Organization

- * www.itp3.uni-stuttgart.de/teaching/qft22
- * 1 Problemset per week, written + oral exercises

80% 66%

2 x blackboard presentations

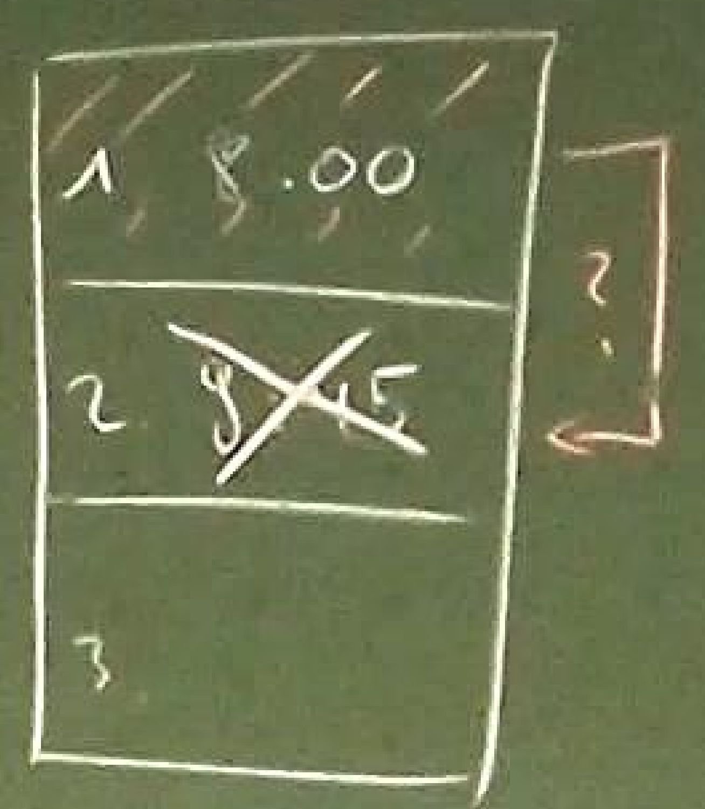
Sign up for exercises.

⇒ lms.itp3.uni-stuttgart.de

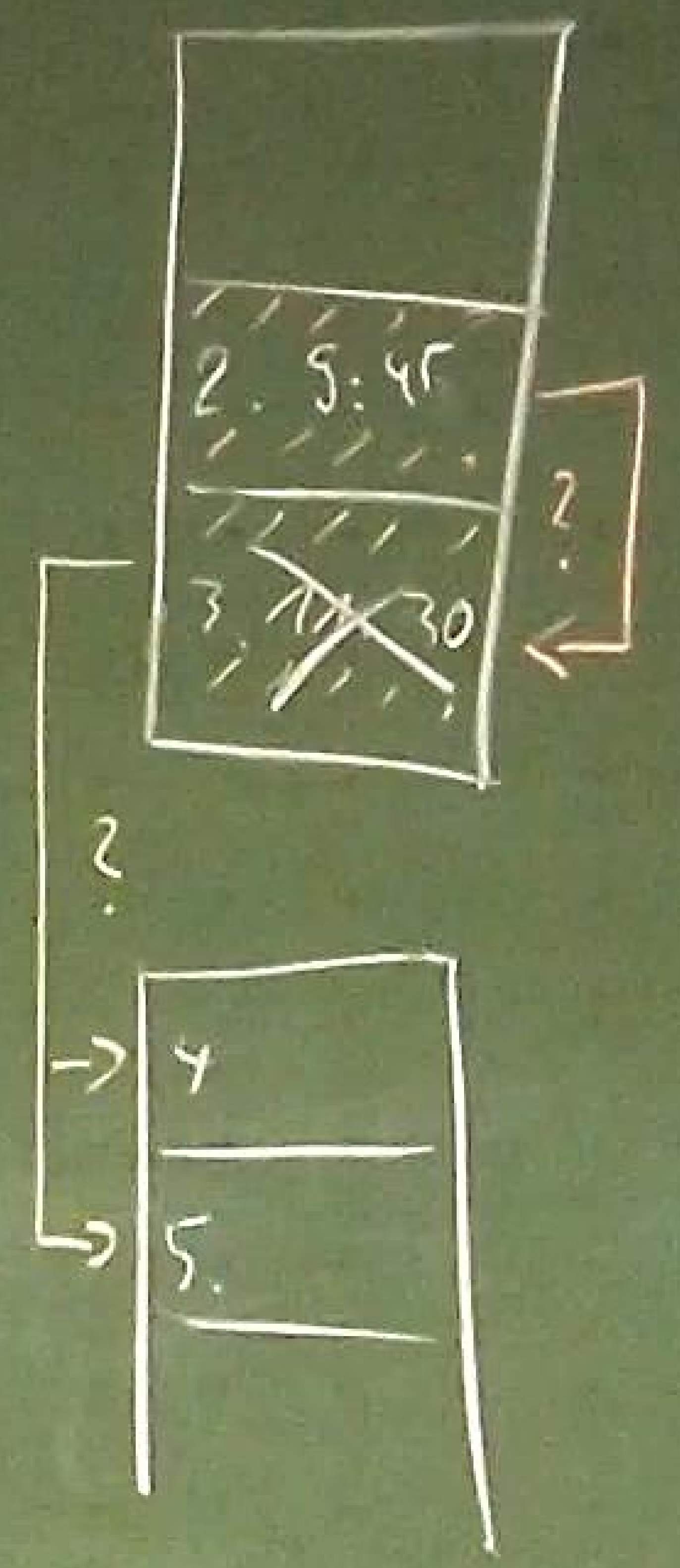
Lecture Key: qft SS22!

- * 2 lectures per week:

Wed



Fri



- * Based on Peskin & Schroeder

- * Requirements.
  - QM (second quantization)
  - SRT (tensor calculus)
  - Complex analysis (residue theorem)


? ? ?
High energy physics
Condensed matter physics
Elementary what ??
(Strings, Spin-foam...?)
→ Emergent Quantum fields
= Elementary Quantum fields
(Dirac fields, Gauge fields..)
→ Emergent particles
= Elementary Particles
(Electrons, Phonons..)
→ Emergent Quantum fields
(Magnetization...)
→ Emergent Quasiparticles
(Magnons...)
1+4
Standard Model
High energies
Small lengths
Short times
10^{28} eV (Planck)
10^9 eV
10^0 eV
Low energies
Large lengths
Long times
This course

1. Elements of classical field theory

1.1. Lagrangian and Hamiltonian formalism

Recap: Classical mechanics of "point"

1. DOF q_i $i = 1 .. N$

2. Lagrangian $L(\{q_i\}, \{\dot{q}_i\}, t)$ $\dot{q}$

$$= T - V$$

3. Action $\underset{q(t)}{S}[q] = \int dt\, L(q(t), \dot{q}(t), t)$

4. Hamilton's principle of least action

$$\frac{\delta S[q]}{\delta q} \overset{!}{=} 0 \Leftrightarrow \delta S = \int dt\, \delta L \overset{!}{=} 0$$

5. Euler-Lagrange equations $(i = 1 .. N)$

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0$$

Analogous: Lagrangian Field Theory

1. ~~x~~ $\phi(x)$ on spacetime
with $\partial_\mu \phi(x)$

2. Lagrangian density
$\mathcal{L}(\phi(x), \partial\phi, x)$

$x \in \mathbb{R}^{1,3}$
$\in \mathbb{R}^4$
$\partial_0 = \partial_t$
∂_i
x y t

→ Lagrangian:

$$L = \int d^3x\, \mathcal{L}(\phi, \partial\phi)$$

3. Action

$$S[\phi] = \int dt\, L = \int d^4x\, \mathcal{L}(\phi(x), \partial\phi(x))$$

4. Action principle:

$$0 \overset{!}{=} \delta S = \int d^4x\, \delta\mathcal{L}$$

$$= \int d^4x \left\{ \underbrace{\frac{\partial \mathcal{L}}{\partial \phi} \underline{\delta\phi}} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right\}$$

$$= \int d^4x \left\{ \swarrow - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \underline{\delta\phi} + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) \right\}$$

$$= \int d^4x \underbrace{\left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right\}} \delta\phi$$

$$+ \int_\partial dA \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \underset{\underset{0}{\shortparallel}}{\delta\phi} \right) \overset{!}{=} 0$$

5. Euler Lagrange equations.

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0}$$

Recap Hamiltonian Mechanics

Lag.

$$L(q, \dot{q}, t) \xrightarrow{\text{Legendre transformation}}$$

$$p \equiv \frac{\partial L}{\partial \dot{q}} \Leftrightarrow \dot{q} = \dot{q}(p)$$

$$H(q, p, t) = p\dot{q} - L(q, \dot{q}, t)$$

Analogous: Hamiltonian Field Theory

1\. x $\quad x = x_i \hat{=} i$

$$\frac{\partial L}{\partial \dot{q}_i} = P_i \hat{=} P(x) = \frac{\partial L}{\partial(\dot\phi(x))}$$

$$= \frac{\partial}{\partial \dot\phi(x)} \int d^3y \, \mathcal{L}(\phi(y), \dot\phi(y))$$

$$= \sum_y d^3y \frac{\partial}{\partial \phi(x)} L$$

$$\delta_{x,y} \frac{\partial \mathcal{L}}{\partial \phi}\Big|_{x=y}$$

$$= \underbrace{\frac{\partial \mathcal{L}}{\partial \dot\phi(x)}}_{} d^3x$$

$\equiv \pi(x)$ Momentum density conjugate of ϕ

② Hamiltonian.

$$H = \sum_x \underbrace{P(x)}_{\pi(x) \cdot d^3x} \dot\phi(x) - \sum_x \mathcal{L}(\phi(x), \dot\phi(x)) \, d^3x$$

$d^3x \to 0$

$$= \int d^3x \{ \underbrace{\pi(x) \dot\phi(x) - \mathcal{L} \, \phi(x), \dot\phi(x))}_{} \}$$

Hamiltonian density $\mathcal{H}(\phi, \pi)$

Example 1.1. Free scalar field

1\. $\phi : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}$

$\phi(x) = \phi(\vec{x}, t)$

2\. L. density:

$$\mathcal{L} = \frac{1}{2} (\partial_t \phi)^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2$$

$$= \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$$

$(\partial_\mu \phi)(\partial^\mu \phi)$

3\. Interpretation:

