Recap

4. Interacting Fields and Feynman Diagrams

4.7. Feynman Rules for QED

Propagators

Fermions

$$a \xrightarrow{\;p\;} b \;=\; \frac{i(\not p + m)_{ba}}{p^2 - m^2 + i\varepsilon}$$

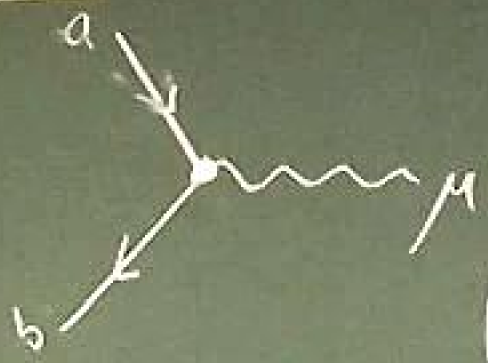
$$\| $$

$$a \longrightarrow b \quad \left(\hat{=}\ \overbracket{\psi_b \bar\psi_a} \right)$$

Photons

$$\mu \overset{q}{\rightsquigarrow} \nu \;=\; \frac{-i g_{\mu\nu}}{q^2 + i\varepsilon} \quad \left(\hat{=}\ \overbracket{A_\mu A_\nu} \right)$$

Vertices



$$= -ie\,\gamma^\mu_{ba}$$

$$\left(\hat{=}\ -ie \int d^4x\, \gamma^\mu_{ba} \right)$$

External legs

Fermions

$$a \xrightarrow{\;p\;} |\, s \;=\; u^s_a(p) \quad \left(\hat{=}\ \overbracket{\psi_a |\vec p, s\rangle} \right)$$

(incoming)

$$s\, | \xrightarrow{\;p\;} a \;=\; \bar u^s_a(p) \quad \left(\hat{=}\ \overbracket{{}_a\langle \vec p, s| \bar\psi_a} \right)$$

(outgoing)

Antifermions

$$a \xrightarrow{\;p\;} |\, s \;=\; \bar v^s_a(p) \quad \left(\hat{=}\ \overbracket{\bar\psi_a |\vec p, s\rangle_b} \right)$$

(incoming)

$$| \xrightarrow{\;p\;} |\, a \;=\; v^s_a(p) \quad \left(\hat{=}\ \overbracket{{}_b\langle \vec p, s| \psi_a} \right)$$

(outgoing)

Photons

$$\overset{q}{\rightsquigarrow} |\, r \;=\; \varepsilon^r_\mu(q) \quad \left(\hat{=}\ \overbracket{A_\mu |\vec q, r\rangle} \right)$$

(incoming)

$$r\, | \overset{q}{\rightsquigarrow} \;=\; \varepsilon^{r*}_\mu(q) \quad \left(\hat{=}\ \overbracket{\langle \vec q, r| A_\mu} \right)$$

(outgoing)

polarization 4-vector

single-photon state

Evaluation

1. Impose momentum conservation at vertices
2. Integrate over all undetermined momenta
3. Compute overall sign of the diagram

