Recap:

4. Interacting Fields and Feynman Diagrams

4.6. Computing S-Matrix Elements from Feynman Diagrams

interacting states

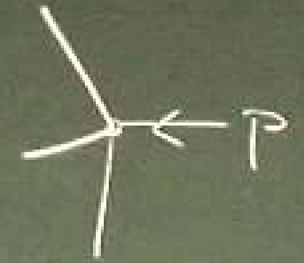
$$\langle \vec{p}_1 \vec{p}_2 | iT | \vec{p}_A \vec{p}_B \rangle$$

$$= \lim_{T \to \infty(1-i\varepsilon)} \left\{ {}_0\langle \vec{p}_1 \vec{p}_2 | T \exp\left[-i \int_{-T}^{T} dt\, H_I(t) \right] | \vec{p}_A \vec{p}_B \rangle_0 \right\}_{\text{fully connected + amputated}}$$

non-interacting states

- Generalized contractions:

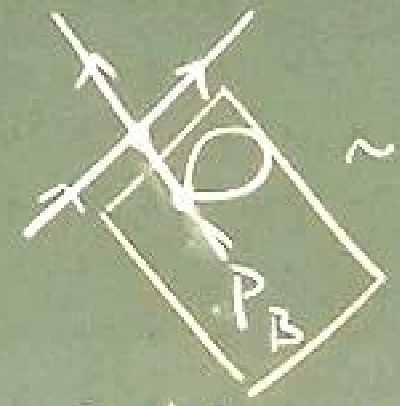
$$\phi_I(x) | \vec{p} \rangle_0 \equiv e^{-ipx} |0\rangle$$



- Generalized Wick's theorem

$$\boxed{\langle \vec{p}_1 \dots | T\{\phi \dots\} | \vec{p}_A \dots \rangle = \sum \text{all full contractions}}$$

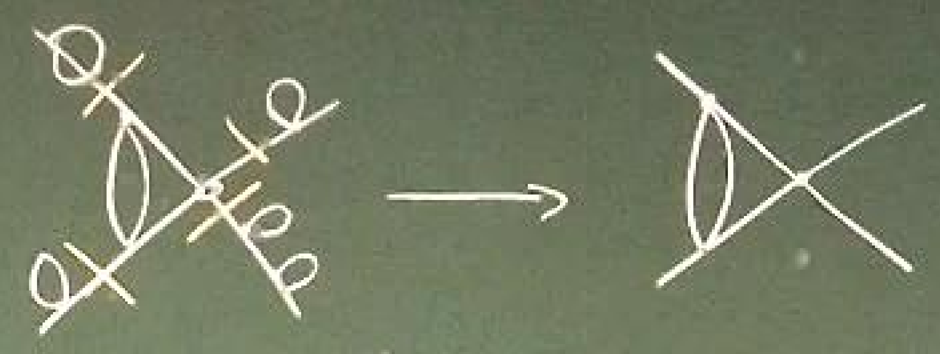
- Unamputated diagrams diverge



$$\sim \frac{1}{p_B^2 - m} = \frac{1}{0} = \infty$$

particle propagation ≠ scattering → drop diagrams

→ Amputated diagrams:



$$\Downarrow$$

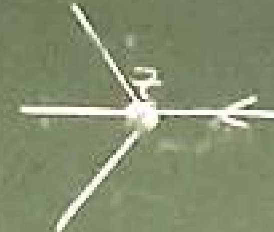
$$\langle p_1 p_2 | iT | p_A p_B \rangle$$

$$= i\mathcal{M}(p_A p_B \to p_1 p_2) \cdot (2\pi)^4 \delta(p_A + p_B - p_1 - p_2)$$

$$= \sum$$ Fully connected + amputated Feynman diagrams (2 incoming + 2 outgoing legs)

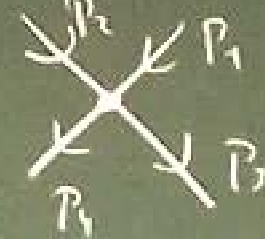
6] Position-space Feynman rules

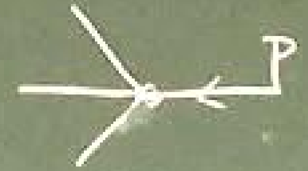
1. Edges: x ——— y $= D_F(x-y)$
2. Vertices:  $= -i\lambda \int d^4 z$

3. External lines:  $= e^{-ip}$

4. Divide by sym factor: $\frac{1}{S} \times \ldots$

7] Momentum-space Feynman rules

1. Edges: $\xrightarrow{p}$ $= \frac{i}{p^2 - m^2 + i\varepsilon}$
2. Vertices:  $= (-i\lambda)(2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4)$

3. External lines:  $= 1$

4. Integrate momenta: $\prod_i \int \frac{d^4 p_i}{(2\pi)^4}$
5. Divide by sym. factor: $\frac{1}{S} \times \ldots$

8] p-Integrations left after using-up all δ-functions = loop momenta

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4.7. Feynman Rules for QED

Setting the Stage

1] Fields.

