Recap:

4. Interacting Fields and Feynman Diag.

4.2 Perturbation Expansion of Corr Func

(Details: P-Set 5)

$$H_{\phi^4} = \underbrace{H_0}_{\text{KG Hamiltonian}} + \underbrace{\int d^3x \frac{\lambda}{4!}\phi^4(\vec{x})}_{\text{Interaction}}$$

Known: $\phi_I(x)$, $|0\rangle$ (free vacuum)

$\langle\Omega| T \phi(x)\phi(y) |\Omega\rangle = ?$

$\langle\Omega|$: interacting vacuum; $\phi(x)$: interacting Heisenberg field; $\phi_I(x)$: Interaction picture field

$$\phi_I(x) = e^{iH_0(t-t_0)}\phi(\vec{x})e^{-iH_0(t-t_0)} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[a_{\vec{p}}e^{-ipx} + a^\dagger_{\vec{p}}e^{+ipx}\right]$$

$$\phi(x) = U^\dagger \phi_I U$$

- $U(t,t') = T\exp\left[-i\int_{t'}^{t} ds\, H_I(s)\right]$, with $H_I = \int d^3x \frac{\lambda}{4!}\phi_I^4(x)$

- $|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \dfrac{U(t_0,-T)|0\rangle}{e^{-iE_0(t_0+T)}\langle\Omega|0\rangle}$, $\langle\Omega|0\rangle \neq 0$

Taylor expand $\Rightarrow$ Perturbation theory

$$\boxed{\Rightarrow \langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T}dt\, H_I(t)\right]\right\}|0\rangle}{\langle 0|T\left\{\exp\left[-i\int_{-T}^{T}dt\, H_I(t)\right]\right\}|0\rangle}}$$

$\rightarrow \langle 0|T\phi_I(x_1)\ldots\phi_I(x_n)|0\rangle = ?$

Wick's theorem (*)

(*) = : all possible contractions :

? = all full contractions

- Contraction: $\phi_I(x)\phi_I(y) = D_F(x-y)$
- Normal order: $:a \ldots a^\dagger: = a^\dagger \ldots a$ (creation ... annihilation)

Systematics for summing all full contractions?

$\rightarrow$ Feynman diagrams

4.4. Feynman Diagrams

1)

$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle \propto \langle 0 | T \{ \phi(x)\phi(y) + \phi(x)\phi(y) \left(-i \int dt\, H_I(t) \right) + \dots \} | 0 \rangle$

2) $\underline{\lambda^0\text{-term}}$ $\langle 0 | T \phi(x)\phi(y) | 0 \rangle = D_F(x-y)$

3) $\underline{\lambda^1\text{-term}}$.

$\langle 0 | T \left\{ \phi(x)\phi(y) \left(\frac{-i\lambda}{4!} \right) \int d^4z\, \phi(z)\phi(z)\phi(z)\phi(z) \right\} | 0 \rangle$

Wick's thm $= 3 \cdot \frac{-i\lambda}{4!} D_F(x-y) \cdot \int d^4z\, D_F(z-z) D_F(z-z)$ $+ 12 \cdot \left(\frac{-i\lambda}{4!} \right) \int d^4z\, D_F(x-z) D_F(y-z) D_F(z-z)$ $\ominus$

$\ominus$

$\rightarrow$ Idea.

Feynman diagram = Graph $\left\{ \begin{array}{lll} \text{edges} & = \text{propagators} & \leftrightarrow D_F \\ \text{internal nodes} \bullet & = \text{vertices} & \leftrightarrow (-i\lambda)\int d^4z \\ \text{external nodes} \, | & = \text{spacetime points} & \leftrightarrow x, y, \dots \end{array} \right\}$ Analytic expression

4) Prefactors:

- Feynman diag. = sum of all identical terms (inc. prefactor)
- $\propto O(\lambda^n)$
  - $\rightarrow$ factor $\frac{1}{n!}$ and n integrals/vertices
  - $\rightarrow$ $n!$ possibilities to interchange vertices
  - $\rightarrow$ ignore $\frac{1}{n!}$
- 4 contractions at each vertex
  - $\rightarrow$ $4!$ poss. to interchange contractions
  - $\rightarrow$ $\frac{1}{4!}$ cancels $4!$
  - $\rightarrow$ vertex $= (-i\lambda) \int d^4z$

