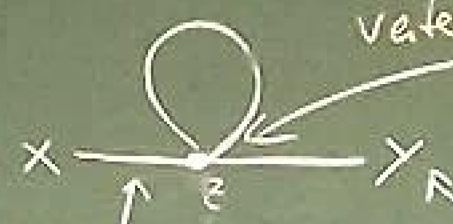
Recap:

4 Interacting Fields and Feynman Diagrams

$$\langle\Omega|T\,\phi(x_1)\phi(x_2)\ldots\phi(x_n)|\Omega\rangle = \left\{\begin{array}{l}\text{Sum of all connected Feynman} \\ \text{diagrams with } n \text{ external points}\end{array}\right\}$$

Feynman diagram $\hat{=}$ Number = Sum of equivalent contractions in Wick's th

Feynman rules:



integral with coupling constant λ

vertex $X = (-i\lambda)\int d^4z$

external point $-y = 1$

$x - z = D_F(x-z)$

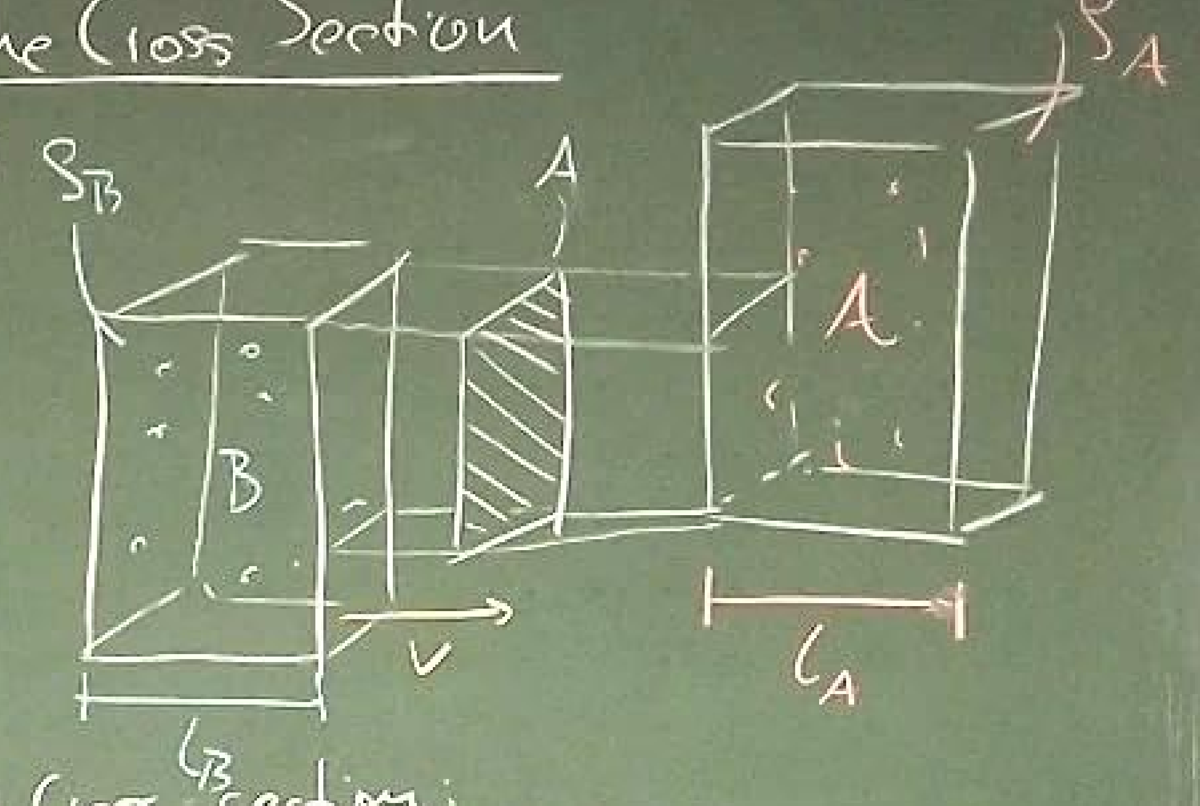
edge propagator

• multiply by sym factor $S(\,\cdot\,) = 2$

4.5. Cross Sections and the S-matrix

The Cross Section

1)



2) Cross section:

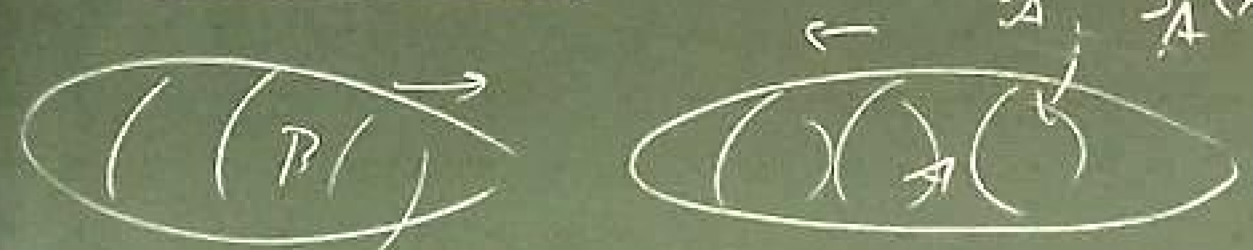
$$\sigma_X \equiv \frac{\#\,\text{scattering events } X}{\rho_A l_A \rho_B l_B A} \Rightarrow [\sigma] = L^2 = \text{Area}$$

→ σ = likelihood of scattering event X

→ Intrinsic property of particles A, B

3| Real experiment

$\rho_A = \rho_A(x, t)$



4| Typically many outcomes X_i

$$e^+e^- \longrightarrow \begin{cases} e^+e^- \\ \mu^+\mu^- \\ \mu^+\mu^-\gamma \\ \vdots \end{cases}$$

5| Differential cross section:

$\forall$ outcome X of n final particles with momenta

$(\vec{p}_1, \dots \vec{p}_n) \in U_p \subseteq \mathbb{R}^{3n}$

$$\sigma_{X|U_p} = \int_{U_p} d^3p_1 \, d^3p_n \underbrace{\frac{d\sigma}{d^3p_1 \cdots d^3p_n}}_{\text{Differential cross section}}$$

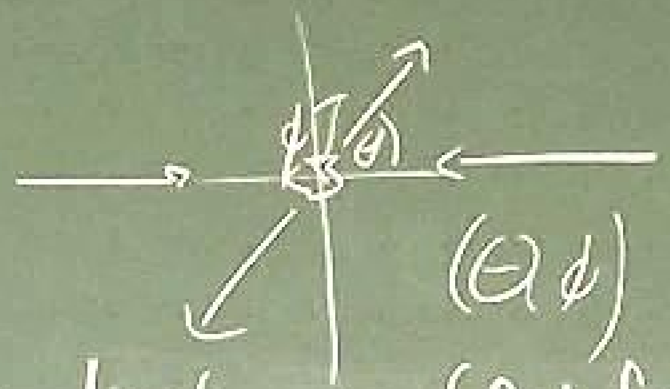
↳ Constrained by 4-momentum conservation.

$$\sum_i p_i = \text{const.}$$

Special case: $n = 2$

→ 6 DOF $(\vec{p}_1, \vec{p})$

→ 2 DOF → Scattering direction in COM frame



$(\theta \, \phi)$

$$\frac{d\sigma}{d^3p_1 d^3p_2} \longrightarrow \frac{d\sigma}{d\Omega} \dots \quad d\Omega = \sin\theta \, d\theta \, d\phi$$

The S-Matrix

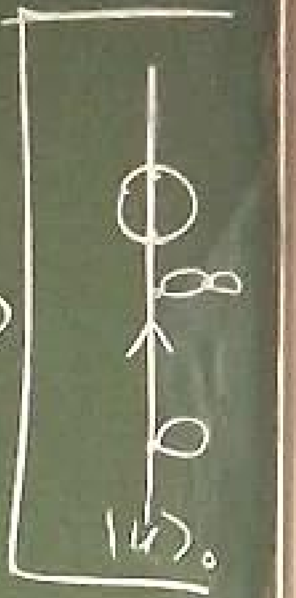
1| $\forall$ One-particle wave packet σ

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \phi(\vec{k}) |\vec{k}\rangle$$

$$\langle\phi|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} |\phi(\vec{k})|^2 = 1$$

