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Problem 2.1: The *classical* complex Klein-Gordon field

[**Written** | 3 pt(s)]

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> **Learning objective**
>
> In the lecture we studied the *real* Klein-Gordon theory. Here we consider its *complex* version as a classical field theory. Its Lagrangian density features a new continuous symmetry. The purpose of this problem is to derive the corresponding Noether current and the conserved charge. Furthermore, you construct the Hamiltonian as preparation for the quantization of the theory.

The classical field theory of a free complex scalar field ϕ with mass $m \geq 0$ is defined by the Minkowski action

$$S[\phi] = \int \mathrm{d}^4x\, \mathcal{L}(\phi, \partial_\mu\phi, \phi^*, \partial_\mu\phi^*) = \int \mathrm{d}^4x\, \left(\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi\right). \tag{1}$$

Here, $\phi(x) \in \mathbb{C}$ is a complex number and $\phi^*(x)$ denotes its complex conjugate; x is a Minkowski four-vector.

a) [1 pt(s)] Write $\phi = u + iv$ and $\phi^* = u - iv$ with real-valued fields $u(x), v(x) \in \mathbb{R}$. Transform the Lagrangian density into the new fields u and v,

$$\mathcal{L}(\phi, \partial_\mu\phi, \phi^*, \partial_\mu\phi^*) = \tilde{\mathcal{L}}(u, \partial_\mu u, v, \partial_\mu v)\,, \tag{2}$$

and show that the equations of motion

$$\partial_\mu\left(\frac{\partial\tilde{\mathcal{L}}}{\partial(\partial_\mu u)}\right) - \frac{\partial\tilde{\mathcal{L}}}{\partial u} = 0 \quad \text{and} \quad \partial_\mu\left(\frac{\partial\tilde{\mathcal{L}}}{\partial(\partial_\mu v)}\right) - \frac{\partial\tilde{\mathcal{L}}}{\partial v} = 0 \tag{3}$$

derived from the real fields are equivalent to the equations of motion

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu \phi)}\right) - \frac{\partial\mathcal{L}}{\partial \phi} = 0 \quad \text{and} \quad \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu \phi^*)}\right) - \frac{\partial\mathcal{L}}{\partial \phi^*} = 0 \tag{4}$$

derived from the complex fields when ϕ and ϕ^* are treated as *independent* variables. Thus it is legit (and convenient) to start from (1) and treat ϕ and ϕ^* as independent fields in the following.

What is the equation of motion that governs the time evolution of the fields ϕ, ϕ^*, u, and v?

b) [1 pt(s)] Calculate the conjugate momenta π and π^* of ϕ and ϕ^*. Perform the Legendre transformation and show that the Hamiltonian is given by

$$H = \int \mathrm{d}^3\boldsymbol{x}\, \left(\pi^*\pi + \nabla\phi^*\nabla\phi + m^2\phi^*\phi\right)\,. \tag{5}$$

c) [1 pt(s)] The Lagrangian $\mathcal{L}(\phi, \partial_\mu\phi, \phi^*, \partial_\mu\phi^*)$ is invariant under the continuous transformation

$$\overline{\phi}(x) = e^{ie\theta}\phi(x) \quad \text{and} \quad \overline{\phi}^*(x) = e^{-ie\theta}\phi^*(x)\,. \tag{6}$$

Here, $\theta \in \mathbb{R}$ is a continuous parameter and $e \in \mathbb{R}$ is an arbitrary but fixed constant. We say that $\mathcal{L}$ features a global $\mathrm{U}(1)$ symmetry with charge e.

Determine its infinitesimal generators $iG_\theta\phi$ and $iG_\theta\phi^*$ and show that the conserved Noether current reads

$$j^\mu(x) = -ie\left[(\partial^\mu\phi^*)\phi - \phi^*(\partial^\mu\phi)\right]\,. \tag{7}$$

Calculate the conserved charge $Q = \int \mathrm{d}^3\boldsymbol{x}\, j^0(t, \boldsymbol{x})$.

