Recap:

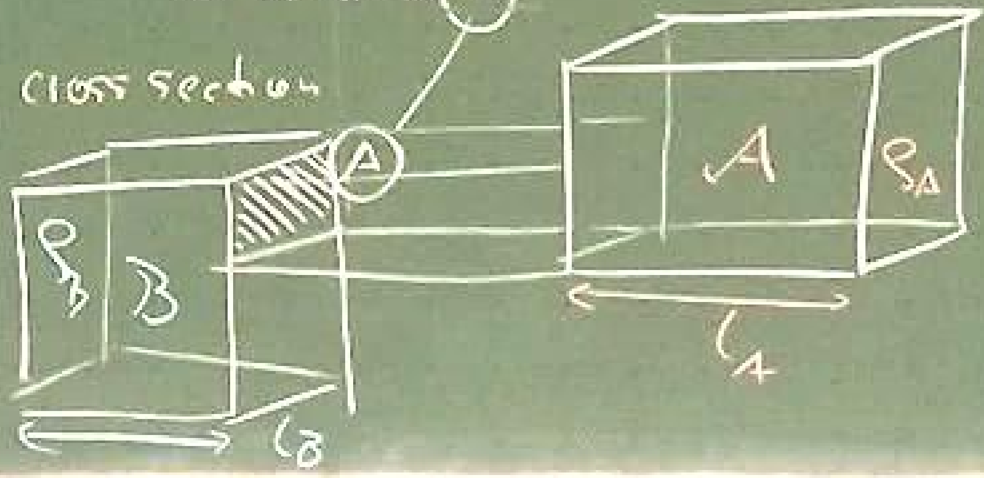
4 Interacting Fields and Feynman Diagrams

4.5 Cross Sections and the S-Matrix

Cross Section:

$$\frac{d\sigma_X}{d^3p_1 \dots d^3p_n} = \frac{\#\,\text{Scattering events } X \text{ with outgoing momenta } \vec{p}_1 \dots \vec{p}_n}{\rho_A l_A \rho_B l_B A}$$

Differential cross section



S- and T-matrix:

$$S_{\vec{p}_1 \dots \vec{p}_n, \vec{k}_A \vec{k}_B} \equiv {}_{\text{out}}\langle \vec{p}_1 \dots \vec{p}_n | \vec{k}_A \vec{k}_B \rangle_{\text{in}}$$

interacting state of two far-separated particles

Heisenberg states

$$= \lim_{T\to\infty} {}_{t_0}\langle \vec{p}_1 \dots \vec{p}_n | e^{-iH2T} | \vec{k}_A \vec{k}_B \rangle_{t_0}$$

reference time

$$\equiv {}_{t_0}\langle \vec{p}_1 \dots \vec{p}_n | S | \vec{k}_A \vec{k}_B \rangle_{t_0}$$

$$\boxed{iT = S - \mathbb{1}}$$

out state ≠ in state

4-Momentum conservation

invariant matrix element

$$\boxed{{}_{t_0}\langle \vec{p}_1 \dots \vec{p}_n | iT | \vec{k}_A \vec{k}_B \rangle_{t_0} = (2\pi)^4 \delta^{(4)}\Big(k_A + k_B - \sum_f p_f\Big) \times i\mathcal{M}(k_A k_B \to \{p_f\})}$$

$$\boxed{d\sigma = \frac{1}{2E_{p_A} 2E_{p_B} |v_A - v_B|} \left(\prod_f \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_{p_f}}\right) \times |\mathcal{M}(p_A p_B \to \{p_f\})|^2 (2\pi)^4 \delta^{(4)}\Big(p_A + p_B - \sum_f p_f\Big)}$$

Differential cross section (LI)

T-Matrix (LI)

p_A^z / E_{p_A} : group velocity — Not LI for boosts in x, y

How to compute the T-Matrix / inv. matrix element perturbatively?