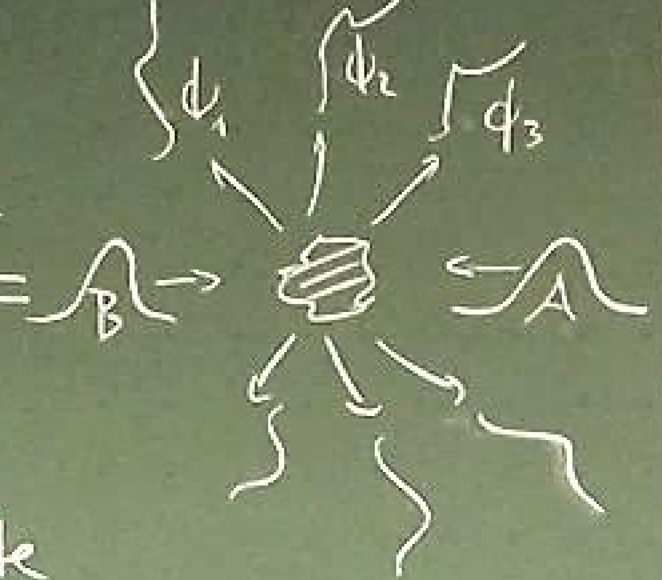
One-particle state of the interacting theory

(cf. $|\vec{k}\rangle_0 = \sqrt{2E_k} \, a_k^\dagger |0\rangle$)



2| We want:

$$P = \left| {}_{out}\langle \phi_1 \ldots \phi_n | \phi_A \phi_B \rangle_{in} \right|^2 =$$



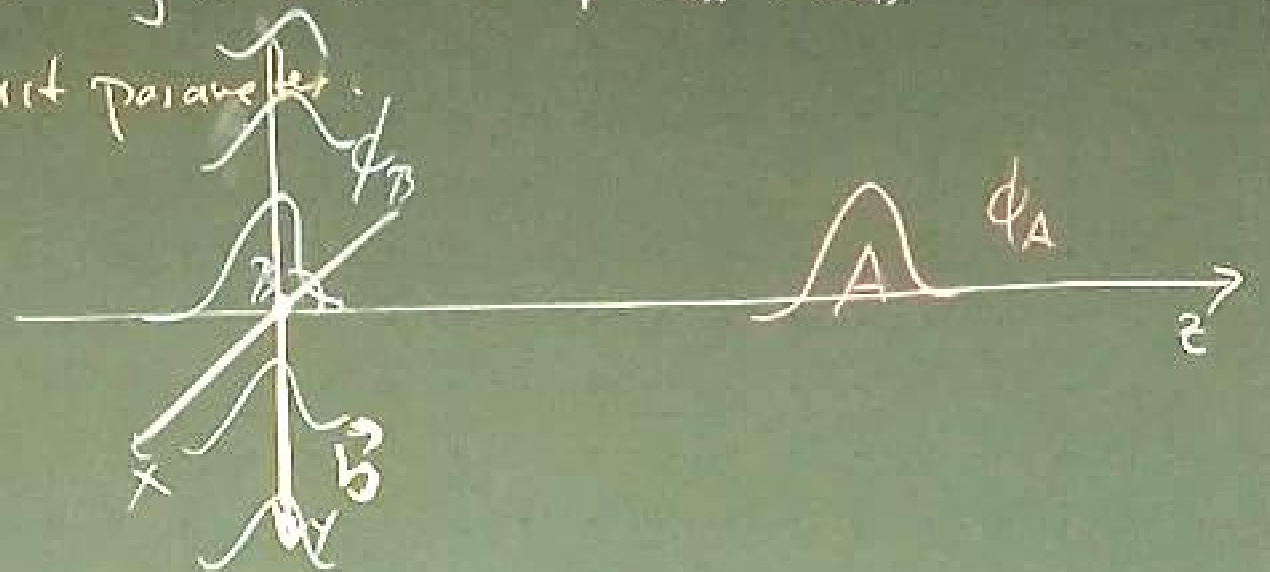
$|\phi_A \phi_B\rangle$: in-state at $T \to -\infty$

$|\phi_1 \ldots \phi_n\rangle$: out-state at $T \to +\infty$

3|

$$|\phi_A \phi_B(\vec{b})\rangle_{in} = \int \frac{d^3k_A}{(2\pi)^3} \int \frac{d^3k_B}{(2\pi)^3} \frac{\phi_A(\vec{k}_A)\,\phi_B(\vec{k}_B)}{\sqrt{2E_{k_A}\, 2E_{k_B}}} e^{-i\vec{b}\vec{k}_B} |k_A k_B\rangle_{in}$$