4. EOM

$$-m^2\phi - \partial_\mu(\partial^\mu\phi) = 0$$

$$\Leftrightarrow \boxed{(\partial_\mu\partial^\mu + m^2)\phi = 0}$$

(classical) Klein-Gordon equation

5. $$\pi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi$$

6. $$\mathcal{H} = \pi\dot\phi - \frac{1}{2}\dot\phi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi$$

$$= \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

1.2 Symmetries and Conservation Laws

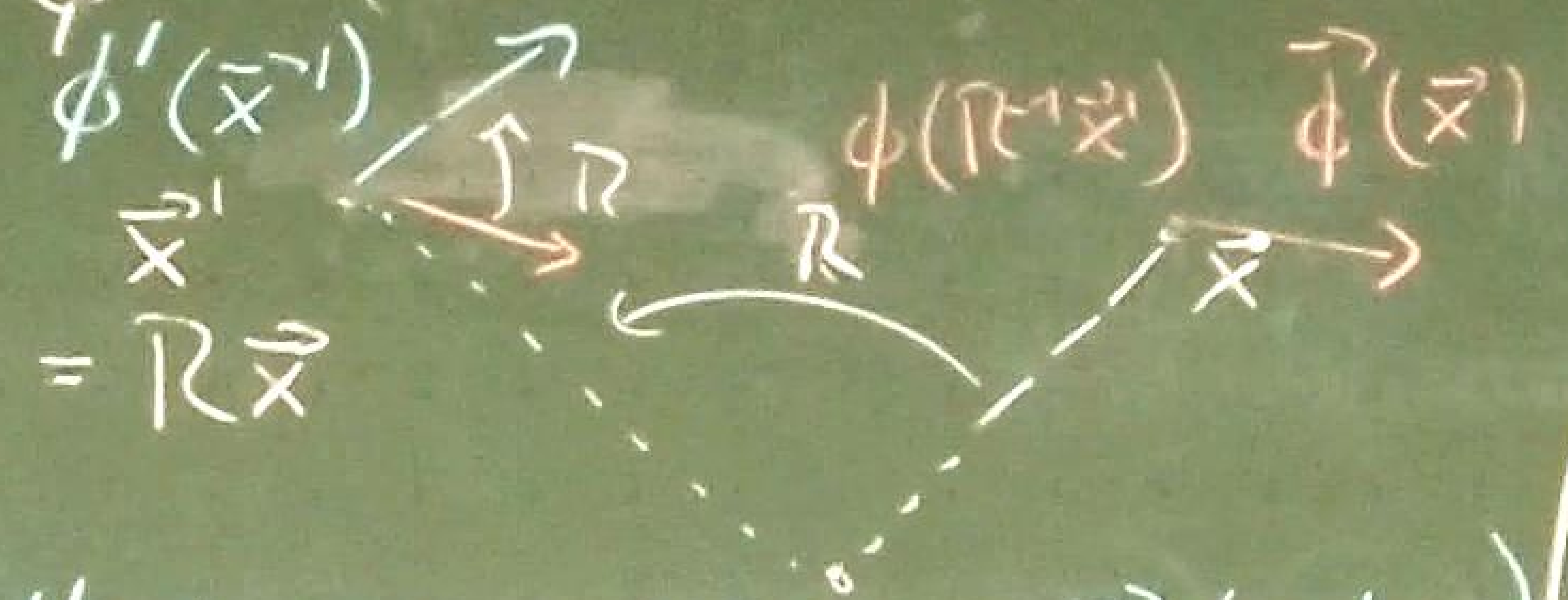
1.X Transformations of coordinates and fields.

$$x \longmapsto x' = x'(x) \quad \text{and}$$

$$\phi(x) \longmapsto \phi'(x') = \mathcal{F}(\phi(x))$$

Example 1.2 Rotation of a vector field $\vec\phi$

i) $\vec\phi = (\phi_1, \phi_2, \phi_3)$, $R \in SO(3)$



$$\vec\phi'(\vec x') = R\vec\phi(\vec x^0) = R\vec\phi(R^{-1}\vec x') \quad \begin{pmatrix} S(x) \\ T(x) \end{pmatrix}$$

2.

$$S' \equiv S[\phi'] = \int d^d x' \mathcal{L}(\phi'(x'), \partial'_\mu\phi'(x'))$$

$$= \int d^d x' \mathcal{L}(\mathcal{F}(\phi(x)), \partial'_\mu \mathcal{F}(\phi(x)))$$

$$= \int d^d x \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}(\mathcal{F}(\phi(x)), \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu\mathcal{F}(\phi(x)))$$

Example 1.3 Translation:

1. $x' := x + a$

$\phi'(x') := \phi(x) = \phi(x'-a)$

$\xrightarrow{\hphantom{xxxxxx}}$ scalar field

2. $\mathcal{F} = \mathbb{1}$ trivial

- $\phi'(x') = \mathcal{F}(\phi(x)) = \phi(x(x'))$

- $\dfrac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu$

3. $S[\phi'] = \int d^dx\, \mathcal{L}(\phi'(x), \partial_\mu \phi'(x)) = \int d^dx\, \mathcal{L}(\phi(x), \partial_\mu \phi(x)) = S[\phi]$

Example 1.4 Scale transformations

scaling dimension

1. $x' = \lambda x$

$\phi' := \lambda^{-\Delta} \phi(x)$

2. $\mathcal{F}(\phi) = \lambda^{-\Delta}\phi$, $\dfrac{\partial x^\nu}{\partial x'^\mu} = \lambda^{-1}\delta^\nu_\mu$

$\hookrightarrow \left|\dfrac{\partial x'}{\partial x}\right| = \lambda^d$

3. Action:

$$\boxed{S[\phi']} = \lambda^d \int d^dx\, \mathcal{L}(\lambda^{-\Delta}\phi(x), \lambda^{-1-\Delta}\partial_\mu \phi)$$

$\mathcal{L}$ FST

$$\overset{m=0}{=} \lambda^{d-2-2\Delta} \int d^dx\, \mathcal{L}(\phi(x), \partial_\mu \phi(x)) = \boxed{\lambda^{d-2-2\Delta} S[\phi]}$$

$\rightarrow S' = S$ iff $\Delta = \frac{d}{2} - 1$

Example 1.5. Phase rotation

1. $x' := x$

   $\phi'(x') := e^{i\theta}\phi(x)$

2. $\mathcal{F}(\phi) = e^{i\theta}\phi$

   $\frac{\partial x'^{\nu}}{\partial x^{\mu}} = \delta^{\nu}_{\mu}, \left|\frac{\partial x'}{\partial x}\right| = 1$

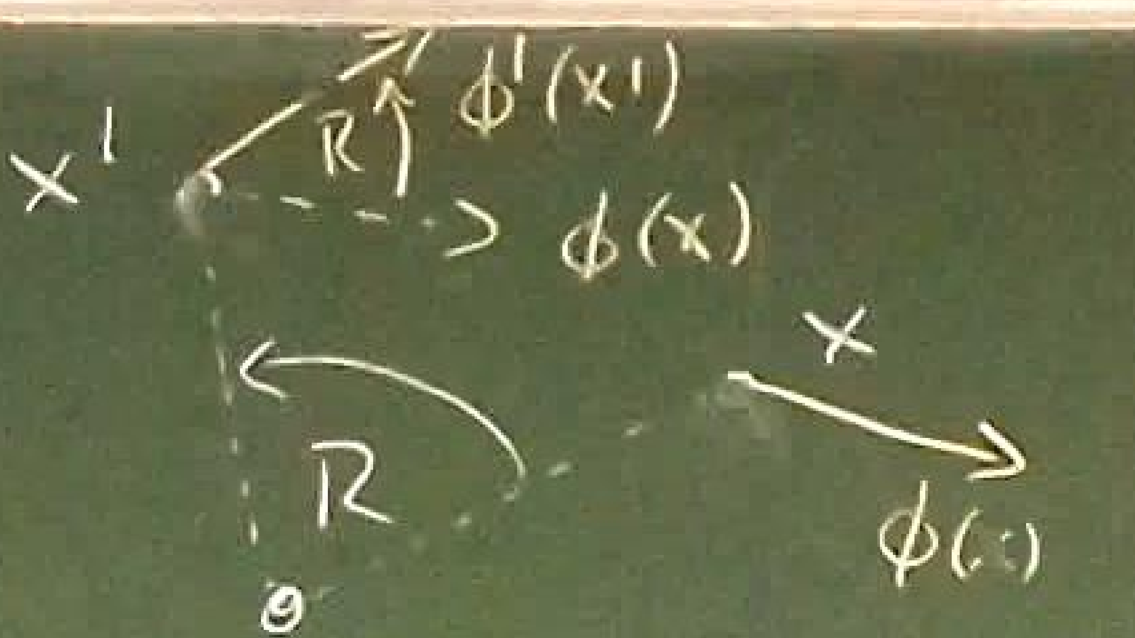
$$\mathcal{L} = \phi\dot{\phi}^* - (\nabla\phi)(\nabla\phi^*)$$

Recap:

- Transformations:

$x \mapsto x' = x'(x)$

$\phi(x) \mapsto \phi'(x') = \mathcal{F}(\phi(x))$



- Action:

$$S' = S[\phi'] = \int d^d x \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}\left(\mathcal{F}(\phi(x)), \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \mathcal{F}(\phi(x))\right)$$

(*)

Infinitesimal Transformations

1. IT:

$$x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a}(x)$$

$$\phi'(x') = \phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x)$$

2. Generator of IT:

$$\delta_\omega \phi(x) = \phi'(x) - \phi(x) \equiv -i\omega_a G_a \phi(x)$$

With:

$$\phi'(x) = \phi(x) - \omega_a \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) + \mathcal{O}(\omega_a^2)$$

$$\Rightarrow \boxed{i G_a \phi = \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \phi - \frac{\delta \mathcal{F}}{\delta \omega_a}}$$

Example 1.6. Translations

1. $x'^\mu = x^\mu + \omega^\mu$

$$= x^\mu + \omega^\nu \frac{\delta x^\mu}{\delta \omega^\nu} \qquad \delta^\mu_\nu$$

2. $\frac{\delta \mathcal{F}}{\delta \omega^\nu} = 0$

3. $i G_\mu \phi = \delta^\nu_\mu \partial_\nu \phi - 0$

$$\Downarrow$$

$$\boxed{G_\mu = -i\partial_\mu = P_\mu}$$

Example 1.7. Scale Transformations

$$G = -i x^\mu \partial_\mu = D$$

Example 1.8. Spatial Rotations

$$G_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + S_{\mu\nu}$$

Noether's Theorem

1\.

Symmetry of the action

$$\Longleftrightarrow \quad S[\phi] = S[\phi']$$

for ω_a independent of x (rigid transformations)

2\. Assume that $\omega_a = \omega_a(x)$ is not rigid

$$(\quad O(\omega_a) = O(\partial\omega_a))$$

3\. Jacobian: $\dfrac{\partial x'^\nu}{\partial x^\mu} = \delta^\nu_\mu + \partial_\mu\left(\omega_a \dfrac{\delta x^\nu}{\delta\omega_a}\right)$

$$\det(\mathbb{1}+A) = 1 + \operatorname{tr}[A] + O(A^2)$$

$$\left|\frac{\partial x'}{\partial x}\right| = 1 + \partial_\mu\left(\omega_a \frac{\delta x^\mu}{\delta\omega_a}\right)$$

4\. Inverse Jacobian matrix.

$$\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu - \partial_\mu\left(\omega_a \frac{\delta x^\nu}{\delta\omega_a}\right)$$

5\. Use (*)

$$S' = \int d^dx \left(1 + \partial_\mu \omega_a \frac{\delta x^\mu}{\delta\omega_a}\right) \times \mathcal{L}\left(\phi + \omega_a \frac{\delta F}{\delta\omega_a}, \left[\delta^\nu_\mu - \partial_\mu\left(\omega_a \frac{\delta x^\nu}{\delta\omega_a}\right)\right] \times \left[\partial_\nu\phi + \partial_\nu\left(\omega_a \frac{\delta F}{\delta\omega_a}\right)\right]\right)$$

6\. Expan in 1st order of $\omega_a, \partial_\mu\omega^a$

7\. $\delta S \equiv \underset{S(\phi')}{S'} - \underset{S(\phi)}{S}$

$\hookrightarrow$ Only terms $\propto \dfrac{\partial\omega_a}{\partial x^\mu}$ remain

8. For generic (non-rigid) transformation:

$$\delta S = \overset{S[\phi']}{S'} - S = -\int d^dx\, j^\mu_a\, \partial_\mu \omega_a$$

with the current:

$$\boxed{j^\mu_a = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}\right) \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta \mathcal{F}}{\delta \omega_a}}$$

9. Integration by parts:

$$\delta S = \int d^dx\, \omega_a (\partial_\mu j^\mu_a) = 0 \quad \forall\, \omega_a(x)$$

10. $\forall\, \phi$ that obey the EOMs

$\longrightarrow$ $\delta S = 0$ for arbitrary variations $\phi' = \phi + \delta\phi$