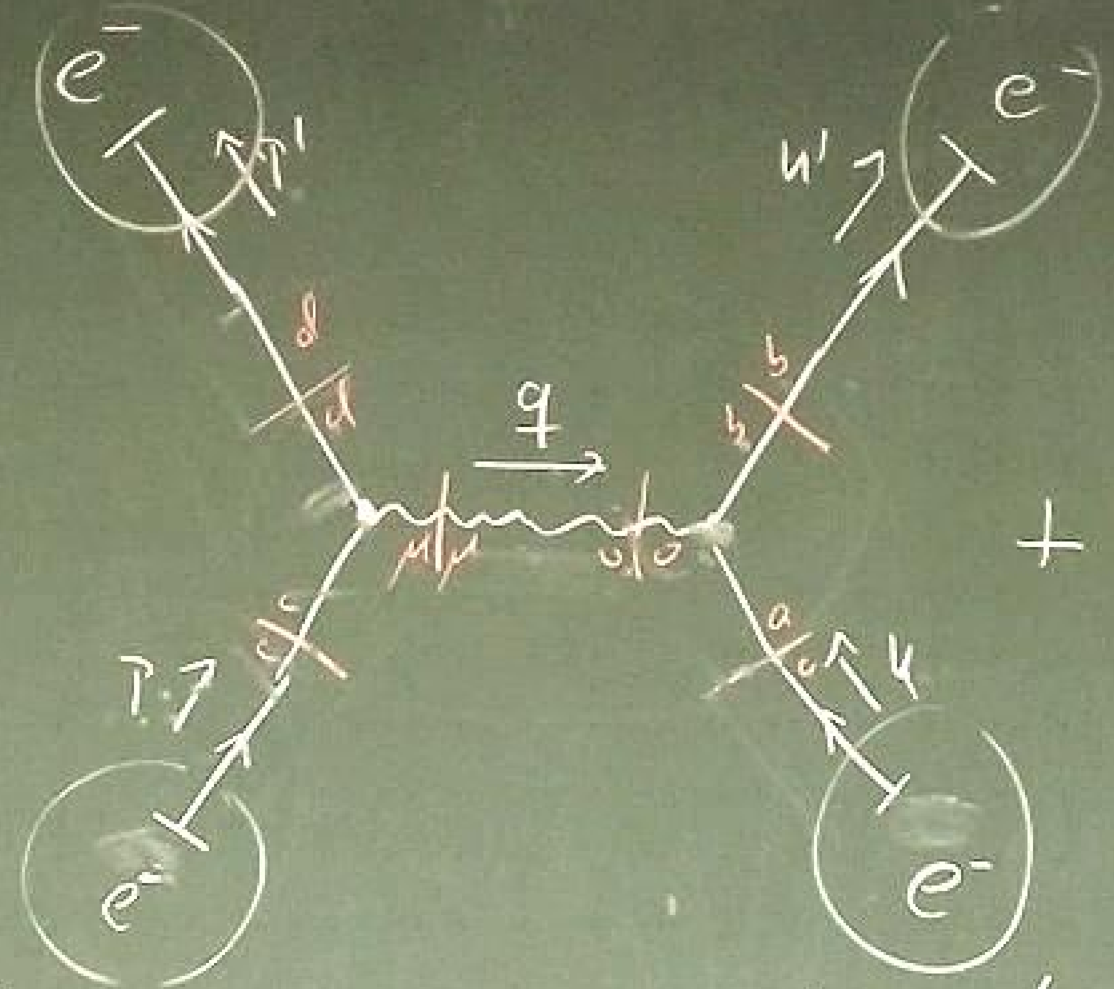
First application: The Coulomb Potential

1) Møller scattering

$$\boxed{e^- + e^- \longrightarrow e^- + e^-}$$

i) Contribution to the tree-level amplitude

$$iM\left(e^-(p)\,e^-(k) \mapsto e^-(p')\,e^-(k')\right) =$$



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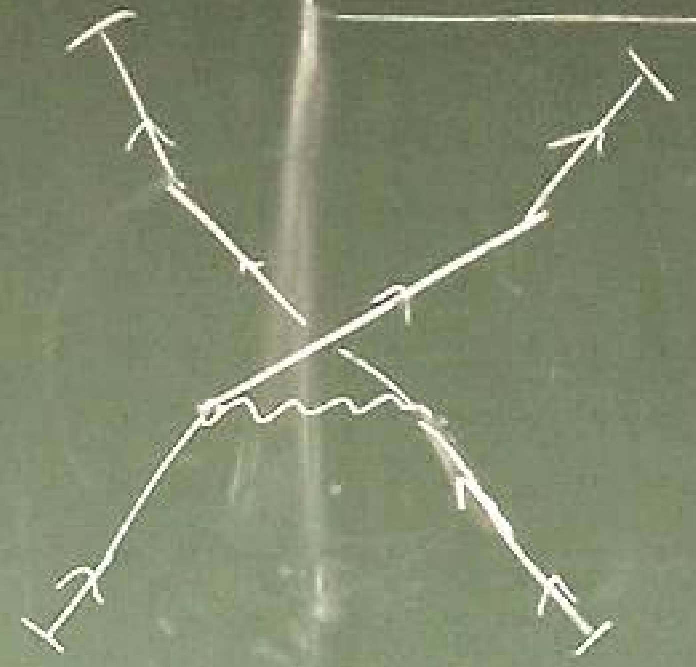