Special cases: (P-Set 7)

12) 2 final particles & center-of-mass frame
n $\qquad \vec{p}_A + \vec{p}_B = 0 = \vec{p}_1 + \vec{p}_2$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{cm} \doteq \frac{1}{2E_{p_A} 2E_{p_B} |v_A - v_B|} \frac{|\vec{p}_1|}{(2\pi)^2 4 E_{cm}} \times |\mathcal{M}(p_A p_B \mapsto p_1 p_2)|^2}$$

solid angle $d\Omega = \sin\theta \, d\theta \, d\varphi$

COM energy: $E_{cm} = \sqrt{(p_A + p_B)^2} = [E_{p_A} + E_{p_B}]_{cm}$

LI

13) For $m_A = m_B = m_1 = m_2$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{cm} \doteq \frac{|\mathcal{M}(p_A p_B \to p_1 p_2)|^2}{64\pi^2 E_{cm}^2}}$$

4.6. Computing S-Matrix elements from Feynman diagrams

Motivation:

1) We want: $\langle p_1 \dots p_n | S | p_A p_B \rangle$
$$= \lim_{T \to \infty} {}_{t_0}\langle p_1 \dots p_n | e^{-iH2T} | p_A p_B \rangle_{t_0}$$

2) Problem:
- $|p_A p_B\rangle_0 = \sqrt{2E_{p_A}}\sqrt{2E_{p_B}}\, a^\dagger_{p_A} a^\dagger_{p_B} |0\rangle$ Eigenstates of H_0
- $|p_A p_B\rangle = ?$

3) Reminder:
$$|\Omega\rangle = \lim_{T \to \infty(1-i\varepsilon)} \left(e^{-iE_0 T} \langle\Omega|0\rangle\right)^{-1} e^{-iHT} |0\rangle$$

4) Assume:
$$|p_A p_B\rangle = \lim_{T \to \infty(1-i\varepsilon)} (\;?\;)\, e^{-iHT} |p_A p_B\rangle_0$$

5]

$$\langle p_1 \dots p_n | S | p_A p_B \rangle$$

$$\propto \lim_{T\to\infty} {}_0\langle p_1 \dots p_n | \underbrace{\left(e^{-iHT(1-i\epsilon)}\right)^{\dagger} e^{-iH2T} e^{-iHT(1-i\epsilon)}}_{e^{-iH2T(1-i\epsilon)}} | p_A p_B \rangle_0$$

$$\propto \lim_{T\to\infty(1-i\epsilon)} \underbrace{{}_0\langle p_1 \dots p_n | e^{-iH2T} | p_A p_B \rangle_0}_{U(T,-T)}$$

$e^{iH_0(T-t_0)}$ $\quad$ $e^{-iH_0(-T-t_0)}$

$$\propto \lim_{T\to\infty(1-i\epsilon)} {}_0\langle p_1 \dots p_n | T \exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right] | p_A p_B \rangle_0$$

6] Correct result:

n=2

$$\langle \vec p_1 \dots \vec p_n | iT | \vec p_A \vec p_B \rangle$$

$$\langle \vec p_1 \vec p_2 |$$

$$= \lim_{T\to\infty(1-i\epsilon)} \left\{ {}_0\langle \vec p_1 \dots \vec p_n | T \exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right] | p_A p_B \rangle_0 \right\}_{fc\,\&\,a}$$

fc: "fully connected"