$\longrightarrow$ In particular true for non-rigid inf. transformations

$$\Rightarrow \boxed{\partial_\mu j^\mu_a(x) = 0 \quad \forall\, x, a}$$

Noether's theorem

11. Conserved charge:

$$\boxed{Q_a = \int_{\text{Space}} d^{d-1}x\, j^0_a}$$

Proof:

$$\frac{dQ_a}{dt} = \int_{\text{Sp}} d^{d-1}x\, \partial_0 j^0_a \overset{\text{Noether}}{=} -\int_{\text{Sp}} d^{d-1}x\, \partial_k j^k_a, \quad k = 1,2,3$$

$$\overset{\text{Gauss}}{=} -\int_{\text{Surface}} d\sigma_k\, j^k_a \underset{!}{=} 0$$

Note 1.1

$$\tilde{j}^\mu_a := j^\mu_a + \partial_\nu B^{\mu\nu}_a \quad \text{with } B^{\mu\nu}_a = -B^{\nu\mu}_a \text{ arbitrary}$$

$$\Rightarrow \partial_\mu \tilde{j}^\mu_a = 0$$

Note 1.2

$$\left.\begin{array}{c}\text{Symmetric Lagrangian}\\ \Downarrow \\ \text{Symmetric Action}\end{array}\right\} \Rightarrow \begin{array}{l}\text{Conserved}\\ \text{currents.}\end{array}$$

$$\Downarrow$$

Symmetric EOM.

$$\mathcal{L} = \lambda^2\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}(\phi)^2\lambda^2$$

$$\rightarrow \quad (\partial^2 + m^2)\phi = 0 \Rightarrow \phi' = \lambda\phi$$

Application: The Energy-Momentum Tensor (EMT)

1. spacetime T $\mathcal{P} = SO(1,3) \ltimes \mathbb{R}^4$

$$x'^\mu = x^\mu + \varepsilon^\mu \rightarrow \frac{\delta x^\mu}{\delta\varepsilon^\nu} = \delta^\mu_\nu$$

2. $\mathcal{L}$ translation invariant action. $S' = S$ $\qquad \frac{\delta \mathcal{F}}{\delta\varepsilon^\nu} = 0$

3. Conserved currents:

$$\underbrace{j^\mu_a = T^\mu{}_\nu} = \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\rho\phi - \delta^\mu_\rho\mathcal{L}\right)\underbrace{\frac{\delta x^\rho}{\partial\varepsilon^\nu}}_{\delta^\rho_\nu}$$

$$= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}$$

$$\boxed{\begin{aligned}T^{\mu\nu} &= g^{\nu\rho}T^\mu{}_\rho\\ &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu - g^{\mu\nu}\mathcal{L}\end{aligned}}$$

Energy-Momentum-Tensor

with $\boxed{\partial_\mu T^{\mu\nu} = 0}$

and four conserved charges.

$$\boxed{P^\nu = \int d^3x\, T^{0\nu}}$$

4. **Energy** $\nu = 0$

$$P^0 = \int d^3x\, T^{00} = \int d^3x \underbrace{\left(\underbrace{\frac{\partial \mathcal{L}}{\partial \dot\phi}}_{\pi}\dot\phi - \mathcal{L}\right)}_{\mathcal{H}}$$

$$= H \quad (\text{: Hamiltonian})$$

5. **Kinetic momentum** $(\nu = i)$

$$P^i = \int d^3x\, T^{0i} = \int d^3x \overset{\pi}{\left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\right)}(-\partial_i \phi)$$

$$= -\int d^3x\, \pi\, \partial_i \phi$$

Note 1.3

In general $T^{\mu\nu} = T^{\nu\mu}$ for canonical EMT

$$\tilde{T}^{\mu\nu} := T^{\mu\nu} + \partial_\rho K^{\rho\mu\nu} \quad \text{with} \quad K^{\rho\mu\nu} = -K^{\mu\rho\nu}$$

(Choose $K^{\rho\mu\nu}$ that

$\tilde{T}^{\mu\nu} = \tilde{T}^{\nu\mu}$ ($\Rightarrow$ Belinfante EMT)

Example 1.9. Electromagnetism in vacuum

1. Four component gauge field $A^\mu = (\phi, A^1, A^2, A^3)$
2. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
3. Lagrangian: $\mathcal{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$
4. Action: $S_{em} = \int d^4x\, \mathcal{L}_{em}$
5. **EOM**: $\boxed{\partial_\mu F^{\mu\nu} = 0}$
6. S_{em}: Lorentz invariant + translational invariant $\hookrightarrow$ EMT = conserved quantities
7. Canonical EMT: $T^{\mu\nu}_{em} = \dfrac{\partial \mathcal{L}_{em}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda - g^{\mu\nu} \mathcal{L}_{em}$
8. $K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu$

$$\Rightarrow \boxed{\tilde{T}^{\mu\nu}_{em} \overset{\circ}{=} \frac{1}{4} g^{\mu\nu} F_{\rho\lambda} F^{\rho\lambda} - F^{\mu\rho} F^{\nu}{}_{\rho}}$$

- $T^{00} = \frac{1}{2}\left(E^2 + B^2\right)$
- $T^{0i} = (\vec{E} \times \vec{B})_i$
- $\tilde{T}^{ij} = \sigma_{ij}$ (Maxwell stress tensor)

$$\tilde{T} = \left(\begin{array}{c|ccc} T^{00} & - & T^{0n} & - \\ \hline - & & & \\ - & & \boxed{\sigma_{ij}} & \\ - & & & \end{array}\right)$$

- $\underbrace{F = \int \underbrace{\rho \ldots \ldots}}_{f}$

2. The Klein-Gordon Field

2.1. Canonical Quantization

1. Theory: $\phi = \phi^*$

i) Real field $\phi(x)$

(ii) Lagrangian: $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$

ii) EOM. $(\partial^2 + m^2)\phi = 0$

$$\text{(iv) Hamiltonian: } \boxed{\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} m \phi^2}$$

2. Canonical quantization.

$$\boxed{\begin{aligned} [\phi(\vec{x}), \phi(\vec{y})] &= 0 \\ [\pi(\vec{x}), \pi(\vec{y})] &= 0 \\ [\phi(\vec{x}), \pi(\vec{x})] &= i\delta^{(3)}(\vec{x} - \vec{y}) \end{aligned}} \quad (3)$$

- $\phi^\dagger = \phi, \quad \pi^\dagger = \pi$

3. Goals:

- Representation of field operators
- Spectrum of Hamiltonian
- Time evolution

4. Motivation.

i) FT KG equation in space.

$$\phi(\vec{x}, t) = \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p}\vec{x}} \tilde{\phi}(\vec{p}, t)$$

$$\xrightarrow{KG} \Big(\partial_t^2 + \underbrace{|\vec{p}|^2 + m^2}_{\omega_p^2}\Big) \tilde{\phi}(p, t) = 0$$

→ Decoupled Harmonic oscillators.

and constraint $\phi^*(\vec{p}, t) = \phi(-\vec{p}, t)$

(ii) $\mathcal{L}$ $\quad H_{SHO} = \frac{1}{2}\tilde{\pi}^2 + \frac{1}{2}\omega_p^2 \tilde{\phi}^2$

$$\tilde{\phi} = \frac{1}{\sqrt{2\omega}}(a + a^\dagger)$$

$$\tilde{\pi} = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$$

wit $[a, a^\dagger] = 1$

$$\Rightarrow H_{SHO} = \omega\left(a^\dagger a + \frac{1}{2}\right)$$

5. Field operators:

$$\left.\begin{aligned}\phi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} + a^{\dagger}_{-\vec{p}}\right) e^{i\vec{p}\vec{x}} \\ &= \int \frac{d^3p}{(2\pi)^3} \left(a_{\vec{p}} e^{i\vec{p}\vec{x}} + a^{\dagger}_{\vec{p}} e^{-i\vec{p}\vec{x}}\right) \\ \pi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} -i\sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}} - a^{\dagger}_{-\vec{p}}\right) e^{i\vec{p}\vec{x}}\end{aligned}\right\} (1)$$

$$\left(\phi^{\dagger} = \phi, \quad \pi^{\dagger} = \pi\right)$$

with momentum modes

$$\boxed{\left[a_{\vec{p}}, a^{\dagger}_{\vec{q}}\right] = (2\pi)^3 \delta(\vec{p} - \vec{q})} \quad (2)$$

Check: (1) + (2) ⇒ (3) $\quad \omega_p = \sqrt{|\vec{p}|^2 + m^2} = E_p$

6. Hamiltonian:

$$H \overset{\circ}{=} \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \omega_p \left(a^{\dagger}_{\vec{p}} a_{\vec{p}} + \frac{1}{2} \underbrace{\cancel{\left[a_{\vec{p}}, a^{\dagger}_{\vec{p}}\right]}}_{\delta(0) = \infty}\right)$$

(→ Normal ordering)

7. Eigenstates + Spectrum:

i) $\xrightarrow{\circ} \left[H, a^{\dagger}_{\vec{p}}\right] = \omega_{\vec{p}} a^{\dagger}_{\vec{p}}$

ii) Vacuum $|0\rangle$ ⟶ Eigenstates

$a_{\vec{p}}|0\rangle = 0 \ \forall \vec{p} \qquad a^{\dagger}_{\vec{p}} a^{\dagger}_{\vec{q}} \ldots |0\rangle$

iii) Energy: $E_{\vec{p}} = \omega_p = \sqrt{|\vec{p}|^2 + \omega^2}$

iv) Kinetic momentum

$$P^i = \int d^3x \, \pi(\vec{x}) \left(-\partial_i \phi(\vec{x})\right) \overset{\circ}{=} \int \frac{d^3p}{(2\pi)^3} p_i \, a^{\dagger}_{\vec{p}} a_{\vec{p}}$$

v) Statistics: $a^{\dagger}_{\vec{p}} a^{\dagger}_{\vec{q}} |0\rangle = a^{\dagger}_{\vec{q}} a^{\dagger}_{\vec{p}} |0\rangle$

→ Excitations $a^{\dagger}_{\vec{p}}$ commute and carry additive energy + momentum.

→ Bosonic particles

8. Normalization:

i) $\forall\ \Lambda = R' L_3(\overset{v/c}{p}) R \in SO^+(1,3)$

$$p' = (\underbrace{E_p'}_{\sqrt{|\vec{p}|^2+m^2}}, \vec{p}\,') = \Lambda p_{(E_p, \vec{p})}$$

ii) Jacobian in space

$$\det\left(\frac{\partial \vec{p}\,'}{\partial \vec{p}}\right) \hat{=} \frac{dp_3'}{dp_3} \hat{=} \frac{E_{p'}}{E_p}$$

$$\rightarrow \underset{\substack{\uparrow \\ \text{not LI}}}{\delta^{(3)}}(\vec{p}-\vec{q}) = \frac{E_{p'}}{E_p}\delta^{(3)}(\vec{p}\,'-\vec{q}\,')$$

$$\rightarrow E_p \delta(\vec{p}-\vec{q}) \ \underline{\text{is}} \ LI$$

iii) Single particle eigenstates.

$$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\, a^\dagger_p |0\rangle$$

$$\Rightarrow \langle \vec{p}|\vec{q}\rangle = (2\pi)^3 2 \underbrace{E_{\vec{p}}\, \delta(\vec{p}-\vec{q})}_{LI}$$

9. Lorentz transformations

$\Lambda \in SO^+(1,3)$

$$U(\Lambda)|\vec{p}\rangle := |\overset{\Lambda^i{}_\mu p^\mu,\ i=1,2,3}{\Lambda \vec{p}}\rangle$$

$$\Leftrightarrow U(\Lambda) a^\dagger_{\vec{p}} U^\dagger(\Lambda) = \sqrt{\frac{E_{\Lambda\vec{p}}}{E_{\vec{p}}}}\, a^\dagger_{\Lambda\vec{p}}$$

10. Interpretation of $\phi(\vec{x})$:

$$\phi(\vec{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{-i\vec{p}\vec{x}} |\vec{p})$$

$|\vec{p}| \ll m \Rightarrow E_p \approx$ const.

