mu+p'^\mu)$

$$\boxed{\Gamma^\mu(p,p') = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)}$$

Form factors



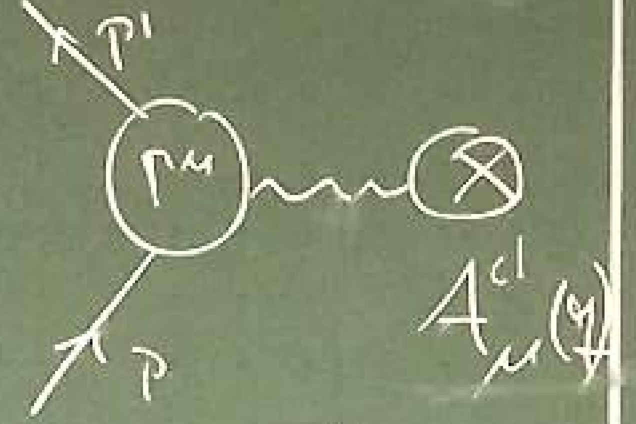
$1+\mathcal{O}(\alpha)$ $\qquad$ $0+\mathcal{O}(\alpha)$

6.3.2 The Landé g-factor

1| ∢ classical, ext field $A^{cl}_\mu(x)$

$$H_{int} = e\int d^3x\, \bar\psi(x)\gamma^\mu\psi(x)A^{cl}_\mu(x)$$

$\to i\mathcal{M}(2\pi)\delta(p'^0-p^0) =$



$$= -ie\,\bar u(p')\Gamma^\mu(p',p)u(p)\cdot A^{cl}_\mu(q=p'-p)$$

2| Electric charge:

i| ∢ $A^{cl}_\mu(x) = (\phi(\vec x),\vec 0)$

$$\Rightarrow A^{cl}_\mu(q) = \left((2\pi)\delta(q^0)\phi(\vec q),\vec 0\right)$$

ii| $i\mathcal{M} = -ie\,\bar u(p')\Gamma^0(p',p)u(p)\cdot\phi(\vec q)$

iii| ∢ $\phi(\vec x)$ slowly varying $\to$ $\phi(\vec q)$ peaked at $\vec q=0$

$\to$ limit $\vec q\to 0$

$$i\mathcal{M} \overset{\vec q\to 0}{\approx} -ie F_1(0)\,\bar u(p')\gamma^0 u(p)\,\phi(\vec q)$$

$$\overset{|\vec p|\ll m^2}{\approx} -ie F_1(0)\,(2m\,\xi'^\dagger\xi)\,\phi(\vec q)$$

iv) → 1st. Born approx:

$$V(\vec{x}) = e\, F_1(0)\, \phi(\vec{x}) \overset{!}{=} e \cdot \phi(\vec{x})$$

$$\Rightarrow \boxed{F_1(0) = 1} \Rightarrow \boxed{F_1^{(n)}(0) = 0 \quad n \geq 1}$$

$$F_1 = \sum_{n=0}^{\infty} F_1^{(n)} \alpha^n$$

$$(\vec{p} + \dots \vec{A}(\vec{x}))^2 = pA + Ap$$

3) Magnetic moment

i) $\measuredangle\ A_\mu^{cl}(x) = (0, \vec{A}(\vec{x})) \Rightarrow A_\mu^{cl}(q) = (0, 2\pi\delta(q^0)\vec{A}(\vec{q}))$

ii) $$iM = ie\, \bar{u}(p') \underbrace{\left[\gamma^i F_1(q^2) + \frac{i\sigma^{i\nu} q_\nu}{2m} F_2(q^2) \right]}_{} u(p)\, A_i^{cl}(\vec{q})$$

vanishes for $q = 0$ if $|\vec{p}|^2 \ll m^2$

→ expand in linear order of $\vec{q}$.

iii) $\measuredangle\ F_1$-term (lin. order of $\vec{p}, \vec{p}'$).

$$\bar{u}(p')\gamma^i u(p) \overset{o}{\approx} 2m\, \xi'^{\dagger} \left(\frac{\vec{p}' \cdot \vec{\sigma}}{2m} \sigma^i + \sigma^i \frac{\vec{p} \cdot \vec{\sigma}}{2m} \right) \xi$$

$\sigma^i \sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$

$$\overset{o}{=} \underbrace{\frac{p'^i + p^i}{2m}}_{} \underbrace{2m\, \xi'^{\dagger} \xi}_{} + \boxed{2m\, \xi'^{\dagger} \left(\frac{-i}{2m} \varepsilon^{ijk} q^j \sigma^k \right) \xi}$$

iv) $\measuredangle\ F_2$-term (expand $\bar{u}$ in lowest order)

$$\frac{i q_\nu}{2m} \bar{u}(p') \sigma^{i\nu} u(p) \overset{o}{\approx} \boxed{2m\, \xi'^{\dagger} \left(\frac{-1}{2m} \varepsilon^{ijk} q^j \sigma^k \right) \xi}$$

$\uparrow$

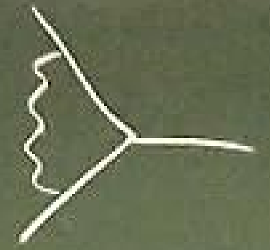
$\Big[\, u(p) \approx \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix},\ [\sigma^i, \sigma^j] = 2i\varepsilon^{ijk}\sigma^k,$
$q_i = -q^i \,\Big]$

v) Summary.