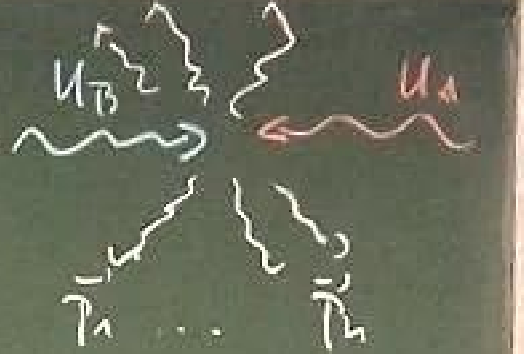
impact parameter



4| Simplification:

$$|\phi_1 \ldots \phi_n\rangle_{out} \to |\vec{p}_1 \ldots \vec{p}_n\rangle_{out}$$

$$\to {}_{out}\langle \vec{p}_1 \ldots \vec{p}_n | k_A k_B \rangle_{in} =$$



5| S-matrix:

"Scattering"

$${}_{out}\langle \vec{p}_1 \ldots \vec{p}_n | \vec{k}_A \vec{k}_B \rangle_{in}$$

$$:= \lim_{T\to\infty} {}_T\langle \vec{p}_1 \ldots \vec{p}_n | \vec{k}_A \vec{k}_B \rangle_{-T}$$

$$= \lim_{T\to\infty} {}_{t_0}\langle \vec{p}_1 \ldots \vec{p}_n | e^{-iH(2T)} | \vec{k}_A \vec{k}_B \rangle_{t_0}$$

$$\equiv {}_{t_0}\langle \vec{p}_1 \ldots \vec{p}_n | S | k_A k_B \rangle_{t_0}$$

$$P(t)|u\rangle_t = u\,|u\rangle_t$$

$$|u_t\rangle = e^{iH(t-t_0)}\,|u\rangle_{t_0}$$

$$= e^{+iH(t-t_0)}\,P_{t_0}\,\underbrace{e^{-iH(t-t_0)}\,e^{iH(t-t_0)}}_{\mathbb{1}}\,|u\rangle_{t_0}$$

$$= u\,|u\rangle_t$$

$$|u_A u_B\rangle_{-T} = e^{iH(-T-t_0)}\,|u_A u_B\rangle_{t_0}$$

$$|p_1 \dots p_n\rangle_{+T} = e^{iH(T-t_0)}\,|p_1 \dots p_n\rangle_{t_0}$$

Example: $\lambda = 0 \Rightarrow S = \mathbb{1}$

6) T-matrix

$$\boxed{S \equiv \mathbb{1} + iT}$$

$\mathbb{1}$: particles miss

iT: non-trivial scattering

7) 4-momentum conservation.

$$\boxed{\langle \vec{p}_1 \dots \vec{p}_n |\, iT\, | \vec{u}_A \vec{u}_B \rangle \equiv \overbrace{(2\pi)^4\,\delta^{(4)}\Big(u_A + u_B - \sum_f p_f\Big)}^{\text{kinematics}} \times i\,\overbrace{\mathcal{M}(u_A u_B \to \{p_f\})}^{\text{dynamics}}}$$

Invariant matrix element

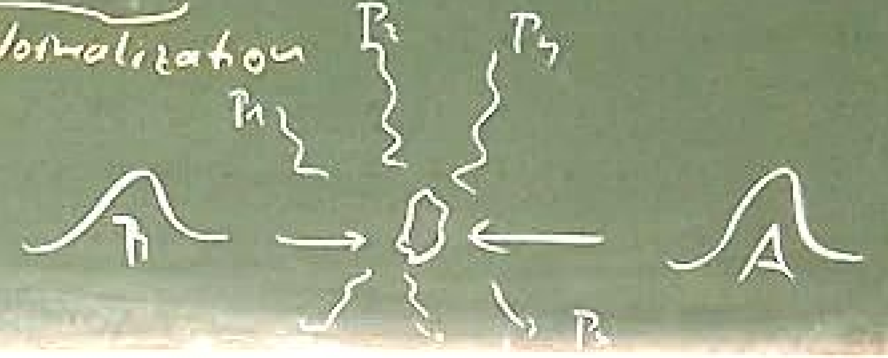
Two questions:

- $\mathcal{M} = ?$ → next time
- $\sigma \overset{?}{=} \sigma(\mathcal{M})$ → now