$\rightarrow \phi(\vec{x})$ creates particle at $\vec{x}$

$(\langle 0|\phi(x)|\vec{p}\rangle = e^{i\vec{p}\vec{x}})$

Note 2.1:

- Projector on SP sector.

$$\mathbb{1}_1 = \int \frac{d^3p}{(2\pi)^3} |\vec{p}\rangle \frac{1}{2E_{\vec{p}}} \langle \vec{p}|$$

$\forall\ f(p)$ LI ($f(p) = f(p')$)

- $\int \boxed{\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}} f(p) \Rightarrow LI$

2.2. The Klein-Gordon Field in Space-Time

1. Heisenberg operators:

$$\phi_{\pi}(\vec{x},t) = \phi_{\pi}(x) = e^{iHt}\,\phi_{\pi}(\vec{x})\,e^{-iHt}$$

2. Heisenberg equation:

$$i\partial_t \underset{\phi,\pi}{O} = [O, H]$$

- $$i\partial_t\phi(x) = \Big[\phi(x), \int d^3y \Big\{\tfrac{1}{2}\overset{(\vec{y},t)}{\pi^2} + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2\Big\}\Big]$$

$$\overset{\circ}{=} i\pi(x) \qquad \boxed{\pi(x) = \partial_t\phi(x)} \quad (*)$$

- $$i\partial_t\pi(x) \overset{\circ}{=} -i(-\nabla^2+m^2)\phi(x) \qquad (**)$$

$$(*) + (**) \Rightarrow \boxed{(\partial_t^2 - \nabla^2 + m^2)\,\phi(x) = 0}$$ Klein Gordon equation

3. Time evolution of modes

$$e^{iHt}\,a_{\vec{p}}^{\dagger}\,e^{-iHt} = a_{\vec{p}}^{\dagger}\,e^{\pm iE_{p}t}$$

4. Field operators:

$$\boxed{\begin{aligned}\phi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}\,e^{-ipx} + a_{\vec{p}}^{\dagger}\,e^{ipx}\right)\Big|_{p^0=E_{\vec{p}}}\\ \pi(x) &= \partial_t\phi(x)\end{aligned}}$$

$$(x-y)^2 \gtrless 0$$

$$\langle 0|\,\phi(x)\,\phi(y)\,|0\rangle$$

Note 2.2

1. • $\phi(x,t) = e^{iHt}\,\phi(\vec{x},0)\,e^{-iHt}$
(Translation in time)

• $\phi(\vec{x}) = e^{-i\vec{P}\vec{x}}\,\phi_s(\overset{\vec{x}}{\vec{0}})\,e^{i\vec{P}\vec{x}}$

• $\phi(x) = e^{iPx}\,\phi(\overset{x}{0})\,e^{-iPx}$

$P^{\mu}x_{\mu}$

Note 2.3

• $e^{\mp ipx} \longleftrightarrow$ pos. (neg.) freq. solutions of KG equation

• $\longleftrightarrow$ annihilation op. $a_{\vec{p}}$ (creation $a_{\vec{p}}^{\dagger}$)

Causality

∡ Amplitude for particle to propagate from y to x.

$$\boxed{D(x-y) = \langle 0|\phi(y)\,\phi(x)|0\rangle \doteq \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\, e^{\underbrace{-ip(x-y)}}}$$

* $D(\Lambda(x-y)) = D(x-y)$

$\Lambda \in SO^+(1,3)$

1] ∡ Tim-like distance:

$x^0 - y^0 = t \qquad \vec{x} - \vec{y} = 0$

$$D(x-y) = \frac{4\pi}{(2\pi)^3}\int_0^\infty dp\, \frac{p^2}{2\sqrt{p^2+m^2}}\, e^{-i\overbrace{\sqrt{p^2+m^2}}^{E}t}$$

$$= \frac{1}{4\pi^2}\int_m^\infty dE\, \sqrt{E^2-m^2}\, e^{-iEt}$$

$\neq 0 \quad \Rightarrow$ Propagation possible

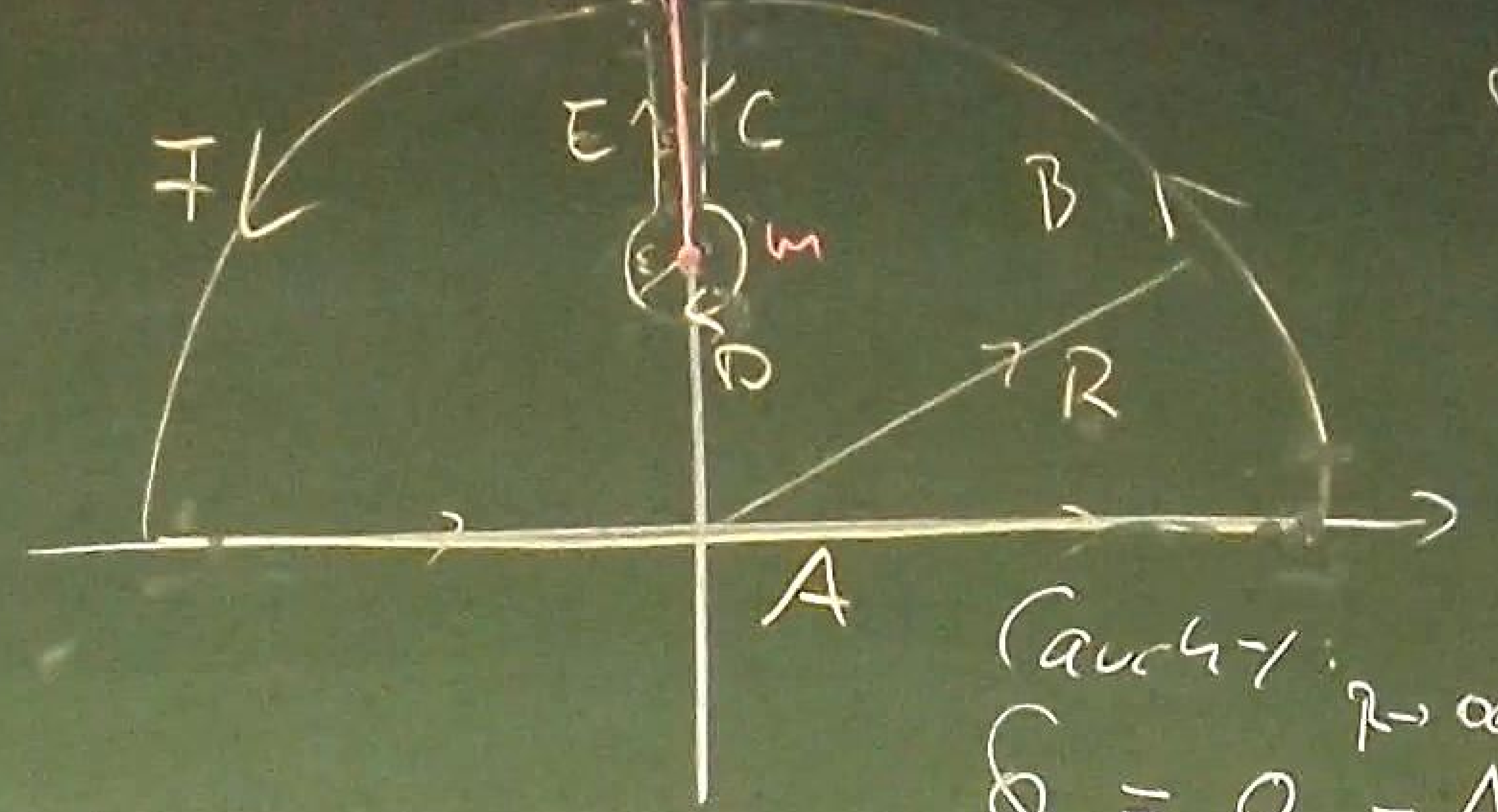
2] ∡ Space like distance:

$x^0 - y^0 = 0$ and $\vec{x} - \vec{y} = \vec{r}$

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\, e^{\underbrace{i\vec{p}\vec{r}}_{p\,r\cdot\overbrace{\cos\theta}^{u}}}$$

$$= \frac{2\pi}{(2\pi)^3}\int_0^\infty dp\, \frac{p^2}{2E_p}\, \frac{e^{ipr} - e^{-ipr}}{ip\cdot r}$$

$$= \frac{-i}{2(2\pi)^2 r}\int_{-\infty}^{\infty} dp\, \frac{p\, e^{ipr}}{\sqrt{p^2+m^2}}$$



$p \in \mathbb{C}$

Cauchy:

$$\oint \ldots \overset{R\to\infty}{=} 0 = A + C + \underset{\underset{C}{\parallel}}{E}$$

$$D(x-y) = -C - E$$

$$= -2C$$

$$= \frac{-i}{(2\pi)^2 r}\int_{im}^{i\infty} dp\, \frac{p\, e^{ipr}}{\sqrt{p^2+m^2}}$$

$$\overset{\rho = -ip}{=} \frac{1}{4\pi^2 r}\int_m^\infty d\rho\, \frac{\rho\, e^{-\rho r}}{\sqrt{\rho^2-m^2}}$$

$$\xrightarrow{r\to\infty} e^{-mr}$$

→ Vanishes exponentially in r

3.] & **Measurements:** A, B
can affect each other iff [A, B] ≠ 0

$$[\phi(x), \phi(y)] \quad \Rightarrow \begin{array}{l} A = \phi(x) \\ B = \phi(y) \end{array}$$

$$\overset{\circ}{=} \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left[e^{-ip(x-y)} - e^{ip(x-y)} \right]}_{\hat{=} \text{ number}} \cdot \mathbb{1}$$