$$= \overset{\pm 1}{\sigma} \cdot \bar{u}_d(p')(-ie\gamma^{\mu}_{dc})u_c(p)\left(\frac{-ig_{\mu\nu}}{q^2+i\epsilon}\right)\bar{u}_b(k')(-ie\gamma^{\nu}_{ba})u_a(k) = \sigma\, \bar{u}(p')\cdot(-ie\gamma^{\mu})\cdot u(p')\left(\frac{-ig_{\mu\nu}}{q^2+i\epsilon}\right)\bar{u}(k')\cdot(-ie\gamma^{\nu})\cdot u(k)$$

- $p - p' = q = k' - k$
- σ = sign

ii) Nonrelativistic limit $|\vec{p}|^2 \ll m^2$

$$u(p) = \begin{pmatrix} \sqrt{p\sigma}\,\xi \\ \sqrt{p\bar{\sigma}}\,\xi \end{pmatrix} \approx \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$p^0 = \sqrt{\vec{p}^2 + m^2} \approx m$$

$$\frac{1}{(p-p')^2} \approx \frac{-1}{|\vec{p}-\vec{p}'|^2}$$

$$\Rightarrow \bar{u}(p')\gamma^\mu u(p) \approx \begin{cases} 2m\,\xi^{+}_{p'}\xi_p & \mu = 0 \\ 0 & \mu = 1,2,3 \end{cases}$$

$$\Rightarrow i\mathcal{M} \approx \sigma \cdot \frac{-ie^2}{|\vec{p}-\vec{p}'|^2}\left(2m\,\xi^{+}_{p'}\xi_p\right)\left(2m\,\xi^{+}_{k'}\xi_k\right)$$

iii) Compare to non-rel scattering theory. (1st. Born approximation)

$$\langle p'|iT|p\rangle \overset{*}{=} -i\hat{V}(\vec{q})\,(2\pi)\,\delta(E_{\vec{p}'}-E_{\vec{p}})$$

$$\vec{q} = \vec{p}' - \vec{p}$$

FT of scattering potential

$$\Rightarrow \hat{V}(\vec{q}) = \sigma\frac{e^2}{|\vec{q}|^2} \Rightarrow V(\vec{r}) \doteq \sigma\frac{e^2}{4\pi|\vec{r}|} = \sigma\frac{\alpha}{r} = \frac{\alpha}{r} \Rightarrow \text{repulsive Coulomb potential}$$

Coulomb potential

iv) Sign of diagram:

$${}_{0a}\langle \vec{p}'\vec{k}'|(\bar{\psi}\psi A)(\bar{\psi}\psi A)|\vec{p}\vec{k}\rangle_{0a}$$

$$= \langle 0|a_{k'}a_{p'}\,\bar{\psi}\psi A\,\bar{\psi}\psi A\,a^{+}_{p}a^{+}_{k}|0\rangle$$

