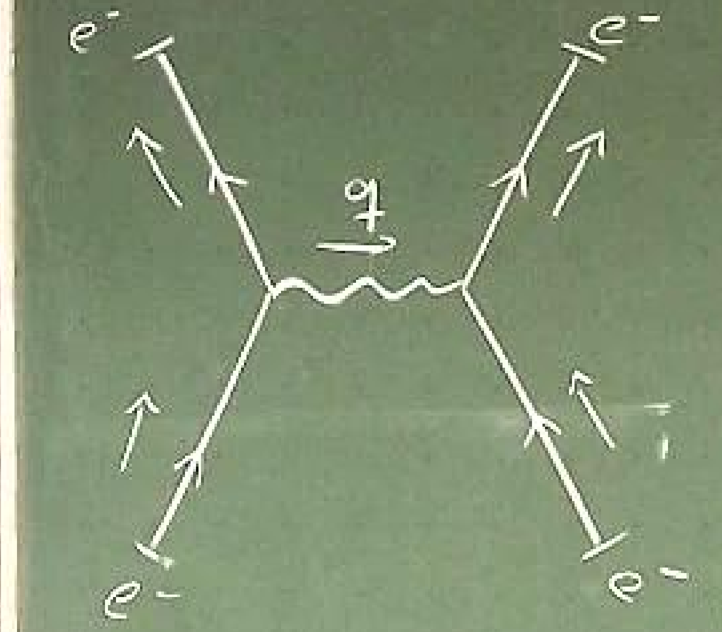
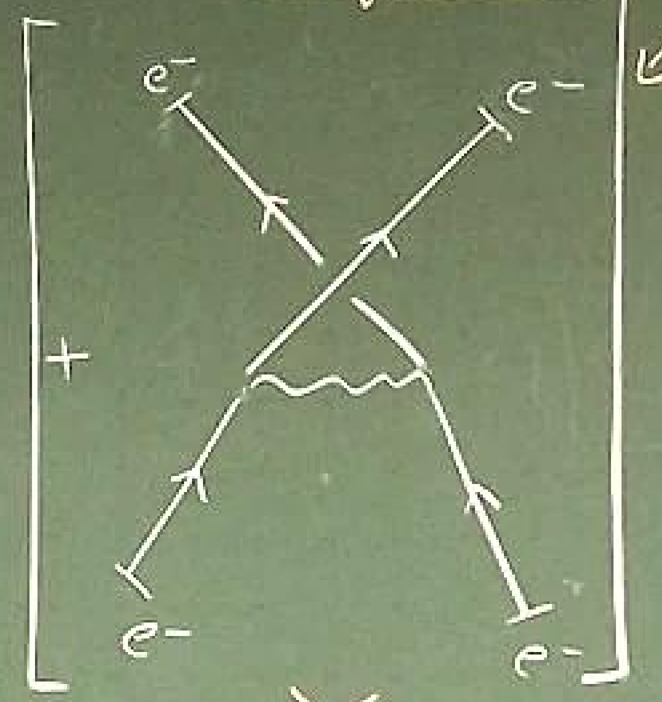
Recap