$$= D(x-y) - D(y-x)$$

- Let $(x-y)^2 < 0$

$\rightarrow \exists\, \Lambda^* \in SO^+(1,3)$

$$\Lambda^*(x-y) = -(x-y)$$

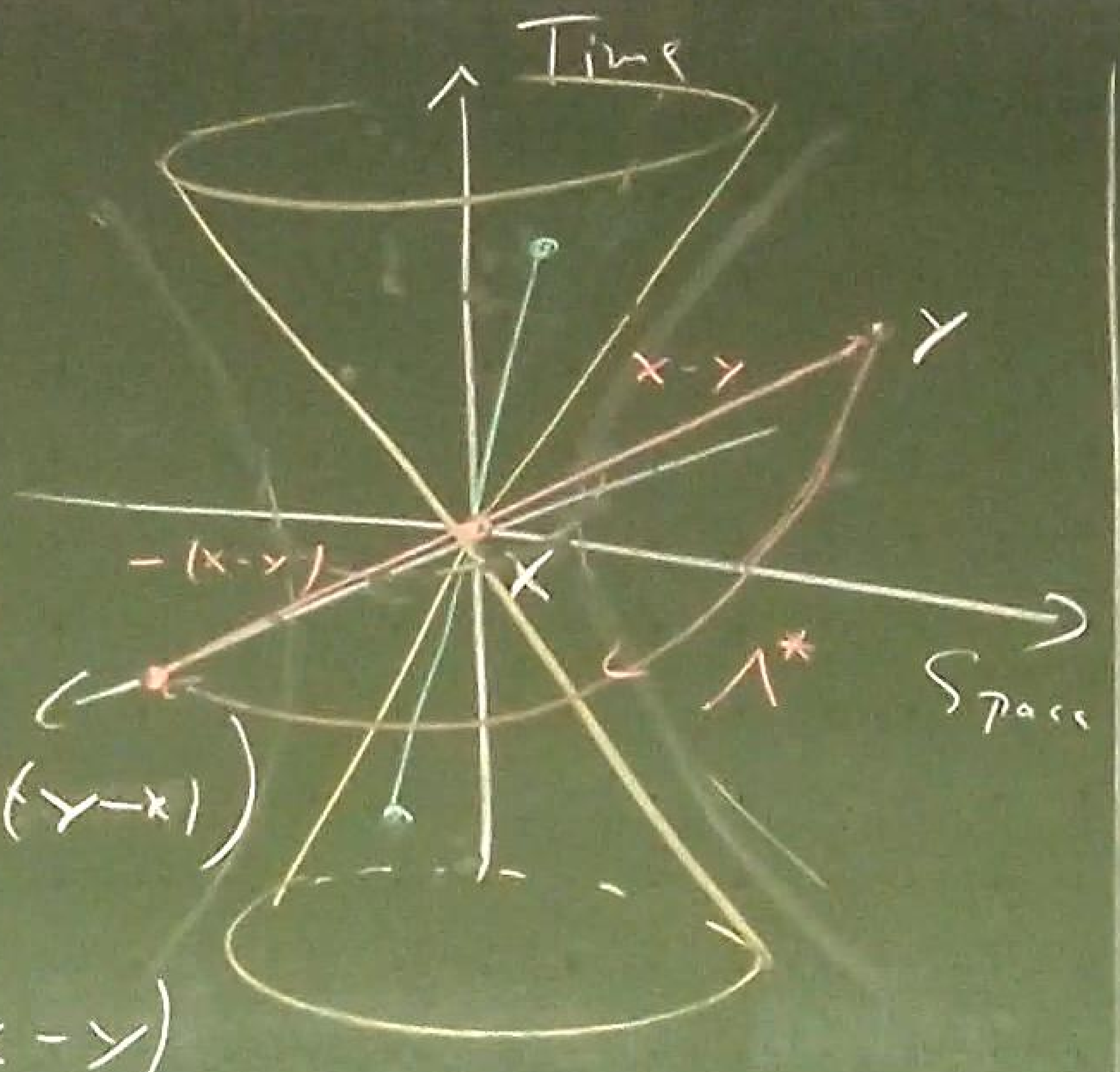
$\Rightarrow$

$$[\phi(x), \phi(y)]$$

$$= D(x-y) - D(\Lambda^*(y-x))$$

$$= D(x-y) - D(x-y)$$

$$= \underline{\underline{0}}$$



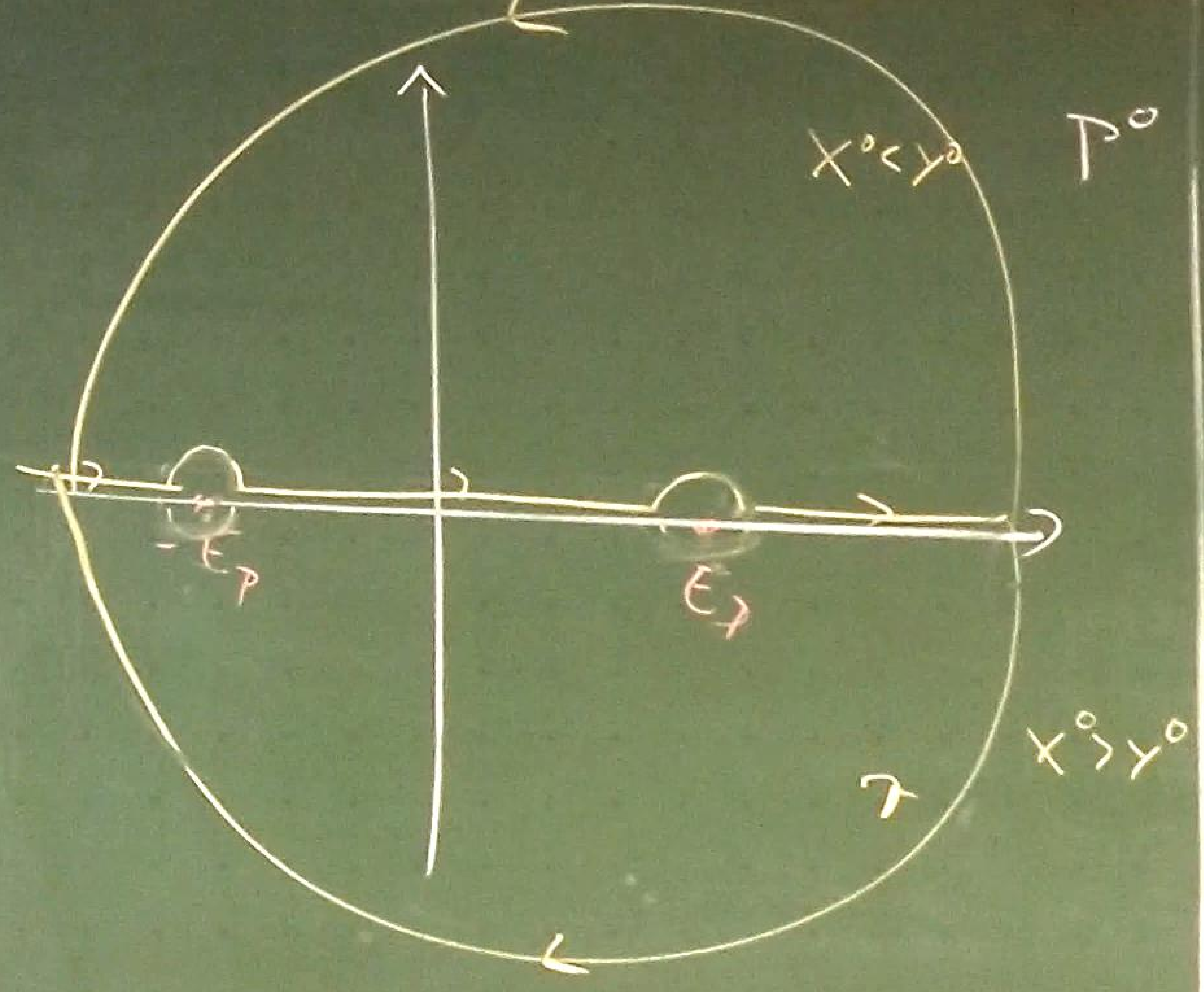
The Propagator

- $\langle 0 | [\phi(x), \phi(y)] | 0 \rangle$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left[e^{-ip(x-y)} - e^{ip(x-y)} \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \left[\frac{e^{-ip(x-y)}}{2E_p} \Big|_{p^0 = E_p} + \frac{e^{-ip(x-y)}}{-2E_p} \Big|_{p^0 = -E_p} \right]$$

$\vec{p} \to -\vec{p}$

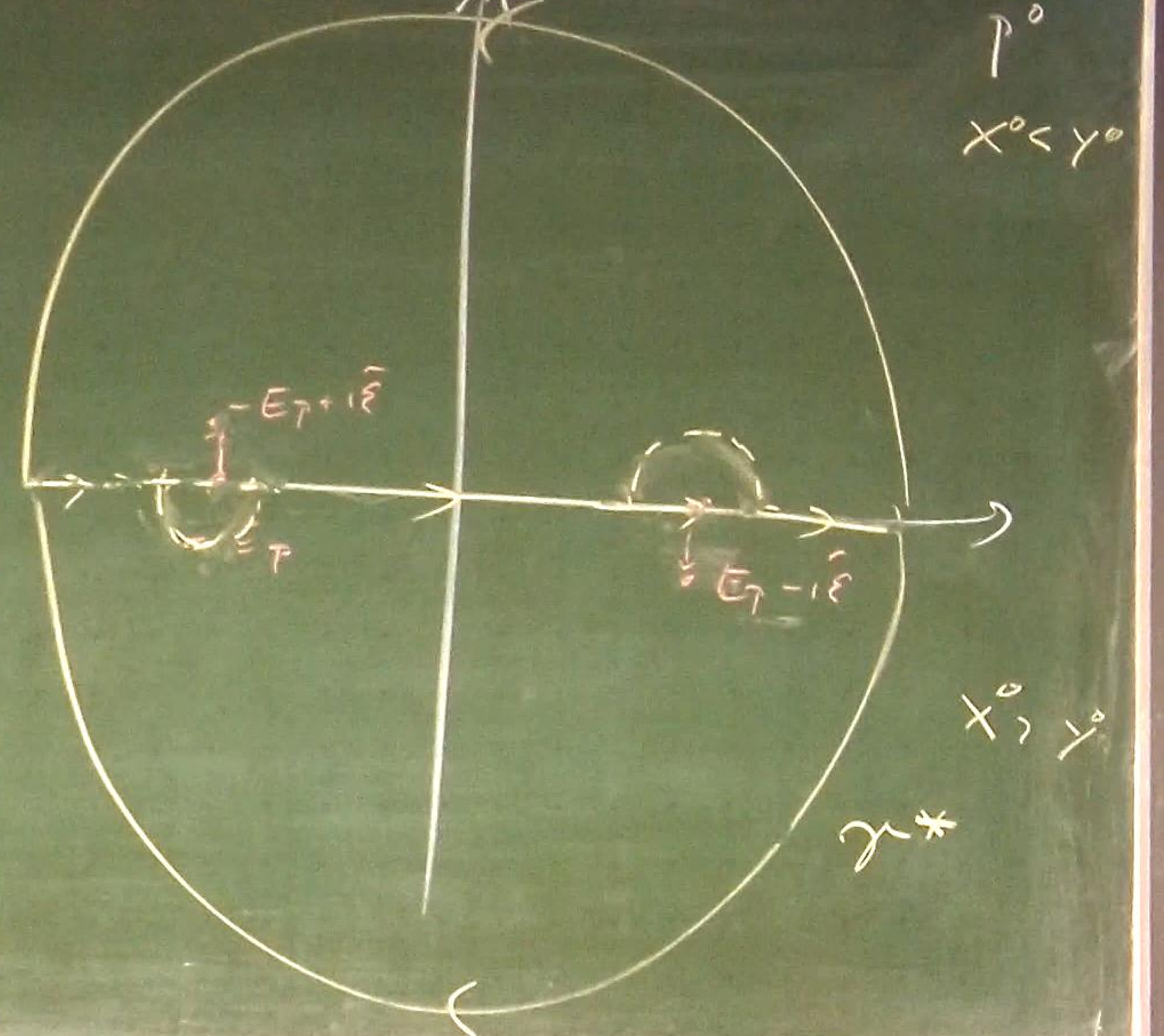


$$\boxed{\begin{aligned}(*) &=: D_R(x-y)\\ &= \theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle\end{aligned}}$$

2] Interpretation:

$$(\partial^2+m^2)D_R(x-y) \overset{\circ}{=} -i\delta^{(4)}(x-y)$$

$\Rightarrow$ Retarded Green's function of KG operator



$$D_F(x-y) \equiv (*)_{\gamma \to \gamma^*}$$

$$= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip(x-y)}$$

Feynman propagator
(of the KG field)

$A(\mathcal{H})$ $\mathcal{H}$

$\phi(x)$ $T(PQ)$

$$\Rightarrow D_F(x-y)$$

$$= \begin{cases} D(x-y) & x^0>y^0 \\ D(y-x) & x^0<y^0 \end{cases}$$

$$= \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle$$

$$\equiv \langle 0|T(\phi(x)\phi(y)|0\rangle$$

Time-ordering operator T

...