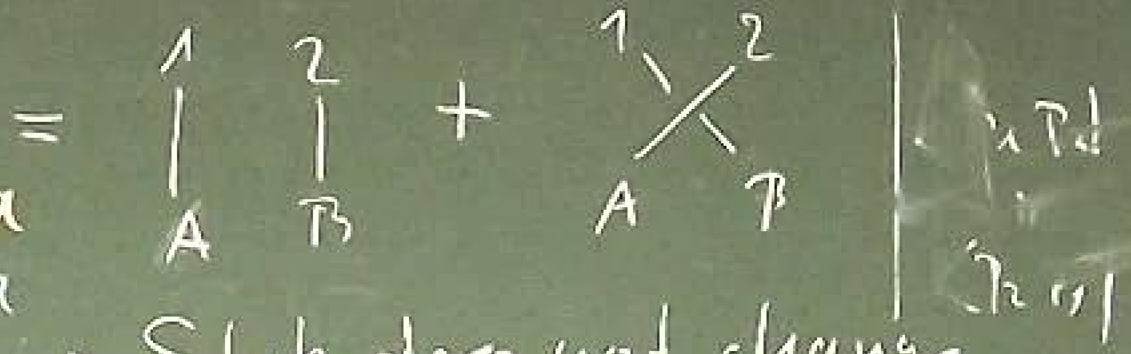
a: "amputated"

Interpretation & Aplication of ϕ^4

1] λ^0-order:

$${}_0\langle p_1 p_2 | p_A p_B \rangle_0 = \sqrt{2E_{p_1} 2E_{p_2} 2E_{p_A} 2E_{p_B}}\, \langle 0 | a_{p_1} a_{p_2} a^\dagger_{p_A} a^\dagger_{p_B} | 0 \rangle \stackrel{!}{=}$$

$$\stackrel{!}{=} 2E_{p_A} 2E_{p_B} (2\pi)^6 \left\{ \delta^{(3)}(\vec p_A - \vec p_1)\, \delta^{(3)}(\vec p_B - \vec p_2) + \delta^{(3)}(\vec p_A - \vec p_2)\, \delta^{(3)}(\vec p_B - \vec p_1) \right\} \quad (*)$$



→ State does not change

→ Contributes to $\mathbb{1}$ in $S = iT + \mathbb{1}$ → not part of fc & a

2) λ^1-order:

i) ${}_0\langle p_1 p_2 | \left(-i\frac{\lambda}{4!}\int d^4x \, \underbrace{T\{\phi_I^4(x)\}}_{\text{Wick's thm}}\right) | p_A p_B\rangle$ (**)

$= \text{—}''\text{—} \; : \phi_I^4(x) : +$ all contractions:

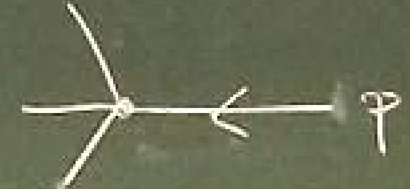
ii) Careful:

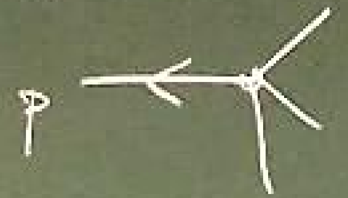
$\phi_I^+(x) = \int \frac{d^3k}{\dots} \, e^{-ikx} a_k \underline{\sqrt{2E_p}\, a_p^\dagger |0\rangle}$

$|p\rangle_0 \hat{=} e^{-ipx} |0\rangle$

${}_0\langle p | \phi_I^-(x) \hat{=} \langle 0 | e^{ipx}$

iii) Definition:

$\phi_I(x)|p\rangle \equiv e^{-ipx}|0\rangle \hat{=}$ 

$\langle p|\phi_I(x) \equiv \langle 0| e^{+ipx} \hat{=}$ 

$\langle p|q\rangle \equiv 2E_p (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}) \hat{=} q \longleftarrow p$

iv) $$\boxed{{}_0\langle p_1 \dots p_n | T\{\phi_a \dots\} | p_A \dots\rangle_0 = \left\{ \begin{array}{l} \text{Sum of all } \underline{\text{full}} \text{ contractions} \\ \text{of fields } \underline{\text{and}} \text{ external momenta} \end{array} \right\}}$$

Example:

${}_0\langle p_1 p_2 | p_A p_B\rangle_0 = \langle 12|AB\rangle + \langle 12|AB\rangle$

$= (*)$

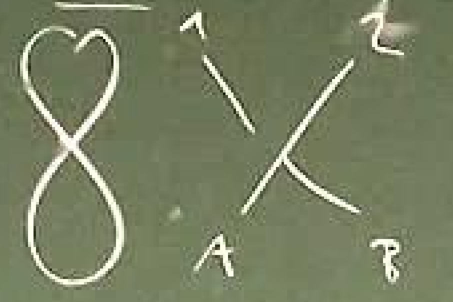
v) Application to (**).