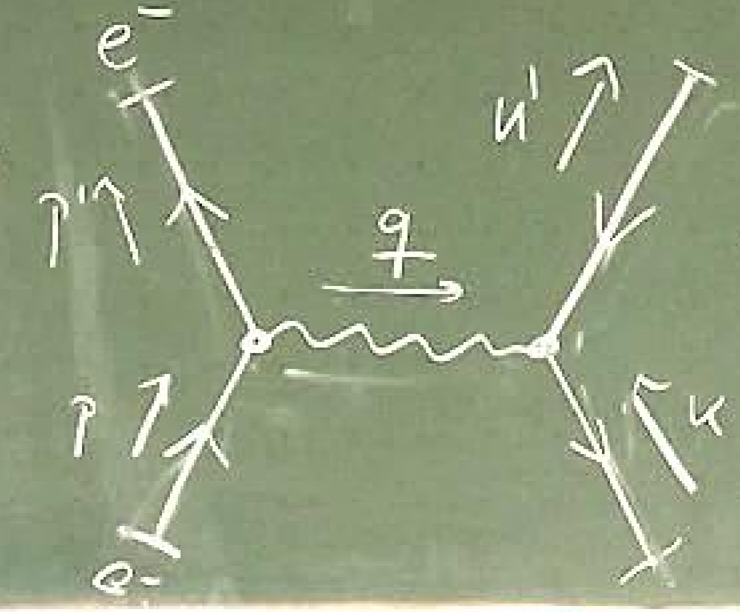
$\rightarrow 1+1+2 = 4$ interchanges

$$(-1)^4 = \sigma = +1$$

2| γ Bhabha Scattering

$$\boxed{e^- + e^+ \longrightarrow e^- + e^+}$$

1| $i\mathcal{M}(e^-(p)\, e^+(k) \longrightarrow e^-(p')\, e^+(k'))$

$=$ 

$= \sigma \cdot \bar{u}(p')(-i\gamma^\mu)u(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{v}(k)(-i\gamma^\nu)v(k')$

$p - p' = q = k' - k$

ii| Non-rel limit. same ($k \leftrightarrow k'$)

iii| Sign

$${}_{ab}\langle p'\, k' |\, \bar{\psi}\, \psi\, A\, \bar{\psi}\, \psi\, A \,|\, p\, k\rangle_{ab}$$

→ $2+1+2 = 5$ interchanges

→ $\sigma = -1$ → attractive Coulomb potential

5 Elementary Processes of QED

5.1. Cross section of $e^+e^- \to \mu^+\mu^-$

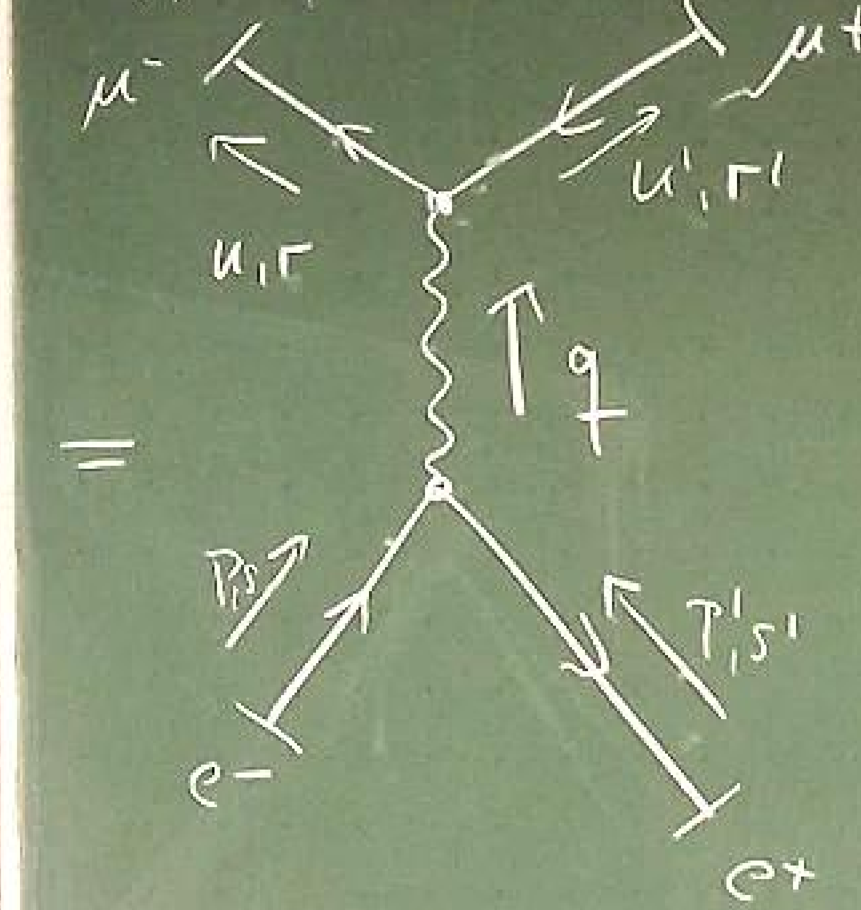
1| γ $\boxed{e^- + e^- \to \mu^- + \mu^-}$