The Propagator

1). $\langle 0|[\phi(x),\phi(y)]|0\rangle = (D(x-y)-D(y-x))$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left[e^{-ip(x-y)}-e^{ip(x-y)}\right]$$

$$\overset{\vec p \to -\vec p}{=} \int \frac{d^3p}{(2\pi)^3}\left[\frac{e^{-ip(x-y)}}{2E_p}\Big|_{p^0=E_p} + \frac{e^{-ip(x-y)}}{-2E_p}\Big|_{p^0=-E_p}\right]$$

Residue Theorem $\uparrow$

$$\overset{x^0>y^0}{=} \int \frac{d^3p}{(2\pi)^3}\int_\gamma \frac{dp^0}{(2\pi i)}\frac{-1}{\underbrace{p^2-m^2}_{(p^0-E_{\vec p})(p^0+E_{\vec p})}}e^{-ip(x-y)}$$

$$E_{\vec p} = \sqrt{\vec p^2+m^2}$$

$$= \int_\gamma \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-ip(x-y)} \quad (*)$$

3. The Dirac Field

3.1. The Dirac Equation

1] Observation I:

i) $\measuredangle$ $x' = \Lambda x$, $\phi'(x') = \phi(x)$

ii) $\measuredangle$ $(\partial^2 + m^2)\phi(x) = 0$

iii) $\phi'(x) = \phi(\Lambda^{-1}x)$ is a new solution

Proof: $(\partial^2 + m^2)\phi'(x) = \dots \overset{!}{=} 0$

2] Observation II:

$\measuredangle$ Vector fields. $\vec{\phi}'(x) = R\vec{\phi}(R^{-1}x)$

$$\phi'_a(x) = M_{ab}(\Lambda)\,\phi_b(\Lambda^{-1}x) \qquad a,b = 1,\dots,n$$

where $\underbrace{M(\Lambda')M(\Lambda)}\,\phi(\Lambda^{-1}\Lambda'^{-1}x) \overset{!}{=} \underbrace{M(\Lambda'\Lambda)}\,\phi((\Lambda'\Lambda)^{-1}x)$

$\longrightarrow$ M n-dimensional representation of the Lorentz group

($M: SO^+(1,3) \longrightarrow \mathrm{End}(V)$)

3] Goal first-order relativistic field equation

$$(i\square^\mu\partial_\mu + \text{const})\,\phi = 0 \qquad \vec{\gamma} = \begin{pmatrix}\gamma^0\\ \vdots\end{pmatrix}$$

$$\boxed{M^{-1}(\Lambda)\,\gamma^\mu M(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu} \quad (*) \Leftarrow$$

4] i) $\measuredangle$ $x' = \Lambda x$, $\phi'(x') = M(\Lambda)\phi(x)$

ii) $\measuredangle$ ϕ: $(i\square^\mu\partial_\mu + \text{const})\,\phi(x) = 0 \quad \forall x$

iii) $\phi'(x) = M(\Lambda)\phi(\Lambda^{-1}x)$ new solution ?

$$(i\square^\mu\partial_\mu + \text{const})\,\phi'(x)$$

$$= \left(i\square^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu + \text{const}\right) M(\Lambda)\,\phi(x)\Big|_{x=\Lambda^{-1}x} \overset{!}{=} 0$$

$$\overset{M^{-1}(\Lambda)}{\Rightarrow} \Big[i\,\underbrace{M^{-1}(\Lambda)\,\square^\mu M(\Lambda)(\Lambda^{-1})^\nu{}_\mu}_{\overset{!}{=}\,\square^\nu}\partial_\nu + \text{const}\Big]\phi(x)\Big|_{x=\Lambda^{-1}x} \overset{!}{=} 0$$

$\rightarrow$ $\square^\mu \equiv \gamma^\mu$ must be n×n matrices

5] $SO^+(1,3)$ is a Lie group.

- $\Lambda = \exp\left[-\frac{i}{2}\boxed{\omega_{\alpha\beta}}\mathcal{J}^{\alpha\beta}\right]$

  $\Lambda \in SO^+(1,3)$

  $\approx \mathbb{1} - \frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}$

- $M(\Lambda) = \exp\left[-\frac{i}{2}\omega_{\alpha\beta} S^{\alpha\beta}\right]$

  $\approx \mathbb{1} - \frac{i}{2}\omega_{\alpha\beta} S^{\alpha\beta}$

$$(\mathcal{J}^{\alpha\beta})_{\mu\nu} = i(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu)$$

- Infinitesimal form of (*):

$$\boxed{\begin{aligned}[\gamma^\mu, S^{\alpha\beta}] &\overset{\circ}{=} (\mathcal{J}^{\alpha\beta})^\mu{}_\nu \gamma^\nu \\ &= i(g^{\alpha\mu}\gamma^\beta - g^{\beta\mu}\gamma^\alpha)\end{aligned}} \quad (1)$$

- $\mathcal{J}^{\alpha\beta} \rightarrow$ Lie algebra of Lorentz group:

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] \overset{\circ}{=} i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho})} \quad (2)$$

$J \in \{\mathcal{J}, S\}$

- Solution: Dirac & γ^μ

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_{n\times n}}$$

Dirac algebra

- If: $\boxed{S^{\mu\nu} := \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$

  $\overset{\circ}{\Rightarrow}$ (1) ✓ (2) ✓

7] Representations

- At least 4-dimensional
- All 4-D reps are unitarily equivalent
- Weyl representation:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

- Henceforth. $\Lambda_{\frac{1}{2}} = M(\Lambda)$ $(n=4)$

8|
$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi_{(x)} = 0}$$

Dirac equation

ψ: Bispinor field

$\psi(x) \in \mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$

9|
$$\underbrace{(-i\gamma^\mu\partial_\mu - m)(i\gamma^0\partial_0 - m)}_{\overset{\circ}{=}(\partial^2 + m^2)}\psi = 0$$

10| Dirac adjoint:

i) First try:
$$\psi'^\dagger\psi' = \psi^\dagger \underbrace{\Lambda_{\frac{1}{2}}^\dagger \Lambda_{\frac{1}{2}}}_{\neq \mathbb{1}} \psi \neq \psi^\dagger\psi$$

ii)
$$\boxed{\bar{\psi} = \psi^\dagger\gamma^0}$$

Dirac adjoint

$$\overset{\circ}{\longrightarrow} \quad \bar{\psi}'\psi' = \bar{\psi}\underbrace{\Lambda_{\frac{1}{2}}^{-1}\Lambda_{\frac{1}{2}}}_{\mathbb{1}}\psi = \bar{\psi}\psi$$

11| Lagrangian:

$$\boxed{\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi}$$

Note 3.1

- $\sigma^\mu = (1, \vec{\sigma})^T$

  $\bar{\sigma}^\mu = (1, -\vec{\sigma})^T$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} -m & i\sigma\partial \\ i\bar{\sigma}\partial & -m \end{pmatrix} \begin{pmatrix} \overset{\mathbb{C}^2}{\psi_L} \\ \underset{\mathbb{C}^2}{\psi_R} \end{pmatrix} = 0$$

ψ_L, ψ_R left / right handed

Weyl spinors

$m=0$ $\quad \left.\begin{aligned} i\bar{\sigma}\partial\psi_L &= 0 \\ i\sigma\partial\psi_R &= 0 \end{aligned}\right\}$ Weyl equations

---

3.7 Free Particle Solutions of the Dirac Equation

1.) $(\partial^2+m^2)\psi=0$

$p^2 \overset{!}{=} m^2 \, , \quad p^0>0$

$E_{\vec p} = p^0 = \sqrt{\vec{p}^2+m^2}$

$$\Psi^\pm(x) = \underset{\in \mathbb{C}^4}{\psi^\pm(p)}\, e^{\mp ipx}$$

2.) $(\pm\gamma^\mu p_\mu - m)\psi^\pm(p)$

$$= \begin{pmatrix} -m & \pm p\sigma \\ \pm p\bar{\sigma} & -m \end{pmatrix}\begin{pmatrix} \psi_L^\pm \\ \psi_R^\pm \end{pmatrix} \overset{!}{=} 0$$

$$\boxed{\begin{aligned} &s=1,2 \\ &\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}}$$

3.) **Note**

- $(p\sigma)(p\bar{\sigma}) = p^2 = m^2$
- Eigenvalues $\sqrt{\vec{p}^2+m^2}$ $p^0 \pm |\vec{p}| \Rightarrow$ $p^0>0$, $m>0$

4.) ∡ $\psi_L^\pm = \sqrt{p\sigma}\,\underset{\in\mathbb{C}^2}{\xi^\pm}$ $\quad \|\xi^\pm\|=1$

$-m\sqrt{p\sigma}\,\xi^\pm \pm p\sigma\,\psi_R^\pm = 0$

$\sqrt{p\sigma}\sqrt{p\bar{\sigma}}=m$

$\Leftrightarrow \quad \psi_R^\pm = \pm\dfrac{m}{\sqrt{p\sigma}}\,\xi^\pm = \pm\sqrt{p\bar{\sigma}}\,\xi^\pm$

5.) **Solutions:**

$$\Psi^+(x) = \underbrace{\begin{pmatrix} \sqrt{p\sigma}\,\xi^s \\ \sqrt{p\bar{\sigma}}\,\xi^s \end{pmatrix}}_{u_s(p)} e^{-ipx}$$

pos. freq. sol

$$\Psi^-(x) = \underbrace{\begin{pmatrix} \sqrt{p\sigma}\,\eta^s \\ -\sqrt{p\bar{\sigma}}\,\eta^s \end{pmatrix}}_{v_s(p)} e^{+ipx}$$

neg. freq. sol

Recap: $\psi^\dagger \gamma^0$ Dirac adjoint

$$\mathcal{L}_{\text{Dirac}} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi$$

$$\Rightarrow \quad (i\gamma^\mu \partial_\mu - m)\Psi = 0$$

$$(i\not{\partial} - m)\psi \overset{!}{=} 0 \in \mathbb{C}^2 \oplus \mathbb{C}^2 \cong \mathbb{C}^4$$

Solutions:

$$\psi^+_{\vec{p}s} = \underbrace{\begin{pmatrix} \sqrt{p\sigma}\,\xi^s \\ \sqrt{p\bar{\sigma}}\,\xi^s \end{pmatrix}}_{u^s(p)} e^{-ipx}$$

$$\boxed{\begin{aligned} \xi^1, \eta^1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \xi^2, \eta^2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}}$$

$$\psi^-_{\vec{p}s} = \underbrace{\begin{pmatrix} \sqrt{p\sigma}\,\eta^s \\ -\sqrt{p\bar{\sigma}}\,\eta^s \end{pmatrix}}_{v^s(p)} e^{+ipx}$$

Relations:

- $\underset{v}{\bar{u}}{}^s = \underset{v}{u}{}^{s\dagger}\gamma^0$
- $\underset{v}{\bar{u}}{}^r \underset{v}{u}{}^s = -2m\delta^{rs}$, $\boxed{(\underset{v}{u}{}^{r\dagger})\,\underset{v}{u}{}^s = 2E_{\vec{p}}\delta^{rs}}$
- $\bar{u}^s v^r = 0$
- $\boxed{u^{r\dagger}(\vec{p})\, v^s(-\vec{p}) = 0}$
- $\not{p} := \gamma^\mu \cdot p_\mu$ Feynman slash notation
- $\sum_s u^s(p)\cdot\bar{u}(p) = \not{p} + m\mathbb{1}$
- $\sum_s v^s(p)\,\bar{v}^s(p) = \not{p} - m\mathbb{1}$

3.3. Dirac Field Bilinears

$\underline{\mathbb{1}}$

- $\boxed{\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \overset{\text{Weyl basis}}{=} \begin{pmatrix} -\mathbb{1} & \\ & \mathbb{1} \end{pmatrix}}$

$\gamma^{5\dagger} = \gamma^5$, $(\gamma^5)^2 = \mathbb{1}$,

$\{\gamma^5, \gamma^\mu\} = 0$ $\quad \mu = 0, 1, 2, 3$

$$S^{\mu\nu} = [\gamma^\mu, \gamma^\nu]$$

$$S = \left(\begin{array}{c|c} \Xi & 0 \\ \hline 0 & \Xi \end{array}\right)$$

2| Bilinears : $\bar{\psi}\Gamma\psi$

$$\boxed{\begin{array}{lll} \Gamma = & \mathbb{1} & \cdot\ \text{Scalar} \\ & \gamma^\mu & \cdot\ \text{vector} \\ & \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] & \cdot\ \text{tensor} \\ & \gamma^\mu\gamma^5 & \cdot\ \text{pseudo vector} \\ & \gamma^5 & \cdot\ \text{pseudo Scalar} \end{array}}$$