Coulomb Potential



Scattering amplitude for indistinguishable fermions



nonrel. limit

$\rightarrow \boxed{V(\vec{r}) = +\frac{e^2}{4\pi r}}$

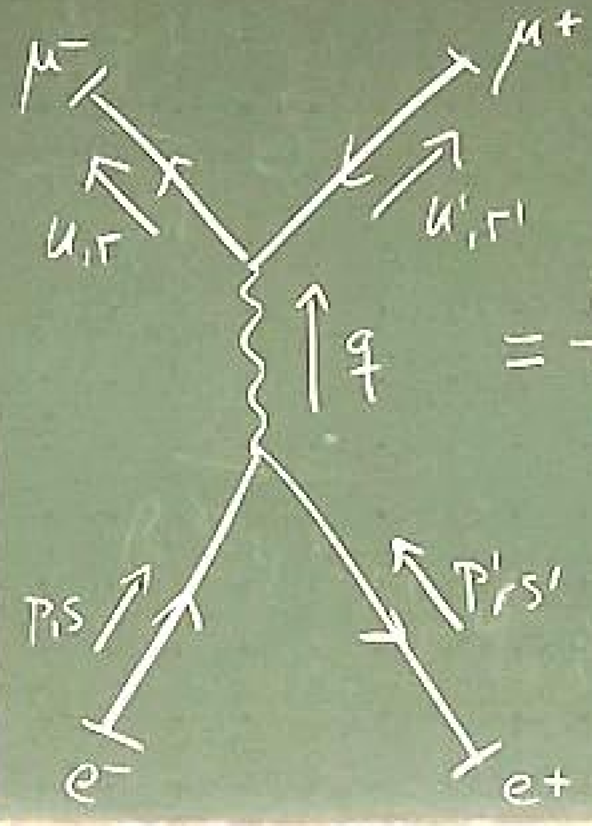
repulsive Coulomb potential

fermions distinguishable

5. Elementary Processes of QED

5.1 Cross section of $e^+e^- \to \mu^+\mu^-$ scattering

3] Tree-level amplitude



$$= \frac{ie^2}{q^2}\left[\bar{v}_e^{s'}(p')\gamma^\mu u_e^s(p)\right]\left[\bar{u}_\mu^r(k)\gamma_\mu v_\mu^{r'}(k')\right]$$

$$p+p' = q = k+k'$$

4] $$|M|^2 = \frac{e^4}{q^4}\underbrace{\left[\bar{v}(p')\gamma^\mu u(p)\,\bar{u}(p)\gamma^\nu v(p')\right]}_{(+)}$$
$$\times\left[\bar{u}(k)\gamma_\mu v(k')\,\bar{v}(k')\gamma_\nu u(k)\right]$$

5] Average in-Spin / Sum over out-Spin.

$$d\sigma \propto \frac{1}{4}\sum_{ss'}\sum_{rr'}\left|M(s,s' \to r r')\right|^2$$

6] Use spin sums.

$$\overbrace{\sum_{ss'} \bar{v}_a^{s'}(p')\gamma^\mu_{ab}u_b^s(p)\,\bar{u}_c^s(p)\gamma^\nu_{cd}v_d^{s'}(p')}^{(+)}$$

$$\overset{\circ}{=} \underbrace{\mathrm{Tr}\left[(\not{p}'-m_e)\gamma^\mu(\not{p}+m_e)\gamma^\nu\right]}_{(*)}$$

8| Trace technology (PSet 7)

9| $(*) = 4\left[p'^{\mu}p^{\nu} + p'^{\nu}p^{\mu} - g^{\mu\nu}(pp' + m_e^2)\right]$

10| $m_e/m_\mu \approx 1/200 \longrightarrow m_e = 0$

$$\rightarrow \boxed{\frac{1}{4}\sum_{ss'rr'} |M|^2 \doteq \frac{8e^4}{q^4}\left[(pk)(p'k') + (pk')(p'k) + m_\mu^2 (pp')\right]}$$

Lorentz scalar

11| Center of mass frame

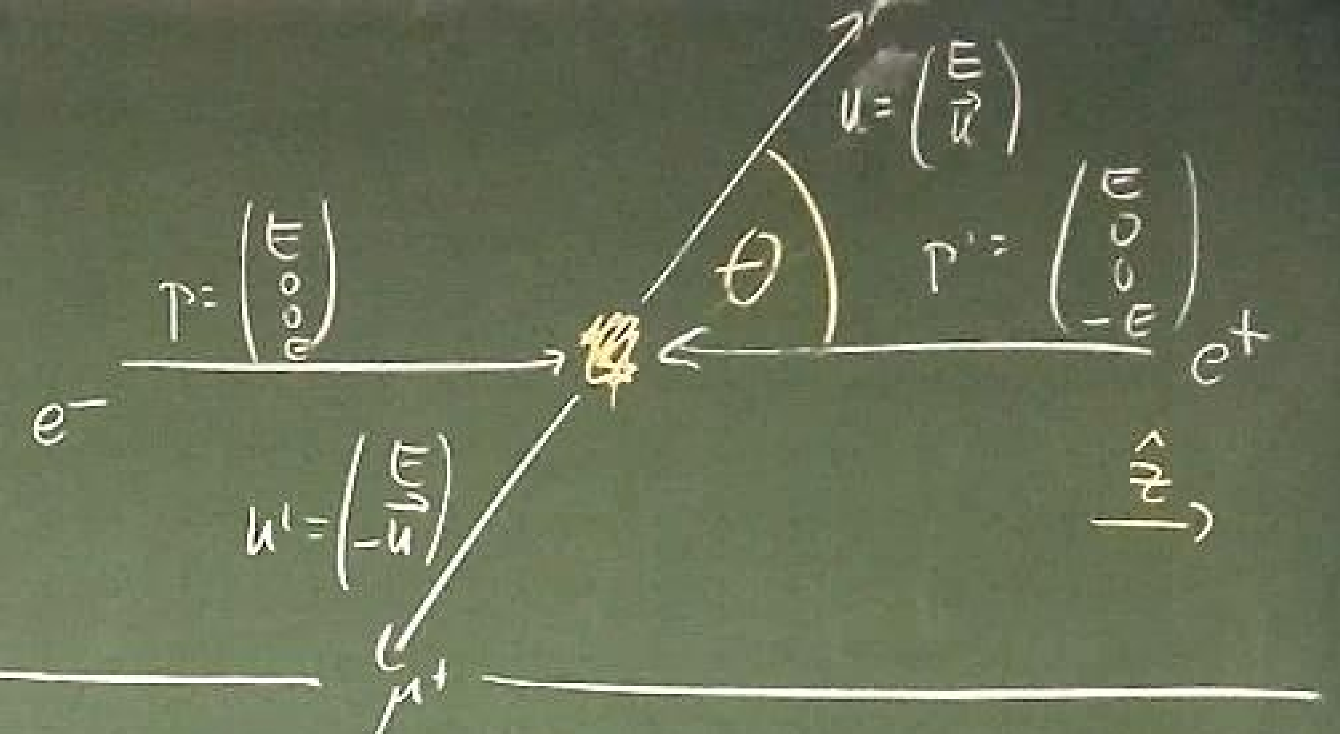
$\vec{p} + \vec{p}\,' = 0 = \vec{k} + \vec{k}\,'$