Fermions: $\psi(x)$ bispinor field

Photons: $A_\mu(x)$ vector field

2] Lagrangian:

mass of the fermions

$$\mathcal{L}_{QED} = \overline{\Psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$- e\,\underbrace{\overline{\Psi}\gamma^{\mu}\psi}_{j^{\mu}}A_{\mu}$$

charge = coupling constant

$$= \overline{\Psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

D_{μ} = covariant derivative

$$D_{\mu} = \partial_{\mu} + ieA_{\mu}$$

3] Hamiltonian:

$$H_{QED} = H_{Dirac} + H_{Maxwell} + e\int d^3x\, \overline{\Psi}\gamma^{\mu}\psi A_{\mu}$$

ψ EOM:

$$(i\not{D} - m)\psi = 0$$

$$\partial_{\mu}F^{\nu\mu} = j^{\mu}$$

Note 4.4 $\mathcal{L}_{QED}$: $U(1)$ gauge theory

$$\psi'(x) = e^{ie\alpha(x)}\psi(x)$$

$$A'_{\mu}(x) = A_{\mu}(x) - \partial_{\mu}\alpha(x)$$

for arbitrary $\alpha : \mathbb{R}^{1,3} \rightarrow \mathbb{R}$

Note 4.5

$$\mathcal{L}^{SM*}_{QED} = \sum_{f} \overline{\Psi}_f(i\not{\partial} - m_f)\psi_f - \frac{1}{4}F_{\mu\nu}F^{\mu}$$

$$- \sum_f g_f \overline{\Psi}_f\gamma^{\mu}\psi A_{\mu}$$

$\{e, \mu, \tau, \nu_e, \nu_{\mu}, \nu_{\tau},$
$u, d, s, c, t, b\}$

Notes on Fermion Sector

Feynman propagator:

$$S_F^{ab}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not p + m)_{ab}}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$

$$= \langle 0 | T \psi_a(x) \bar{\psi}_b(y) | 0 \rangle$$

1) Time ordering.

$\psi \in \{\Psi, \bar{\Psi}\}$

signum of σ

$$T\{\psi_{\sigma 1} \dots \psi_{\sigma N}\} \equiv \boxed{(-1)^{\sigma}} \psi_1 \dots \psi_N$$

$\uparrow$ Permutation $\in S_N$

for $x_1 \geqslant \dots \geqslant x_N$

2) Normal ordering:

$X \in \{a_p^s, b_p^s, a_p^{s\dagger}, b_p^{s\dagger}\}$

$$:X_1 \dots X_n: \equiv \boxed{(-1)^{\#}} \text{(creation op)} \times \text{(annihilation op)}$$

number of operator interchanges

3) Contractions

$$\boxed{\overbracket{\psi_a(x)\psi_b}(y) \equiv T\{\psi_a \psi_b\} - :\psi_a \psi_b:}$$

$\Downarrow$

- $\overbracket{\Psi_a(x) \bar{\Psi}_b(y)} \overset{\circ}{=} S_F^{ab}(x-y)$
- $\overbracket{\Psi_a(x) \Psi_b(y)} \overset{\circ}{=} 0$
- $\overbracket{\bar{\Psi}_a(x) \bar{\Psi}_b(y)} \overset{\circ}{=} 0$

4) Contraction + Normal order

$$:A \, \overbracket{\psi_a(x) \, B \, \psi_b(y)} \, C:$$

$$\equiv \boxed{(-1)^{\#}} \overbracket{\psi_a(x) \psi_b(y)} \; :ABC:$$

number of interchanges of ψ_a with A and ψ_b wit AB.

5) Wick's theorem:

$$\boxed{T\{\psi_a(x_1) \psi_b(x_2) \dots\} \overset{*}{=} :\psi_a(x_1) \psi_b(x_2) \dots + \text{all possible contractions}:}$$

Notes on the Photon Sector

1] Observation: A^μ has 4 DOF
but photons have only
2 polarizations

2] Problem: Gauge invariance
→ Unphysical DOF
→ Fix gauge to quantize
only physical DOF

3] Different solutions

- Coulomb gauge: $\nabla\vec{A} = 0$ (not LI)
- Lorenz gauge: $\partial_\mu A^\mu = 0$ (LI)
  ↳ Gupta-Bleuler formalism Lorentz
- Faddeev-Popov procedure
  (→ Path integrals)

4] Motivation:

i] Lorenz gauge: $\partial_\mu A^\mu = 0 \xrightarrow{\text{EOM}} \partial^2 A^0 = 0$

ii] Expand field in classical solutions:

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} \sum_{r=0}^{3}\Big[a_p^r\, \varepsilon_\mu^r(p)\, \underline{e^{-ipx}} + a_p^{r\dagger}\, \varepsilon_\mu^{r*}(p)\, \underline{e^{ipx}}\Big]$$

- $p^2 = 0 \Leftrightarrow p^0 = E_p = |\vec{p}|$
- ε_μ^r: polarization 4-vectors.