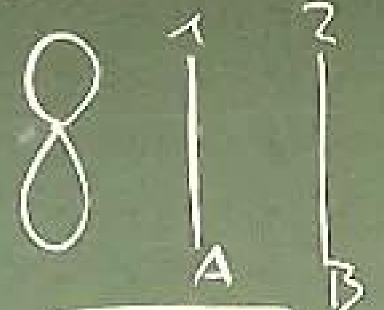
(**) = ?

v)

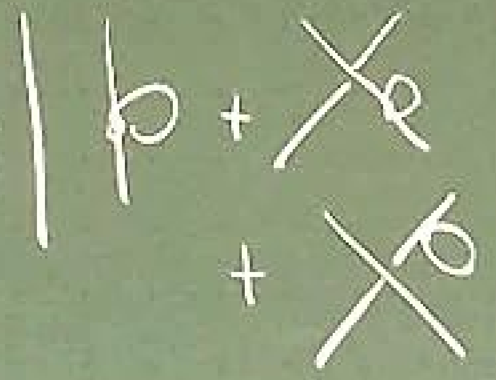
$$-i\frac{\lambda}{4!}\int d^4x \; {}_0\langle p_1 p_2 | T\, \phi_I^4(x) \, | p_A p_B \rangle_0$$

$$= \langle 1 2 | \phi\phi\phi\phi | A B \rangle +$$

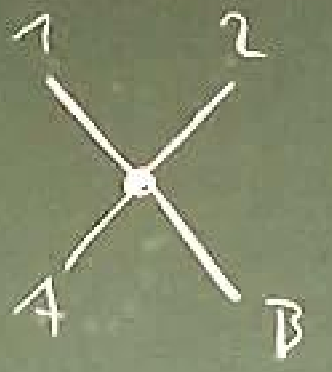




$$+ \langle 1 2 | \phi\phi\phi\phi | A B \rangle +$$



$$+ \langle 1 2 | \phi\phi\phi\phi | A B \rangle$$



→ Terms with $\phi\phi\phi\phi$ or $\phi\phi\phi\phi$ do not contribute to T

→ Only fully connected diagrams contribute to T

vi)

$$\langle p_1 p_2 | iT | p_A p_B \rangle \approx$$

$$= \cancel{4!}\left(-i\frac{\lambda}{\cancel{4!}}\right)\int d^4x \; e^{-i(p_A+p_B-p_1-p_2)x}$$

$$= -i\lambda\,(2\pi)^4\,\delta^{(4)}(p_A+p_B-p_1-p_2)$$

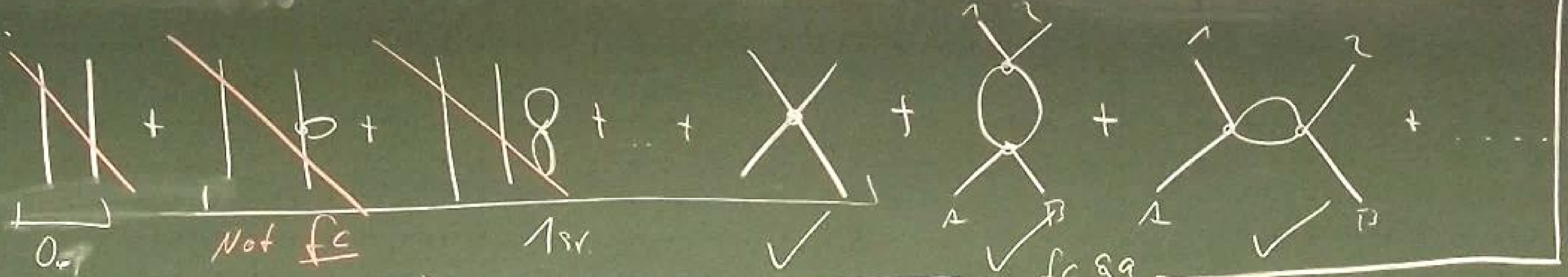
$$\overset{\text{def}}{=} i\mathcal{M}\,(2\pi)^4\,\delta^{(4)}(p_A+p_B-p_1-p_2)$$