$$iM \overset{q \to 0}{\approx} -ie\, \xi'^{\dagger} \left\{ \frac{-1}{2m} \sigma^k \left[F_1(0) + F_2(0) \right] \right\} \xi \times \underbrace{\left[-i\varepsilon^{ijk} q^j A_{cl}^i(\vec{q}) \right]}_{B_{cl}^k(\vec{q})} (2m)$$

$$\vec{B}_{cl} = \nabla \times \vec{A}_{cl} \Rightarrow B^k = \varepsilon^{ijk} \partial_j A^i \Rightarrow B^k(\vec{q}) = -i\varepsilon^{ijk} q^j A_{cl}^i(\vec{q})$$

vi) → 1st. Born approx.:



$$V(\vec{x}) = -\langle\vec{\mu}\rangle\cdot\vec{B}$$

$$\Downarrow$$

$$\langle\vec{\mu}\rangle = \underbrace{\frac{e}{2m}}_{\mu_B}\, 2\left[F_1(0)+F_2(0)\right]\, \underbrace{\xi'^{\dagger}\overbrace{\boxed{\frac{\sigma^k}{2}}}^{\vec{S}}\xi}_{\langle\vec{S}\rangle}$$

$$\equiv g\cdot\mu_B\langle\vec{S}\rangle$$

$\frac{e}{2m}$: Bohr magneton

⇒ Landé g-factor:

$$\boxed{g = 2\left[F_1(0)+F_2(0)\right] = 2 + 2F_2(0)}$$

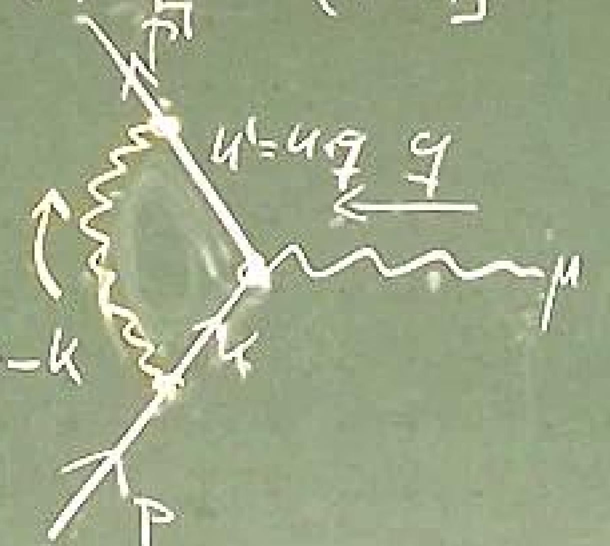
$$\boxed{= \underbrace{2}_{\text{Dirac equation}} + \underbrace{2\alpha F_2^{(1)}(0)}_{\text{Anomalous magnetic moment}} + O(\alpha^2)}$$

6.3.3. Evaluation

1) Scattering amplitude:

$$u(p')\,\alpha\left[\Gamma^{(1)}(p',p)\right]^{\mu}u(p)$$

=



$\tilde{q} = p - k$

$$= \int\frac{d^4k}{(2\pi)^4}\,\bar{u}(p')(-ie\gamma^{\sigma})\frac{i(k'+m)}{k'^2-m^2+i\epsilon}\gamma^{\mu}\frac{i(k+m)}{k^2-m^2+i\epsilon}(-ie\gamma^{\rho})u(p) \times \frac{-ig_{\rho\sigma}}{\tilde{q}^2+i\epsilon}$$

Contraction identities $\gamma^{\nu}\gamma^{\mu}\gamma_{\nu} = -2\gamma^{\mu}$

$$iM \overset{q\to 0}{\approx} -ie\,\overbrace{\xi'^{\dagger}\left\{\frac{-1}{2m}\sigma^k\left[F_1(0)+F_2(0)\right]\right\}\xi}^{\langle\vec{\mu}\rangle} \times \underbrace{\left[-i\epsilon^{ijk}q^j A^i_{cl}(\vec{q})\right]}_{B^k_{cl}(\vec{q})}\cdot(2m)$$

$$\vec{B}_{cl} = \nabla\times\vec{A}_{cl} \Rightarrow B^k = \epsilon^{ijk}\partial_j A^i \Rightarrow B^k(\vec{q}) = -i\epsilon^{ijk}q^j A^i_{cl}(\vec{q})$$

$$\doteq 2ie^2\int\frac{d^4k}{(2\pi)^4}\frac{\bar{u}(p')\left[\cancel{k}\gamma^\mu\cancel{k'}+m^2\gamma^\mu-2m(k+k')^\mu\right]u(p)}{\underbrace{(\hat{q}^2+i\varepsilon)}_{A_1}\underbrace{(k'^2-m^2+i\varepsilon)}_{A_2}\underbrace{(k^2-m^2+i\varepsilon)}_{A_3}}$$

2) Feynman parameters:

$$\frac{1}{A_1\cdots A_n}=\left(\prod_{i=1}^{n}\int_0^1 dx_i\right)\delta\left(\sum_i x_i-1\right)\frac{(n-1)!}{[x_1A_1+\dots+x_nA_n]^n}$$

Feynman parameters

3) $$\frac{1}{A_1A_2A_3}=\int_0^1 dx\,dy\,dz\;\delta(x+y+z-1)\frac{2}{\boxed{D^3}}$$

$$\boxed{D\doteq k^2+2k(yq-zp)+yq^2+zp^2-(x+y)m^2+i\varepsilon}$$

$x+y+z=1$
$\hat{q}=p-k$
$k'=k+q$

$\ell\equiv k+yq-zp$

$$=\boxed{\ell^2-\Delta+i\varepsilon}$$

$$\boxed{\Delta=-xyq^2+(1-z)^2m^2>0}$$