$p^2 = m_e^2 = 0$

$E^2 - |\vec{p}|^2 = 0$

$|\vec{p}| = E$

- wlog. $p = \begin{pmatrix} E \\ E\hat{z} \end{pmatrix}, \quad p' = \begin{pmatrix} E \\ -E\hat{z} \end{pmatrix}$
- $|\vec{k}| = \sqrt{E^2 - m_\mu^2}$
- $\vec{k}\cdot\hat{z} = |\vec{k}|\cos\theta$



$\Downarrow$

- $q^2 = (p + p')^2 = 4E^2$
- $pp' = 2E^2$
- $pk = p'k' = E^2 - E|\vec{k}|\cos\theta$
- $pk' = p'k = E^2 + E|\vec{k}|\cos\theta$

$$\Rightarrow \quad |\overline{M}|^2 = \frac{1}{4}\sum_{ss'}\sum_{rr'} |M|^2 \doteq e^4\left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

12| (4.116)

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{2E_p 2E_{p'} |v_p - v_{p'}|} \frac{|\vec{u}|}{(2\pi)^2 4 E_{cm}} |\bar{M}|^2 = \frac{\alpha^2}{4E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left[\left(1+\frac{m_\mu^2}{E^2}\right)+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]}$$

$E_{cm} = 2E$

$|v_p - v_{p'}| = \left|\frac{p^3}{E_p} - \frac{p^{3'}}{E_{p'}}\right| = 2$

13|

$$\boxed{\sigma_{total} \overset{\circ}{=} \frac{4\pi\alpha^2}{3E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{m_\mu^2}{2E^2}\right)}$$

14| Discussion:

- $\underbrace{2E}_{= E_{cm}} \geq 2m_\mu c^2$

If: $E_{cm} < 2m_\mu c^2$ $|M|^2$

→ no pair production possible

5.2 Summary

1| Draw relevant Feynman diagrams

2| Feynman rules → M

3| $|\bar{M}| = \sum_{spins} |M|^2$ (Spin sum relation)

4| Evaluate traces

5| Fix reference frame, express 4-momenta in kinematical variables ($E, \theta, \ldots$)

6| Plug $|\bar{M}|^2$ in Eq (4.114)

(integrate over momenta that are not measured)