5] Results:

i] Constraints on external (physical photons)

$$\boxed{\varepsilon^\mu(p) = \begin{pmatrix} 0 \\ \vec{\varepsilon}(p) \end{pmatrix} \text{ and } \underset{\substack{\uparrow \\ \text{transvers polarization}}}{\vec{p}\cdot\vec{\varepsilon} = 0}}$$

→ Two. r, s = 1, 2 independent bosonic modes for each momentum

$$\boxed{[a^r_p, a^{s\dagger}_q] = (2\pi)^3 \delta_{rs} \delta^{(3)}(\vec{p}-\vec{q})}$$

ii) Propagator:

$$\boxed{\langle 0|T A_\mu(x) A_\nu(y)|0\rangle \overset{*}{=} \int \frac{d^4q}{(2\pi)^4} \frac{-i g_{\mu\nu}}{q^2+i\epsilon} e^{-iq(x-y)}}$$

Feynman rules:

1) Expectations:

a) 2 fields → 2 propagators

- Fermion line.

- Photons.

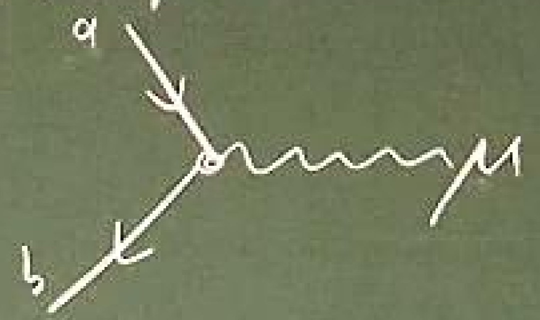
→ Two particle types: · (anti-) fermions
· photons

→ · Fermion / Antifermion: $|\vec{p}, s\rangle_{a,b}$ ⌐ s = 0, 1 spin

· Photons. $|\vec{p}, r\rangle$ ↑ r = 1, 2. polarization

b) Interaction: $\mathcal{L}_{int} \sim \underline{\bar{\psi}_b(x)} \gamma^\mu_{ba} \underline{\psi_a(x)} \underline{A_\mu(x)}$

→ Vertices have degree 3.



Feynman rules for QED: (momentum space)

1 **Propagators:**

Fermions:

$$= \frac{i(\not{p}+m)_{ba}}{p^2-m^2+i\varepsilon}$$

$\Rightarrow$

Photons:

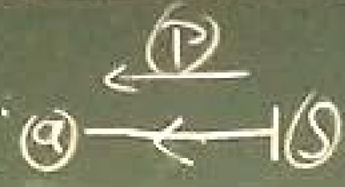
$$= \frac{-ig_{\mu\nu}}{q^2+i\varepsilon}$$

Vertices:

$$= -ie\gamma^{\mu}_{ba}$$

External legs:

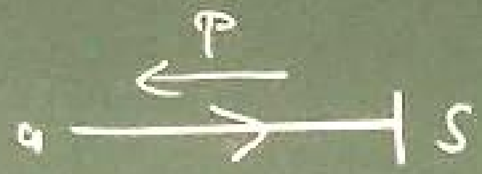
Ferm.:



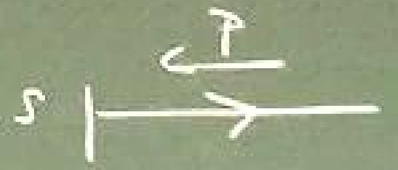
$= u^{\textcircled{s}}_{\textcircled{a}}(\textcircled{p}) \quad \overbrace{\Psi_a\;|\vec{p},s\rangle}_{(a)}$

$= \bar{u}^s_a(p) \quad {}_a\overbrace{\langle\vec{p},s|\;\bar{\Psi}}$

Antifermion:

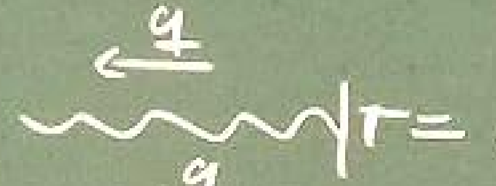


$= \bar{v}^s_a(p)$



$= v^s_a(p)$

Photons:



$= \varepsilon^r_\mu(q)$

$= \varepsilon^{r*}_\mu(q)$

Evaluation:

1. Impose mom. cons. at all vertices
2. Integrate over all undet. momenta
3. Compute the overall sign