- Symmetries of diagrams reduce number of distinct contractions

→ divide by symmetry factor S

↳ Example:

$$S\left(x \text{—O—} y\right) = 2$$

$$S(8) = 2 \cdot 2 \cdot 2 = 8$$

$\phi\phi\phi\phi$

$\phi\phi\phi\phi$

Therefore:

$$3 \cdot \frac{1}{4!}$$

$$x \text{——} y \;\; 8 = \frac{1}{8} D_F(x-y)\,(-i\lambda)\int d^4z\, D_F(z-z)\, D_F(z-z)$$

$$12 \cdot \frac{1}{4!}$$

$$x \text{—O—} y = \frac{1}{2}(-i\lambda)\int d^4z\, D_F(x-z)\, D_F(z-z)\, D_F(z-y)$$

$$\boxed{\langle 0|T\,\phi(x)\phi(y)\, e^{-i\int dt\, H_I(t)}|0\rangle = \sum \left\{ \begin{array}{l} \text{Feynman diagrams with} \\ \text{two external points } x \text{ and } y \end{array} \right\}}$$

$$x \text{——} y + x \text{——} y \;\, 8 + x \text{—O—} y + \dots$$

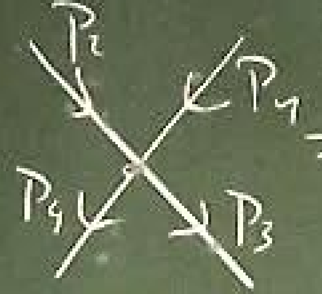
with position/real-space Feynman rules for ϕ^4-Theory.

1. For each propagator: $x \xrightarrow{z} y = D_F(x-y)$
2. — " — vertex: $\times = (-i\lambda)\int d^4z$
3. — " — external point: $x \text{—} \dots = 1$
4. Divide by sym factor: $\frac{1}{S} \times \dots$

b) In momentum space:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \underbrace{\frac{i}{p^2-m^2+i\varepsilon}}_{\text{F.T. in momentum space}} e^{-ip(x-y)}$$

Assign orientations to edges.

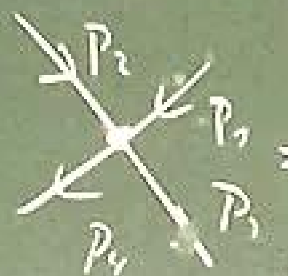


$= (-i\lambda)\int d^4z \;\dots\dots = (-i\lambda)(2\pi)^4 \, \delta(p_1+p_2-p_3-p_4)$

→ Momentum conservation at vertices

→ Momentum-space Feynman rules

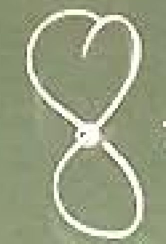
1. Propagators: $\xrightarrow{p} = \frac{i}{p^2-m^2+i\epsilon}$
2. Vertex:



$= (-i\lambda)(2\pi)^4 \times \delta(p_1+p_2-p_3-p_4)$

3. External points: $x \xleftarrow{p} \; = e^{-ipx}$
4. Integrate momenta: $\prod_i \int \frac{d^4p_i}{(2\pi)^4}$
5. Divide by sym factor: $\frac{1}{S}\times$

7) Problem:

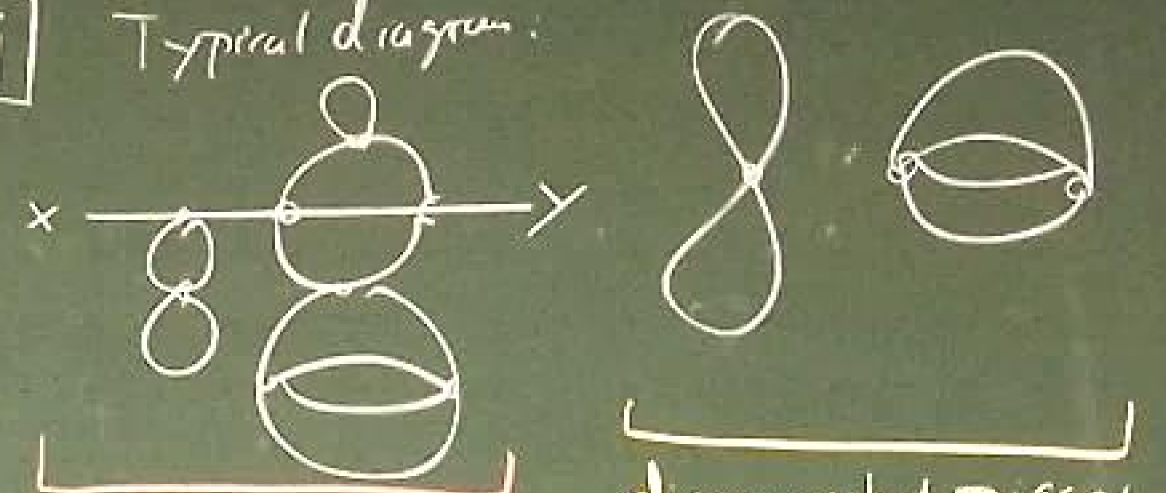


$$= \frac{1}{8}(-i\lambda)\underbrace{\overbrace{\int d^4z}^{\int_{-T}^{T}dt\int d^3x}}_{2T\,\cdot\,\text{volume of space}} \underbrace{D_F(0)D_F(0)}_{\text{const.}}$$

⇒ Infinite!

8) Exponentiation of disconnected diagrams

i) Typical diagram:



ii) $V = \{V_1, V_2, \dots\} = \left\{\begin{array}{l}\text{all disconnected}\\ \text{FD without}\\ \text{external points}\end{array}\right\}$

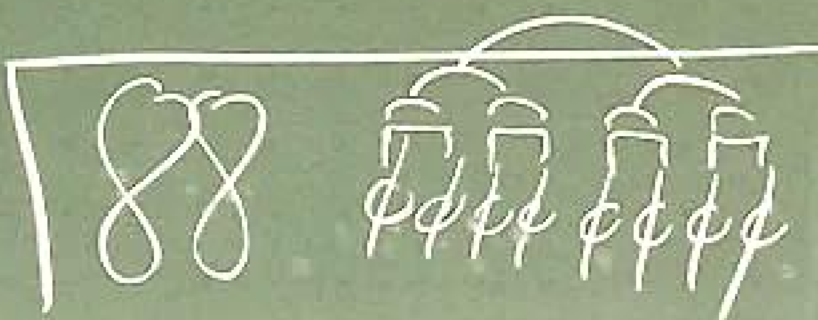
$\mathcal{F}^{xy} = \left\{\begin{array}{l}\text{all connected FD}\\ \text{with ext. points } x, y\end{array}\right\}$

→ FD $F = \{ F^{xy}, \underbrace{V_1, \dots, V_1}_{n_1}, \underbrace{V_2, \dots, V_2}_{n_2} \dots \}$

iii) Amplitude:

$$F = F^{xy} \cdot \prod_i \frac{1}{n_i!}(V_i)^{n_i}$$

Sym. factor



iv) $\langle 0 | T\{\phi(x)\phi(y)\, e^{-i\int dt\, H_I(t)}\} |0\rangle$

$$= \sum_{F^{xy} \in \mathcal{F}^{xy}} \sum_{n_1, n_2, \dots} F^{xy} \underbrace{\prod_i \frac{1}{n_i!}(V_i)^{n_i}}$$

$$= \left(\sum_{F^{xy}} F^{xy}\right)\cdot\left(\sum_{n_1,n_2,\dots}\prod_i \frac{1}{n_i!}(V_i)^{n_i}\right)$$

$$= —''— \left(\prod_i \underbrace{\sum_{n_i} \frac{1}{n_i!}(V_i)^{n_i}}_{e^{V_i}}\right)$$

$$= —''— \cdot \exp\Big[\sum_i V_i\Big]$$

$$\langle 0|T\phi(x)\phi(y)\, e^{-i\int dt\, H_I}|0\rangle = \sum(\mathcal{F}^{xy}) \times e^{\sum(V)}$$

9) Denominator

$$\langle 0|T e^{-i\int dt\, H_I(t)}|0\rangle = e^{\sum(V)}$$

10) Two-point correlator:

$$\boxed{\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \sum(\mathcal{F}^{xy}) = \left\{\begin{array}{l}\text{Sum of all connected}\\ \text{FD with two ext. points } x, y\end{array}\right\}}$$