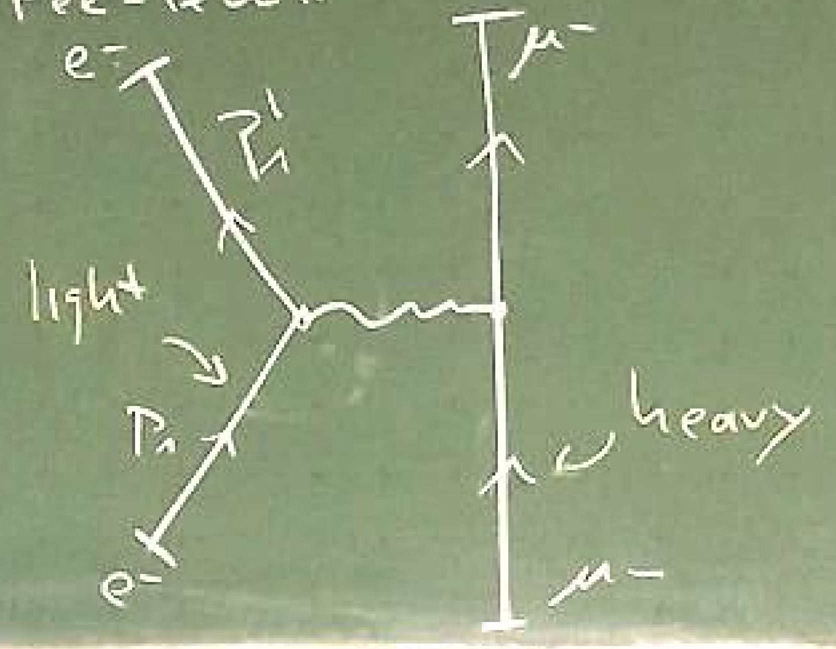
6) Radiative Corrections of QED

6.1. Overview

1) e^- scattering of very heavy particle.

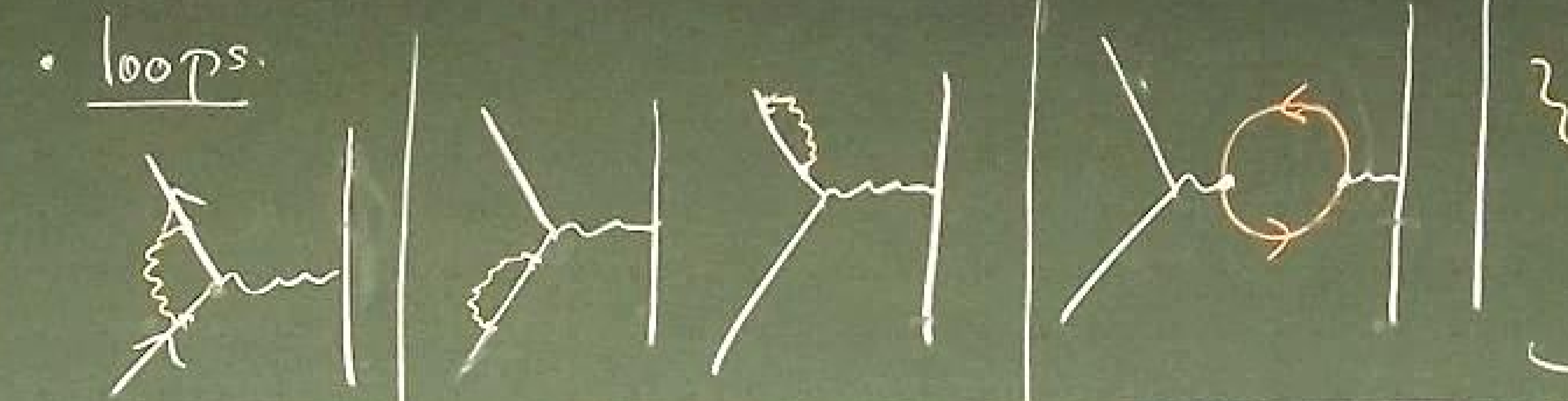
$$e^- + \mu^- \to e^- + \mu^-$$

2) Tree-level.



3) Radiative corrections:

• loops.