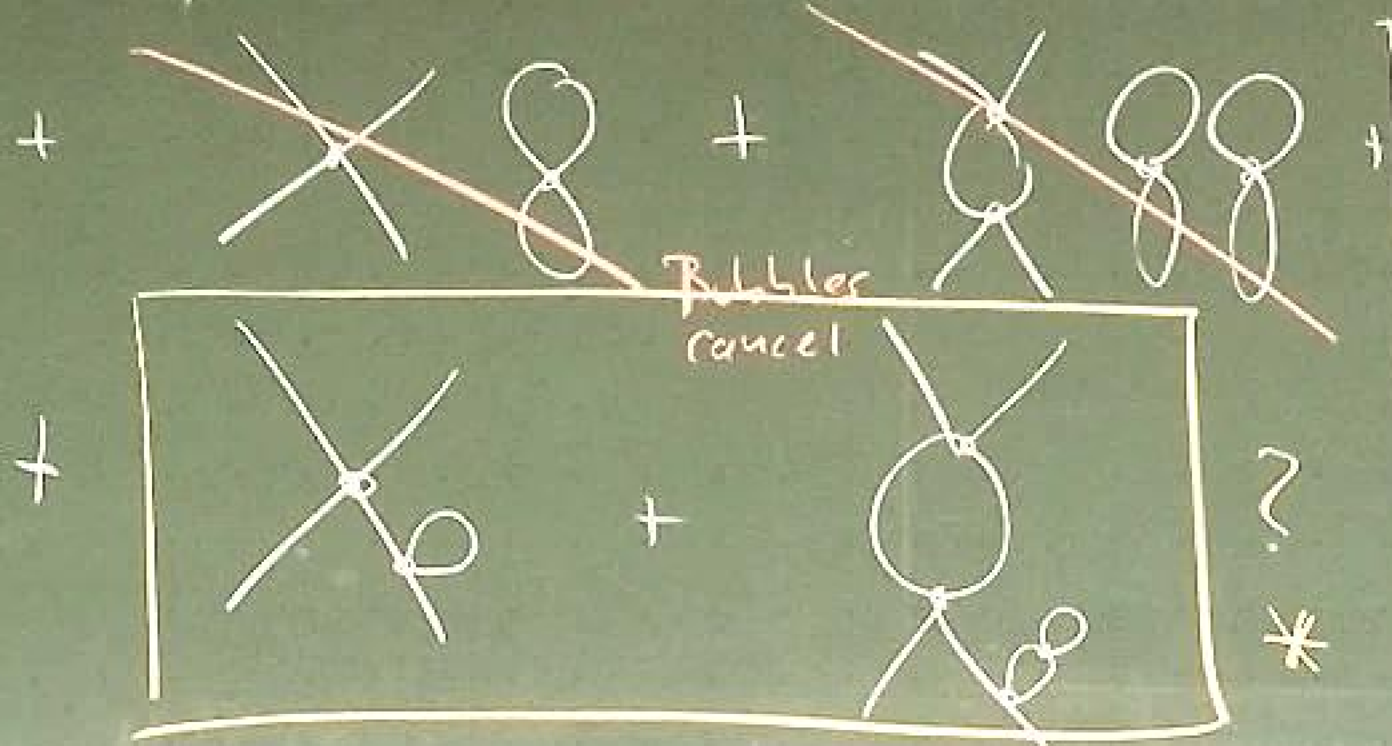
$$\rightarrow \mathcal{M}(p_A p_B \rightarrow p_1 p_2) = -\lambda + O(\lambda^2)$$