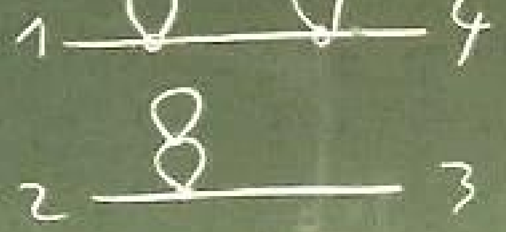
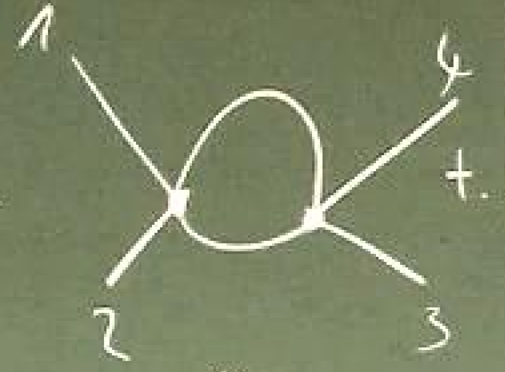
11) Generalization:

$$\boxed{\langle\Omega|T\phi(x_1)\dots\phi(x_n)|\Omega\rangle = \left\{\begin{array}{l}\text{Sum of all connected}\\ \text{FD with } n \text{ external points}\end{array}\right\}}$$

Note 7.2

- Connected diagrams

$\langle \Omega | T \phi_1 \phi_2 \phi_3 \phi_4 | \Omega \rangle$

$= \dots +$  $+$  $+ \dots$

connected diagram (disconnected graph) — // — (connected graph)

- Disconnected diagrams = "Vacuum bubbles"

- $\lim_{T\to\infty(1-i\epsilon)} \langle 0 | T\{\phi_I(x)\phi_I(y) e^{-i\int H}\} | 0 \rangle$

$= \langle \Omega | T \phi(x)\phi(y) | \Omega \rangle \times \lim_{T\to\infty(1-i\epsilon)} \left(\underbrace{e^{-iE_0 2T} |\langle 0|\Omega\rangle|^2}_{e^{\sum(V)}} \right)$

$= \sum (\mathcal{T}^{xy})$

$\sum V_i$

$V_i = (2T \cdot V) \cdot \tilde{V}_i$ — Vol. of space

$= \underbrace{2T \cdot V \sum \tilde{V}_i}$

$\Rightarrow \frac{E_0}{V} = \boxed{\sum_j \tilde{V}_j}$

10^{60}

10^{120} eV

ρ_{vac} "

$\rho_{vac}(R^{\mu\nu} + g R + \boxed{\Lambda} g_{\mu\nu}) = T^{\mu\nu}$

$\Big] \{ \Omega \rangle$

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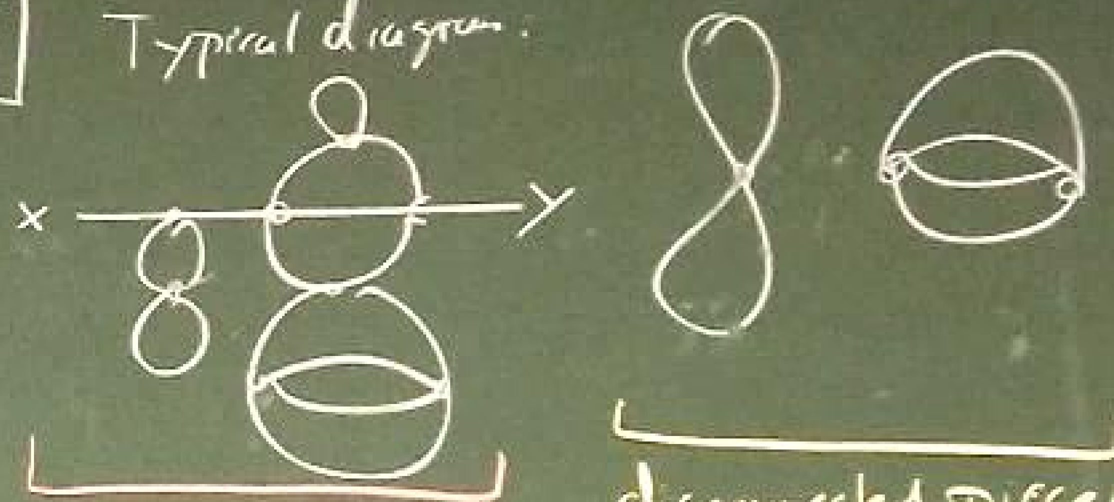
8) Exponentiation of disconnected diagrams

i) Typical diagram:



ii) $\mathcal{V} = \{V_1, V_2, \dots\} = \left\{ \begin{array}{l} \text{all disconnected} \\ \text{FD without} \\ \text{external points} \end{array} \right\}$

$\mathcal{T}^{xy} = \left\{ \begin{array}{l} \text{all connected FD} \\ \text{with ext. points } x, y \end{array} \right\}$