8) $\propto$ Prob to scatter into $d\mathcal{V}_p = \prod_f d^3p_f$

$$dP(AB_5 \to 1 \dots n)$$

$$= \underbrace{\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}}_{\text{Normalization}}\,\Big|{}_{\text{out}}\langle \vec{p}_1 \dots \vec{p}_n |\, \phi_A\, \phi(5)\rangle_{\text{in}}\Big|^2$$

= 

g) $\measuredangle$ Single particle A and many particles $B_{\vec{b}}$

Area density $n_B(x,y)$

$$d(\#\text{scattering events}) = \int_{x,y} d^2b\, n_B\, dP(AB_{\vec{b}} \to 1\ldots n)$$

$$\to d\sigma = \frac{d(\#\text{scattering events})}{\underbrace{\rho_B l_B}_{n_B}\underbrace{\rho_A l_A A}_{1}} = \frac{\#}{n_B}$$

$$= \int d^2b\, dP(AB_{\vec{b}} \to 1\ldots n)$$

$$= \left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_{p_f}}\right)\left|d^2b \prod_{i=A,B}\left(\int \frac{d^3u_i}{(2\pi)^3}\frac{\phi_i(u_i)}{\sqrt{2E_{u_i}}}\int \frac{d^3q_i\,\phi_i^*(q_i)}{(2\pi)^3\sqrt{2E_{q_i}}}\right)\right.$$

$$\times e^{i\vec{b}(\vec{q}_B-\vec{u}_B)} \underbrace{\left({}_{out}\langle\{\vec{p}_f\}|\{u_i\}\rangle_{in}\right)}_{(2\pi)^4\delta^{(4)}(\sum u_i - \sum p_f)\, i\mathcal{M}(\{u_i\}\to\{p_f\})} \underbrace{\left({}_{out}\langle\{\vec{p}_f\}|\{q_i\}\rangle_{in}\right)^*}_{(2\pi)^4\delta^{(4)}(\sum q_i - \sum p_f)\,(-i)\mathcal{M}^*(\{q_i\}\to\{p_f\})}$$

$$q_B^\perp = (q_B^x, q_B^y)$$

$$(2\pi)^2\delta^{(2)}(u_B^\perp - q_B^\perp)$$

$\to$ Evaluate ~~six~~ q_i-integrals.

i) $q_B^\perp = u_B^\perp$

ii) $q_A^\perp = u_A^\perp$

iii) $\measuredangle$ $q_A^z\, q_B^z$-integrals.

$$\int dq_A^z\, dq_B^z\, \delta(q_A^z + q_B^z - \sum_f p_f^z) \times \delta(E_A + E_B - \sum_f E_f)$$

$$\sqrt{q_A^2 + m^2}$$

$$\hat{=} \frac{1}{\left|\underbrace{\frac{q_A^z}{E_A}}_{=v_A} - \underbrace{\frac{q_B^z}{E_B}}_{v_B}\right|}$$

$$v_g = \frac{\partial E}{\partial q} = \frac{q}{E}$$

$$\left.\begin{array}{c} u^z_{A,B} = q^z_{A,B} \\ \hline \Downarrow \\ \vec{u}_{A/B} = \vec{q}_{A/B} \end{array}\right\}$$

10] $\phi_i(\vec{u}_i)$ peaked at p_i $i = A, B$

$$d\sigma = \left(\prod_f \frac{d^3 p_f}{(2\pi)^3}\frac{1}{2E_{\vec{p}_f}}\right)\frac{|\mathcal{M}(p_A p_B \to p_f)|^2}{2E_{\vec{p}_A}2E_{\vec{p}_B}|v_A - v_B|}$$

$$\times \int \frac{d^3 u_A}{\cdots}\int \frac{d^3 u_B}{\cdots}\, |\phi_A(u_A)|^2 |\phi_B(u_B)|^2$$

$$(2\pi)^4 \delta^{(4)}(\underbrace{u_A + u_B}_{p_A + p_B} - \Sigma p_f)$$

11] Finite momentum resolution of detectors $(\Sigma' p_f)$

$$\boxed{d\sigma = \frac{1}{2E_{\vec{p}_A}2E_{\vec{p}_B}|v_A - v_B|}\left(\prod_f \frac{d^3 p_f}{(2\pi)^3}\frac{1}{2E_{\vec{p}_f}}\right) \times |\mathcal{M}(p_A p_B \to \{p_f\})|^2 \cdot (2\pi)^4 \delta^{(4)}(p_A + p_B - \Sigma p_f)}$$