$$\rightarrow \boxed{\sigma_{total} = \frac{\lambda^2}{32\pi E_{cm}^2}}$$

3/ Higher orders

$\langle p_1 p_2 | iT | p_A p_B \rangle =$





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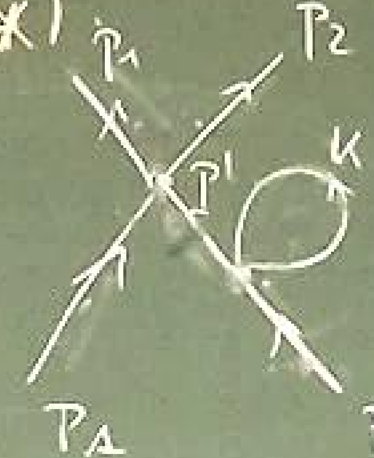


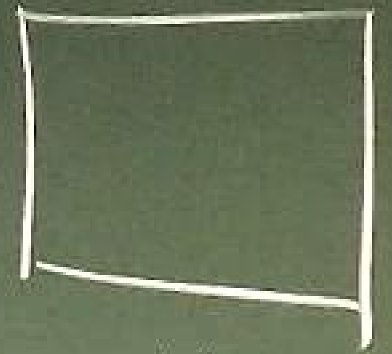
$$\doteq \frac{1}{2}\int \frac{d^4p'}{(2\pi)^4}\,\frac{i}{p'^2-m^2}\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2}\,(-i\lambda)(2\pi)^4\delta^{(4)}(p_A+p'-p_1-p_2)$$

$$\times(-i\lambda)(2\pi)^4\delta^{(4)}(p_B-p')$$

$$\sim \frac{i}{p_B^2-m^2} = 0$$

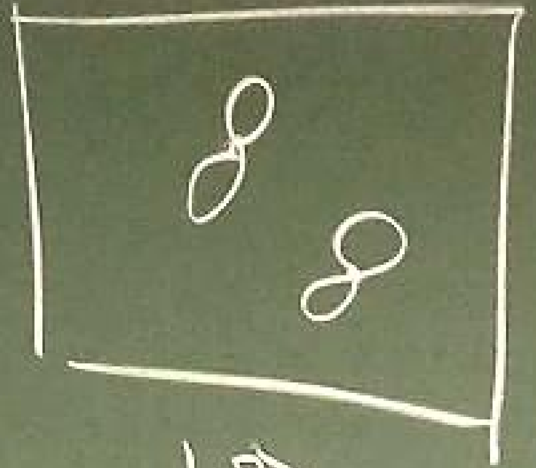
$E_{p_B}^2-\vec{p}_B^{\,2} = p_B^2 = m^2$ since on-shell!