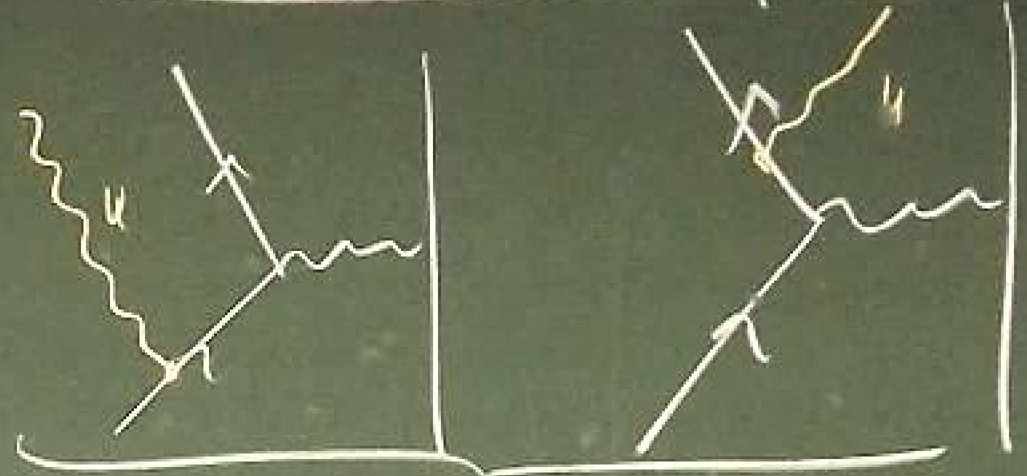


Vertex correction	External leg corrections	Vacuum polarization
• UV-divergence	• UV —"—	• UV "
• IR-divergence	• IR —"—	• Spoilers:

• IR-diver cancel with Bremsstrahlung

• UV-div. cancel in observable quantities

• extra final-state photons



• IR-divergence

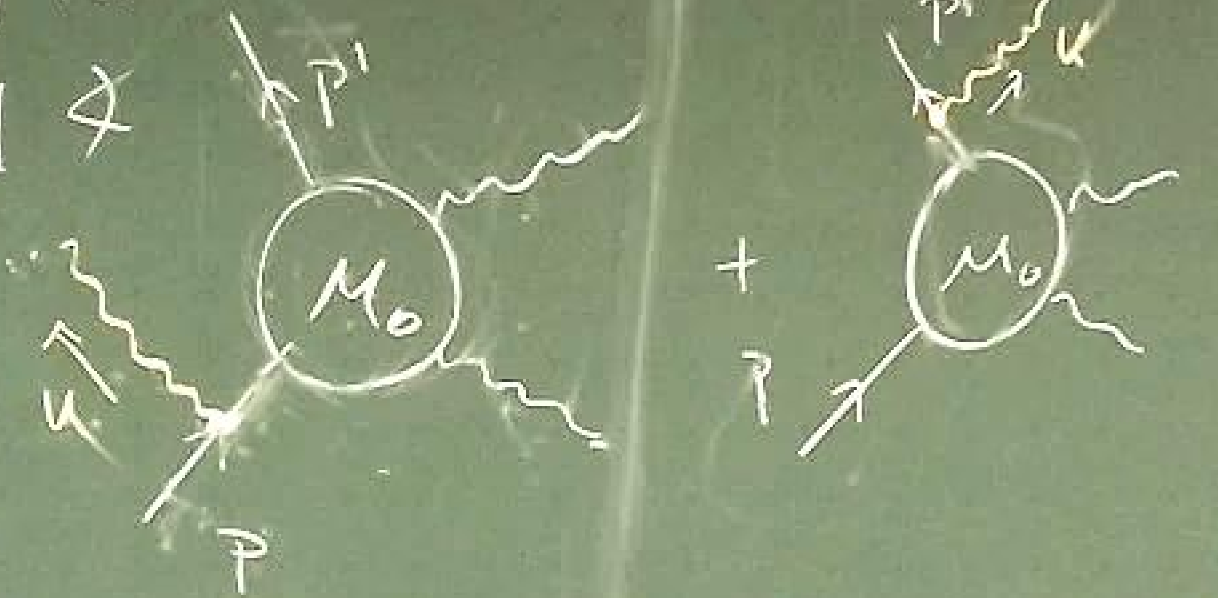
→ Bremsstrahlung

6.2. Soft Bremsstrahlung

1] "Soft": Low-energy photons ($k \approx 0$)

2] Can be derived vom Maxwell's equations

3] $\not k$



M_0: (unknown) interaction amplitude
(4x4 matrix)

$$iM = (-ie)\,\bar u(p') M_0(p',p-k) \frac{i(\not p - \not k + m)}{(p-k)^2-m^2+i\epsilon}\gamma^\mu u(p) \cdot \epsilon^*_\mu(k)$$
$$+ (-ie)\,\epsilon^*_\mu(k)\,\bar u(p')\gamma^\mu \frac{(\not p' + \not k + m)}{(p'+k)^2-m^2+i\epsilon} M_0(p'+k,p)\, u(p)$$

4] Simplifications:

- $(p-k)^2 - m^2 = -2pk$

  $(p'+k)^2 - m^2 = 2p'k$

- Soft photons: $|\vec k| \ll |\vec p - \vec p'| = |\vec q|$

$\to M_0(p', p-k) \approx M_0(p',p)$ | $\to \not p - \not k \approx \not p$

$M_0(p'+k,p) \approx M_0(p',p)$

- Dirac algebra:

$(\not p + m)\gamma^\mu \epsilon^*_\mu u(p) \doteq 2p^\mu \epsilon^*_\mu u(p)$

$\bar u(p')\gamma^\mu \epsilon^*_\mu (\not p' + m) \doteq \bar u(p')\, 2p'^\mu \epsilon^*_\mu$

$(\not p u = m u)$

5] $$iM = \overbrace{\bar u(p') M_0(p',p)\, u(p)}^{\text{elastic scattering}} \cdot \underbrace{\left[e\left(\frac{p'\cdot\epsilon^*}{p'k} - \frac{p\cdot\epsilon^*}{pk}\right)\right]}_{\text{Bremsstrahlung}}$$

6| Scattering cross section.

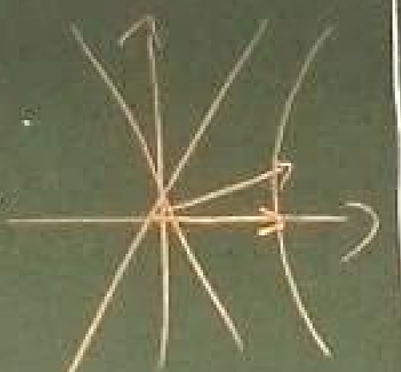
$$d\sigma(p \to p' + \gamma) = d\sigma(p \to p') \underbrace{\int \frac{d^3 k}{(2\pi)^3} \sum_r \frac{e^2}{2|\vec{k}|} \left| \frac{p'\varepsilon^r}{p'k} - \frac{p\varepsilon^r}{pk} \right|^2}_{dP_k(p \to p')}$$

$$\left. dP_k(p \to p') \right| \quad \tilde{k} = \frac{k}{|\vec{k}|} = \begin{pmatrix} 1 \\ \hat{k} \end{pmatrix}$$

7| Evaluation.

$$\int dP_k = \frac{\alpha}{\pi} \int_0^\infty dk \, \frac{1}{k} \boxed{\left(\int \frac{d\Omega_k}{4\pi} \sum_r \left| \frac{p'\varepsilon^r}{p'\tilde{k}} - \frac{p\varepsilon^r}{p\tilde{k}} \right|^2 \right) \to I(p,p')}$$

$$= \frac{\alpha}{\pi} I(p,p') \left[\underbrace{\log(\infty)}_{\text{Prob 1}} - \underbrace{\log(0)}_{\text{Prob 2.}} \right]$$



8| Approximations

i| Prob. 1: Soft photon approx invalid for $k > |\vec{q}| = |\vec{p} - \vec{p}'|$

$\to$ Upper cutoff of $|\vec{q}|$

ii| Problem 2. IR-divergence

iii| Relativistic limit: $E_{p,p'} \gg m$

$$I(p,p') \overset{*}{\approx} 2 \log\left(\frac{-q^2}{m^2} \right)$$

$$-q^2 = -(p'-p)^2 \geq 0$$

Solution: Regularization with finite photon mass $\mu > 0$

$$\frac{1}{k} = \frac{1}{E_k} \longmapsto \frac{1}{\sqrt{k^2 + \mu^2}}$$

$$\to \int_0^{|\vec{q}|} dk \, \frac{1}{\sqrt{k^2+\mu^2}} = \log\left(\frac{\sqrt{\mu^2 + \vec{q}^2} + |\vec{q}|}{\mu} \right)$$

$$\overset{\mu \to 0}{\sim} \log\left(2 \frac{|\vec{q}|}{\mu} \right) \sim \log\left(\frac{|\vec{q}|}{\mu} \right) = \frac{1}{2} \log\left(\frac{|\vec{q}|^2}{\mu^2} \right)$$

9| Result

$$\boxed{d\sigma(p \to p' + \gamma) \approx d\sigma(p \to p') \cdot \frac{\alpha}{\pi} \underbrace{\log\left(\frac{-q^2}{m^2} \right) \log\left(\frac{-q^2}{\mu^2} \right)}_{\text{Sudakov double logarithm}}}$$