2| $q_e = q_\mu = e, \quad m_e \ll m_\mu$

$$\mathcal{L}^{e.m}_{QED} = \sum_{f:e,\mu}\left[\bar{\psi}_f(i\partial\!\!\!/ - m)\psi_f - q_f\bar{\psi}_f\gamma^\mu\psi_f A_\mu\right] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

3| Tree-level amplitude:

$i\mathcal{M}(e^-(p)e^+(p') \to \mu^-(k)\mu^+(k'))$

$=$ 

$= \bar v_e^{s'}(p')(-ig_e\gamma^\mu)u_e^s(p)\left(\frac{-ig_{\mu\nu}}{q^2+i\varepsilon}\right)\bar u_m^r(k)(-ig_m\gamma^\nu)v_m^{r'}(k')$

$= \frac{ie^2}{q^2}\left(\bar v(p')\gamma^\mu u(p)\right)\left(\bar u(k)\gamma_\mu v(k')\right)$

$p+p'=q=k+k'$

4| We want: $d\sigma \propto |\mathcal{M}|^2 \to \mathcal{M}^*$

Use $(\bar v\gamma^\mu u)^* \doteq \bar u\gamma^\mu v$

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\left[(\bar v(p')\gamma^\mu u(p))(\bar u(p)\gamma^\nu v(p'))\right]\cdot\left[(\bar u(k)\gamma_\mu v(k'))(\bar v(k')\gamma_\nu u(k))\right]$$

5| Typical collider setups:

- e^+, e^- unpolarized → Average over s, s'
- Detectors cannot resolve spin → Sum over r, r'

→ $d\sigma \propto \frac{1}{4}\sum_{ss'}\sum_{rr'}|\mathcal{M}(ss'\to rr')|^2$

6| Use spin sum relations: $\sum_s u^s \bar{u}^s = \not{p} + m$
$\qquad\qquad\qquad\qquad\quad v \; \bar{v} \qquad (\not{p} - m)$

$$\sum_{ss'} \underbrace{\bar{v}_a^{s'}(p')}_{(\not{p}' - m_e)} \gamma^\mu_{ab} \underbrace{u_b^s(p)\, \bar{u}_c^s(p)}_{(\not{p} + m_e)} \gamma^\nu_{cd} \underline{v_d^{s'}(p')}$$

$$\overset{6}{=} \mathrm{Tr}\left[(\not{p}' - m_e) \gamma^\mu (\not{p} + m_e) \gamma^\nu \right]$$

7| $\to \frac{1}{4} \sum_{ss'rr'} |M|^2 = \frac{e^4}{4q^2} \underbrace{\mathrm{Tr}\left[(\not{p}' - m_e) \gamma^\mu (\not{p} + m_e) \gamma_\nu\right]}_{a} \cdot \underbrace{\mathrm{Tr}\left[(\not{k} + m_m) \gamma_\mu (\not{k}' - m) \gamma_\nu\right]}_{b}$

8| Trace technology.

Trace identities:

- $\mathrm{Tr}[\text{odd } \# \, \gamma] = 0$

$\mathrm{Tr}[\gamma^\mu \gamma^\nu] = 4 g^{\mu\nu}$

$\vdots$

Contraction identities:

$\gamma^\mu \gamma_\mu = 4 \cdot \mathbb{1}$

$\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu$

9| $\to$
$(a) = 4\left[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(pp' + m_e^2)\right]$
$(b) = 4\left[k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}(kk' + m_m^2)\right]$

10| $m_e / m_m \approx \frac{1}{200}$, $m_e = 0$

$$\boxed{\frac{1}{4} \sum_{ss'rr'} |M|^2 \overset{0}{=} \frac{8e^4}{q^2} \left[(pk)(p'k') + (pk')(p'k) + m_m^2 (pp') \right]}$$