→

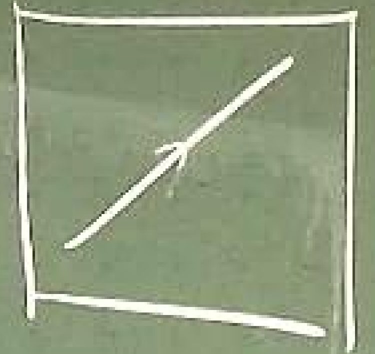


$|0\rangle$

$\xrightarrow{+H_{int}}$

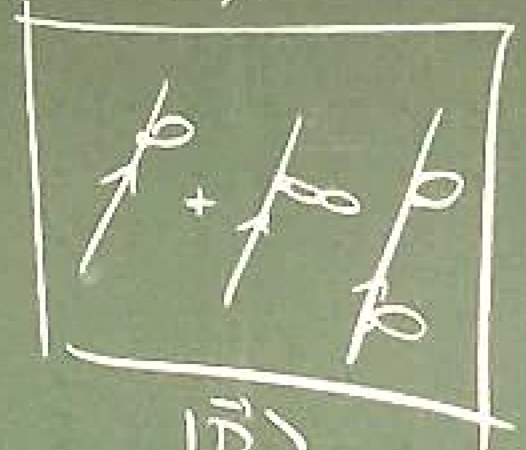


$|\Omega\rangle$

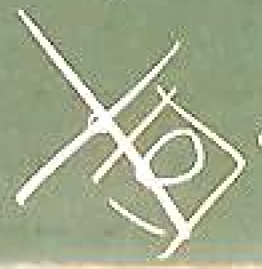


$|\vec{p}\rangle_0$

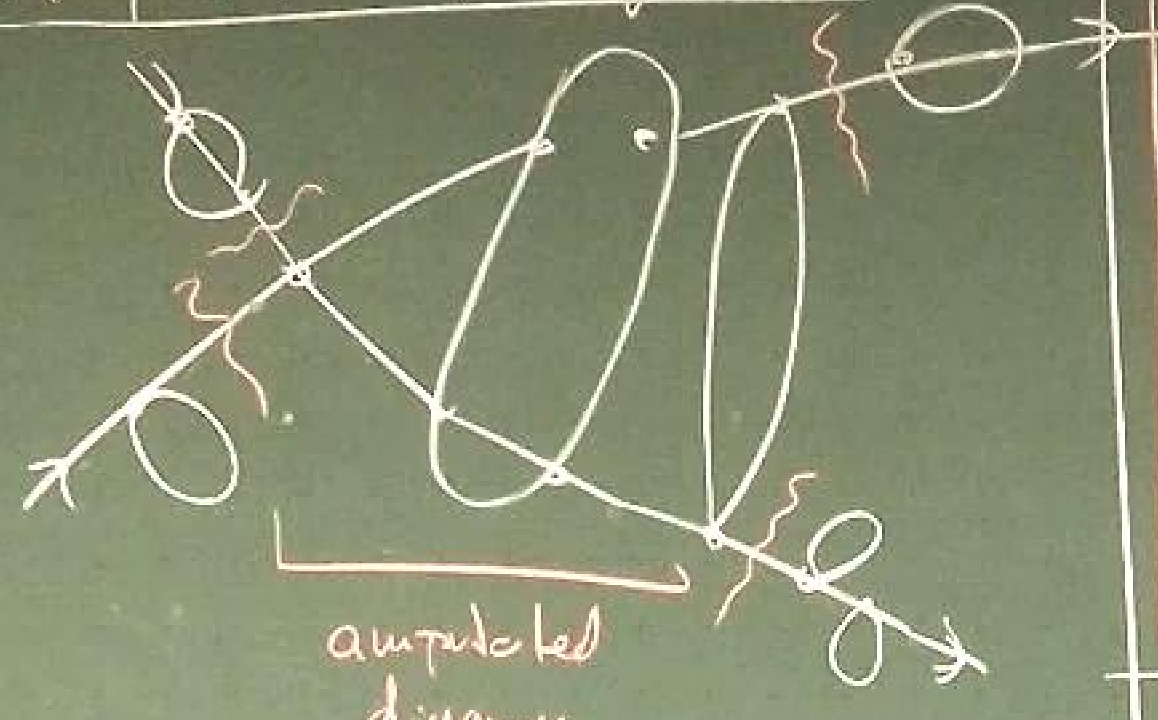
$\xrightarrow{+H_{int}}$



$|\vec{p}\rangle$

→  ← not related to scattering

4) Amputation of diagrams



5) →

$$iM\cdot(2\pi)^4\,\delta^{(4)}(p_A+p_B-p_1-p_2)$$

$= \Big\{$ Sum of all fully connected and amputated Feynman diagrams with p_A, p_B incoming and p_1, p_2 outgoing $\Big\}$