Problem 2.2: The *quantized* complex Klein-Gordon field [**Oral** | 3 pt(s)]

ID: ex_quantized_complex_klein_gordon_field:qft22

Learning objective

Here we quantize the complex Klein-Gordon theory along the lines of the real theory shown in the lecture. The most important difference is that the complex Klein-Gordon field gives rise to *two* types of particles with opposite Noether charge: *particles* and *antiparticles*.

We quantize the classical Hamiltonian (5) by raising the complex fields ϕ and π to operators with the momentum mode expansions

$$\phi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}e^{i\boldsymbol{p}\boldsymbol{x}} + b^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\boldsymbol{x}}\right)\,, \tag{8a}$$

$$\pi(\boldsymbol{x}) = (-i)\int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\sqrt{\frac{E_{\boldsymbol{p}}}{2}}\left(b_{\boldsymbol{p}}e^{i\boldsymbol{p}\boldsymbol{x}} - a^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\boldsymbol{x}}\right)\,. \tag{8b}$$

The complex conjugate fields are identified with adjoint operators: $\phi^* \to \phi^\dagger$ and $\pi^* \to \pi^\dagger$. Here, $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$ and the momentum modes obey the usual commutation relations,

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) \quad \text{and} \quad \left[b_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\,\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\,; \tag{9}$$

all other commutators vanish. The quantized Hamiltonian of the complex Klein-Gordon field follows from (5) and reads

$$H = \int \mathrm{d}^3\boldsymbol{x}\, {:}\pi^\dagger\pi + \nabla\phi^\dagger\nabla\phi + m^2\phi^\dagger\phi{:} \tag{10}$$

where ${:}a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}{:} = a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}$ and ${:}a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}}{:} = a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}$ denotes *normal ordering* (the same holds for $b_{\boldsymbol{p}}$), that is, ${:}\bullet{:}$ reorders products $\bullet$ of mode operators such that creation operators are on the left.

a) [1 pt(s)] First, check the canonical commutation relations

$$[\phi(\boldsymbol{x}), \pi(\boldsymbol{y})] = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \quad \text{and} \quad [\phi(\boldsymbol{x}), \phi(\boldsymbol{y})] = 0 = [\pi(\boldsymbol{x}), \pi(\boldsymbol{y})]\,. \tag{11}$$

What is the difference between the “real” and the “complex” Klein-Gordon field operators?

Diagonalize the quantized Hamiltonian H in terms of momentum modes $a_{\boldsymbol{p}}$ and $b_{\boldsymbol{p}}$. What happens if the normal ordering in (10) is omitted, and is the quantization well-defined without it? Compare your result with the *real* Klein-Gordon theory.

b) [1 pt(s)] Calculate the time-dependent field operators $\phi(x) = \phi(t, \boldsymbol{x})$ and $\pi(x) = \pi(t, \boldsymbol{x})$.

Hint: Show that $e^{iHt} a_{\boldsymbol{p}} e^{-iHt} = e^{-iE_{\boldsymbol{p}}t}\, a_{\boldsymbol{p}}$ (and similarly for $b_{\boldsymbol{p}}$).

Use your results to show that $\pi(x) = \partial_0 \phi^\dagger(x)$ and that $\phi(x)$ satisfies the Klein-Gordon equation. How does this relate to the classical results in Problem 1?

c) [1 pt(s)] Quantize the conserved charge Q from Problem 1 c) and show that it takes the form

$$Q = ie \int \mathrm{d}^3\boldsymbol{x} :\pi\phi - \phi^\dagger\pi^\dagger: = -e \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \left(a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}} - b_{\boldsymbol{p}}^\dagger b_{\boldsymbol{p}}\right) . \tag{12}$$

Prove that Q is conserved, i.e., show that $[Q, H] = 0$.

Explain why normal ordering (:•:) is again necessary for Q to be well-defined as a quantum operator. Why is it reasonable to call $a_{\boldsymbol{p}}^\dagger$ *particles* and $b_{\boldsymbol{p}}^\dagger$ *antiparticles*? What is the interpretation of e?