Example:

$$(j^\mu)' = \bar{\psi}'\gamma^\mu\psi' = \bar{\psi}\underbrace{\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}}_{\Lambda^\mu{}_\nu\gamma^\nu}\psi$$

$$= \Lambda^\mu{}_\nu \underbrace{\bar{\psi}\gamma^\nu\psi}_{j^\mu} = \Lambda^\mu{}_\nu j^\mu$$

3.4 Quantization of the Dirac Field

1| $\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$

2| Can momentum: $\Pi_a = i\psi_a^*$

3| Hamiltonian:

$$H = \int d\vec{x}^3\, \psi^\dagger \underbrace{[-i\vec{\alpha}\nabla + m\beta]}_{\hookrightarrow\ H_D}\psi$$

$$\vec{\alpha}^i = \gamma^0\gamma^i$$

$$\beta = \gamma^0$$

$$\Big[\underset{\pm E}{i\gamma^0\partial_0} + \underbrace{i\vec{\gamma}\nabla - m}_{\gamma^0 H_D}\Big]\psi = 0 \qquad \psi \sim e^{\pm \underbrace{ipx}_{E_p t}}$$

$$\Rightarrow H_D u^s(p) e^{i\vec{p}\vec{x}} = E_p \text{ —"— }$$

$$H_D v^s(p) e^{-i\vec{p}\vec{x}} = - E_p \text{ —"— }$$

5] Mode expansion:

$$\psi(\vec{x}) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \Big[a^s_{\vec{p}}\, u^s(\vec{p})\, e^{i\vec{p}\vec{x}} + b^s_{\vec{p}}\, v(\vec{p})\, e^{-i\vec{p}\vec{x}} \Big]$$

6]

$$H_D \psi(\vec{x}) = \sum_s \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{E_{\vec{p}}}{2}} \Big[a^s_{\vec{p}}\, u^s_{\vec{p}}\, e^{ipx} - b^s_{\vec{p}}\, v^s(\vec{p})\, e^{-i\vec{p}\vec{x}} \Big]$$

$$H = \int d^3x\, \psi^\dagger H_D \psi \overset{\circ}{=} \sum_s \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \Big[a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \Big]$$

First try: Commutator

7]

- $[\psi_a(\vec{x}), \Pi_b(\vec{y})] = i\delta_{ab}\,\delta^{(3)}(\vec{x}-\vec{y})$
- $[\psi_a(\vec{x}), \psi^\dagger_b(\vec{y})] = \delta_{ab}\,\delta^{(3)}(\vec{x}-\vec{y})$

8] Mode algebra $\xrightarrow{\circ}$

$$[\underset{b}{a^r_{\vec{p}}}, \underset{b}{a^{s\dagger}_{\vec{q}}}] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{q})$$

→ Irred. Representation:

Bosonic Fock space

9] Problem: $(b^{s+}_{\vec{p}})^N |0\rangle$

Energy: $-N E_{\vec{p}} \xrightarrow{N\to\infty} -\infty$

$(H = a^+a \boxed{-} b^+b)$

↳ No stable vacuum state

10] Fix (?): $b \longleftrightarrow b^+$

i) $\psi(\vec{x}) = [a^s + b^+ \quad]$

ii) $H = \overset{+}{a}a - b^+_{\vec{p}} b^s_{\vec{q}}$ + const

iii) $[b^r_{\vec{p}}, b^{s+}_{\vec{q}}] = \boxed{-}(2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{q})$

v) $[H, b^{s+}_{\vec{p}}] = \underbrace{E_{\vec{p}}}_{>0} b^+_{\vec{p}}$

b^+ creates a particle with positive energy

→ $H \geq 0$

vi) But:

$$|| b^{s+}_{\vec{p}} |0\rangle ||^2 = \langle 0| b b^+ - b^+ b |0\rangle$$

$$= \langle 0|[b, b^+]|0\rangle$$

$$= - \underbrace{\langle 0|0\rangle}_{\delta^{(3)}(0)} < 0$$

11 • instability of vacuum

• loss of unitarity

→ No consistent quantization!

Second try: Anticommutator

7] $$\boxed{\{\psi_a(\vec{x}), \psi^+_b(\vec{y})\} = \delta_{ab} \delta^{(3)}(\vec{x}-\vec{y})}$$

8] $\xrightarrow{\circ}$

$$\boxed{\{\underset{b}{a^s_{\vec{p}}}, \underset{b}{a^{r+}_{\vec{q}}}\} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{q})}$$

→ Fermionic Fock space

9) Problem: $b^{s\dagger}_{\vec p}|0\rangle$: Energy

$-E_{\vec p}$

→ Still no stable vacuum state

10) Fix: $b \leftrightarrow b^\dagger$

i) $H = \sum_s \int \frac{d^3p}{(2\pi)^3} E_p \left(a^{s\dagger}_{\vec p} a^s_{\vec p} - b^s_{\vec p} b^{s\dagger}_{\vec p}\right)$

$$= \sum_s \int \frac{d^3p}{(2\pi)^3} E_p \left[a^{s\dagger}_{\vec p} a^s_{\vec p} + b^{s\dagger}_{\vec p} b^s_{\vec p}\right]$$

ii) The mode algebra is invariant under $b \leftrightarrow b^\dagger$

→ Unitarity is preserved and Hamiltonian is lower-bounded

---

11) Heisenberg picture:

- $e^{iHt} \underset{b}{a^s_{\vec p}} e^{-iHt} \overset{\circ}{=} \underset{b}{a^s_{\vec p}} e^{-iE_{\vec p}t}$

- $\psi(x) = e^{iHt} \psi(\vec x) e^{-iHt}$

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a^{s\dagger}_{\vec p}\, \bar u^s(p)\, e^{\overset{+}{-}ipx} + b^{s(\dagger)}_{\vec p}\, \bar v^s(p)\, e^{\overset{-}{}ipx}\right]$$

$\bar\psi(x)$

$$\rightarrow Q = \int d^3x\, \psi^\dagger \psi \underset{\substack{\uparrow \\ \text{up to } \infty}}{\overset{\circ}{=}} \sum_s \int \frac{d^3p}{(2\pi)^3} \left(\underline{a^{s\dagger}_{\vec p} a^s_{\vec p}} - \underline{b^{s\dagger}_{\vec p} b^s_{\vec p}}\right)$$

Continuous Symmetries:

- Time translation ↪ Hamiltonian
- Spatial translations ↪ Momentum operator

$$\vec P \overset{\circ}{=} \int d^3x\, \psi^\dagger (-i\nabla) \psi \overset{\circ}{=} \sum_s \int \frac{d^3p}{(2\pi)^3}\, \vec p \left(a^{s\dagger}_{\vec p} a^s_{\vec p} + b^{s\dagger}_{\vec p} b^s_{\vec p}\right)$$

- Rotations ↪ Angular momentum operator
- Global phase rotation $e^{i\alpha}\psi$

↪ Conserved current $j^\mu = \bar\psi \gamma^\mu \psi$

↪ Conserved charge: →

Excitations = Particles

$a^{s\dagger}_{\vec p}|0\rangle$: **Fermion**
- energy $E_{\vec p}$
- momentum $\vec p$
- spin $J=\frac{1}{2}$ (polarizations)
- Charge $Q=+1$

$b^{s\dagger}_{\vec p}|0\rangle$: **Antifermion**
- energy $E_{\vec p}$
- momentum $\vec p$
- spin $J=\frac{1}{2}$ (polarization ... s)
- Charge $\underline{Q=-1}$

Lorentz transformations

1| $\Lambda \in SO^+(1,3)$

$|\vec p, s\rangle := \sqrt{2E_{\vec p}}\, a^{s\dagger}_{\vec p}|0\rangle$

$|\vec p, s\rangle \longmapsto U(\Lambda)|\vec p, s\rangle$

representation of $SO^+(1,3)$ on Fock space

$$\boxed{U(\Lambda)\Psi(x)U^{-1}(\Lambda) = \Lambda^{-1}_{\frac{1}{2}}\Psi(\Lambda x)}$$

2| $\measuredangle$ Special rel.

→ Spin pol do not mix

$$\boxed{U(\Lambda)\, a^s_{\vec p}\, U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\vec p}}{E_{\vec p}}}\, a^s_{\Lambda\vec p}}$$

3|

$$\underline{\langle \vec p, s, \vec q, r\rangle} = \underbrace{2E_{\vec p}(2\pi)^3\delta^{(3)}(\vec p\,'-\vec q\,')\delta^{rs}}$$

$$= \langle \vec p\,', s|\vec q\,', r\rangle = \langle \vec p, s|\overbrace{\underline{U^\dagger(\Lambda)}\,\underline{U(\Lambda)}}^{\underline{1}}|\vec q, r\rangle$$

→ $U(\Lambda)$ are $\underline{\text{unitary}}$

$SO^+(1,3) \longrightarrow \mathcal{A}(\mathcal{H})$

Fock space

4|
- $\boxed{\Lambda}$ acts on 4-vectors in $\mathbb{R}^{1,3}$

  $D=4 \longrightarrow$ $\underline{\text{not unitary}}$
- $\boxed{\Lambda_{\frac{1}{2}}}$ acts on bispinors $\mathbb{C}^2\oplus\mathbb{C}^2$

  $D=4 \longrightarrow$ $\underline{\text{not unitary}}$
- $\boxed{U(\Lambda)}$ acts on states in fermionic Fock space

  $D=\infty \longrightarrow$ $\underline{\text{unitary}}$

Spin-Statistics Theorem

- Observation:

- KG field ϕ: Spin 0 → commutator
  (bosonic particles)
- Dirac field ψ: Spin $\frac{1}{2}$ → anticommutators
  (fermionic particles)

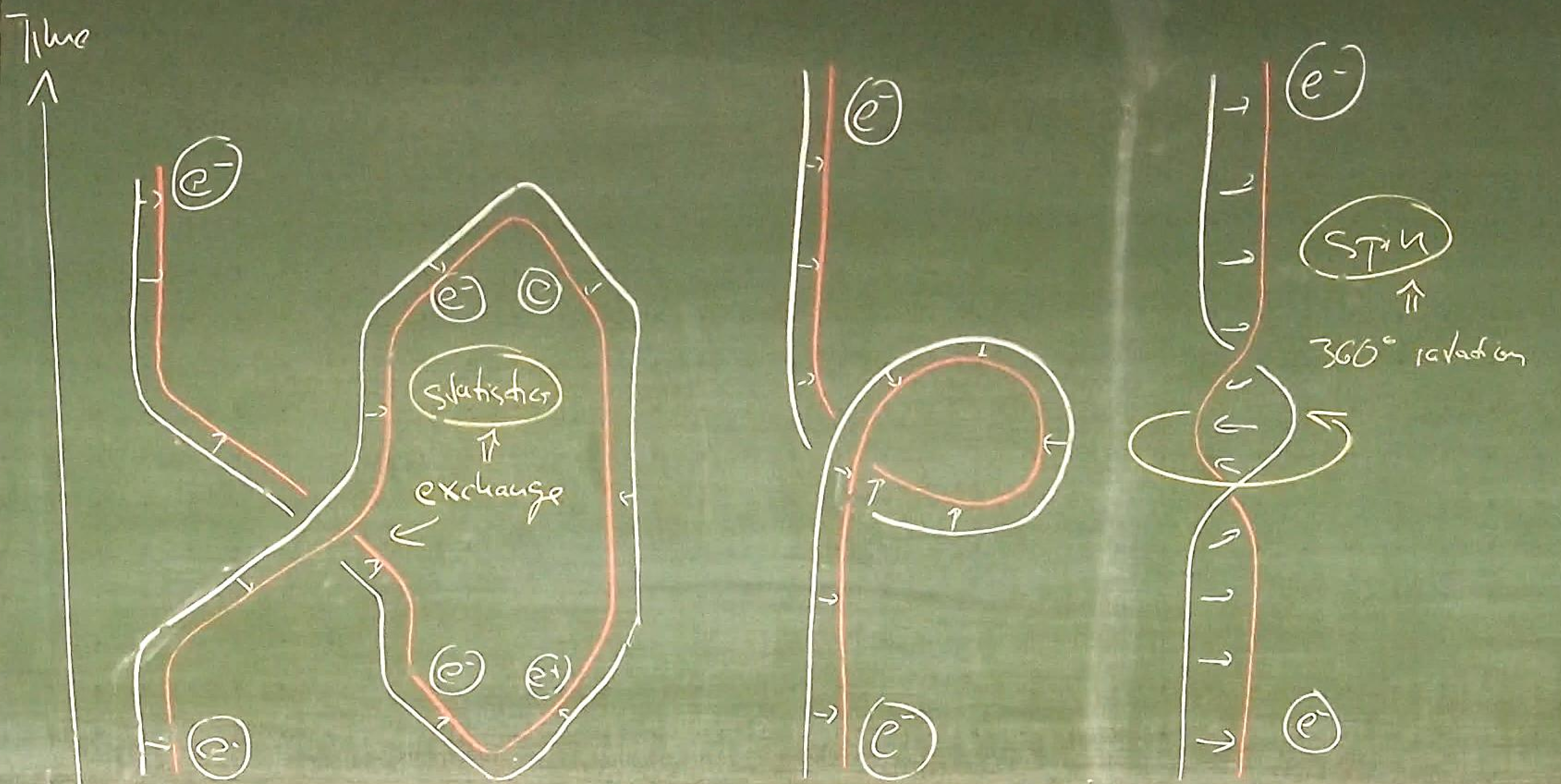
Spin statistics theorem:

- Lorentz invariance
- Causality
- Positive energies
- Positive norms

⇒ Integer Spin ⟷ Bosons

Half-Integer Spin ⟷ Fermions

"Proof by picture"

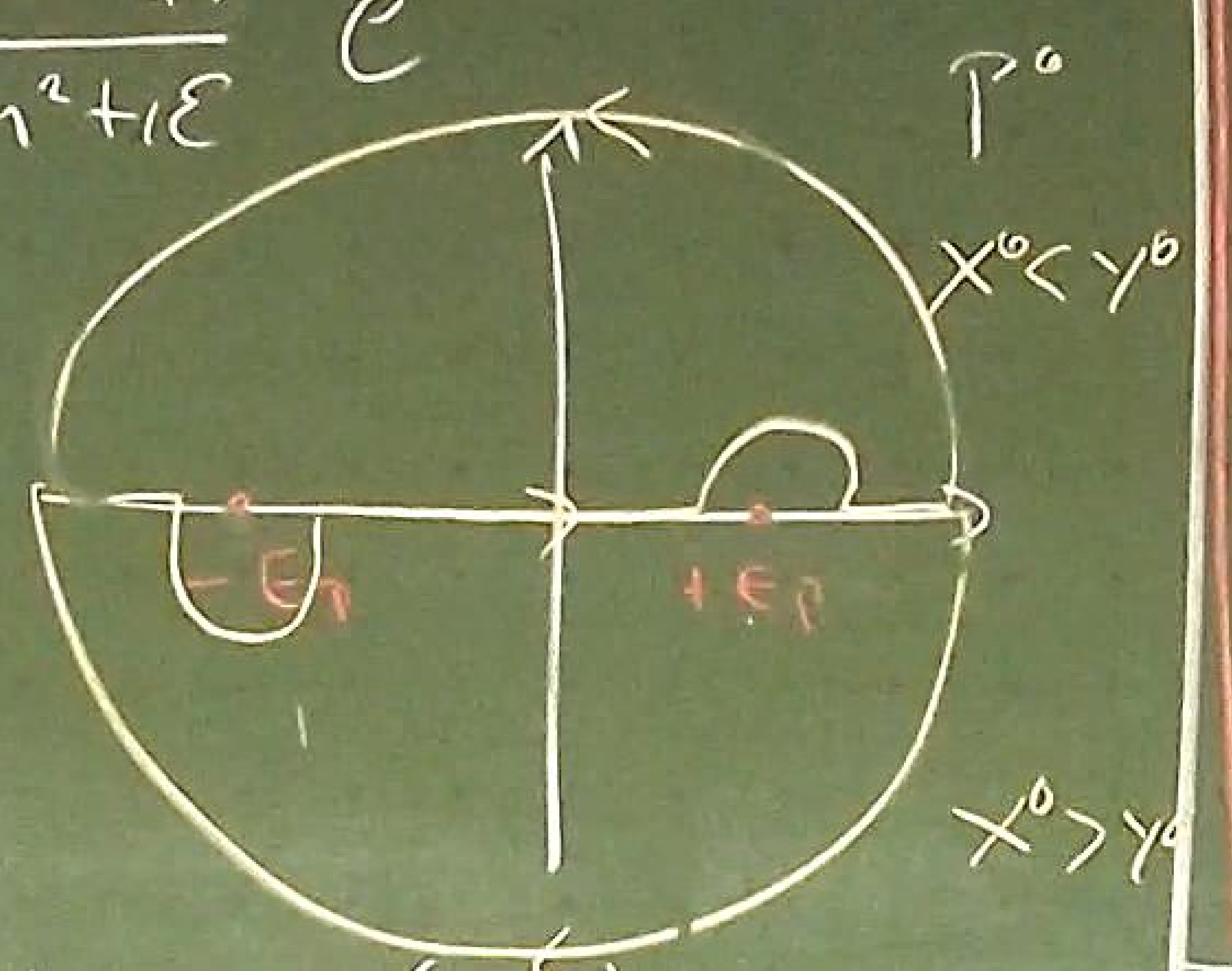


Dirac Propagator

1)

$$\cdot \langle 0|\psi_a(x)\bar\psi_b(y)|0\rangle = \underbrace{\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}}_{D(x-y)}\underbrace{\sum_s u_a^s(p)\bar u_b^s(p)}_{(\not p+m)_{ab}} \overset{x^0>y^0}{=} \int\frac{d^4p}{(2\pi)^3}\frac{1}{2E_p}\frac{i(\not p+m)_{ab}}{p^2-m^2+i\epsilon}e^{-ip(x-y)}$$

$$= (i\not\partial_x+m)_{ab}D(x-y) \qquad (\not p+m)_{ab} =$$



$$\cdot \langle 0|\bar\psi_b(y)\psi_a(x)|0\rangle = \int\text{''}\dots e^{-ip(y-x)}\underbrace{\sum_s v_a^s\bar v_b^s}_{(\not p-m)_{ab}} \overset{x^0<y^0}{=} \boxed{-}\int\frac{d^4p}{(2\pi)^3}\frac{i(\not p+m)_{ab}}{p^2-m^2+i\epsilon}e^{-ip(x-y)}$$

$$= -(i\not\partial_x+m)D(y-x)$$

2) Feynman propagator:

$$\boxed{\begin{aligned} S_F^{ab}(x-y) &= \int\frac{d^4p}{(2\pi)^4}\frac{i(\not p+m)_{ab}}{p^2-m^2+i\epsilon}e^{-ip(x-y)} \\ &= \begin{cases} \langle 0|\psi_a(x)\bar\psi_b(y)|0\rangle & x^0>y^0 \\ \boxed{-}\langle 0|\bar\psi_b(y)\psi_a(x)|0\rangle & x^0<y^0 \end{cases} \\ &\equiv \langle 0|T\psi_a(x)\bar\psi_b(y)|0\rangle \end{aligned}}$$

Note. $T\,\psi(t_2)\psi(t_1) \overset{t_1>t_2}{=} -\psi(t_1)\psi(t_2)$

Causality

1) Measurable Operators:

$$\hat{O}(x) = \sum \prod_{\text{even}} \{\psi_i^{(+)}, \partial\psi_i^{(+)} \dots\}$$

Example: * $j^\mu = \overline{\psi}\gamma^\mu\psi$ ✓

* $\psi_a + \psi_a^\dagger$ ✗

2) Causality $\Leftrightarrow \{\psi_a(x), \overline{\psi}_b(y)\} = 0$ for $(x-y)^2 < 0$

(Superselection)

We find:

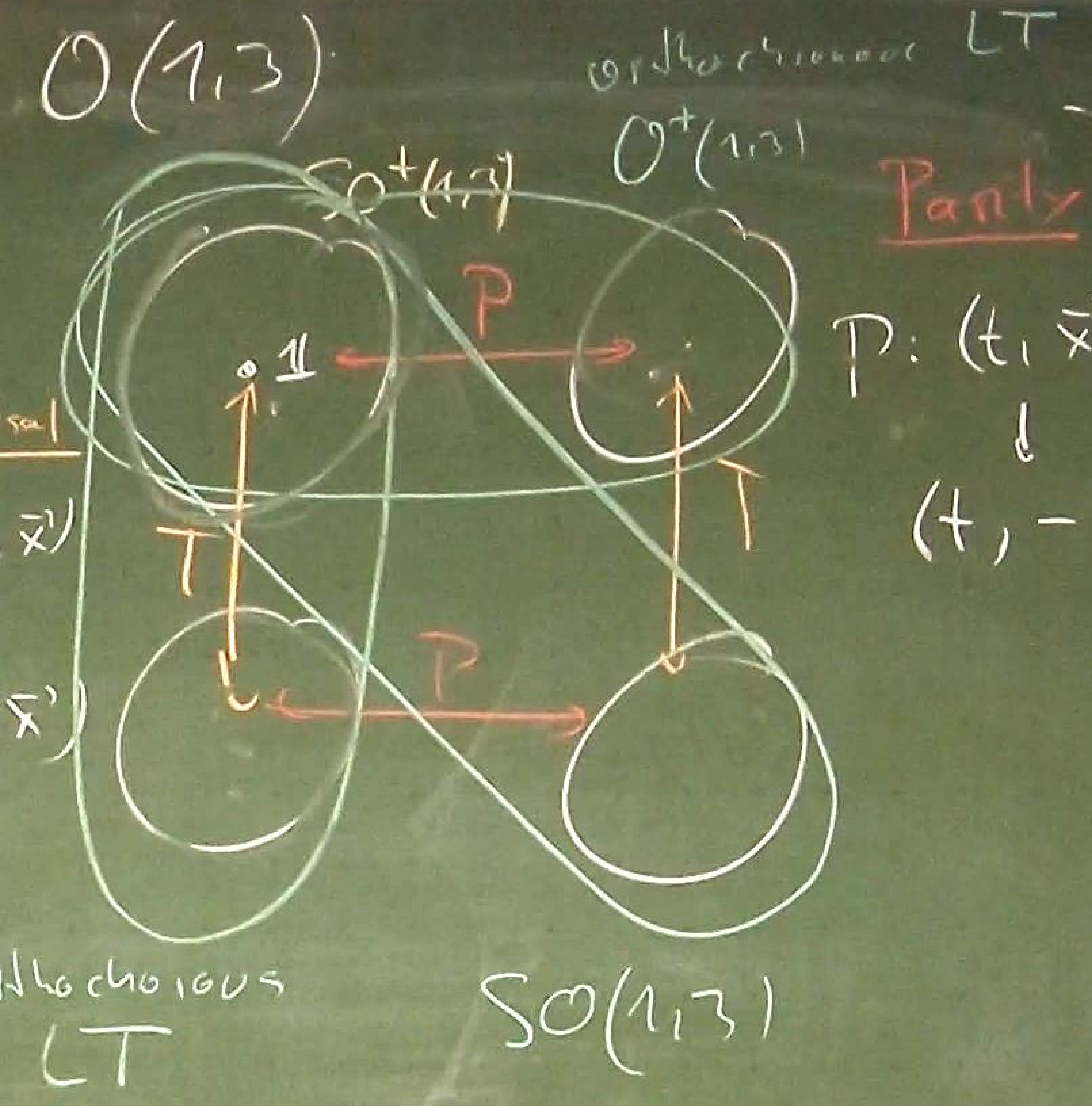
$$\{\psi_a(x), \overline{\psi}_b(y)\} \overset{\circ}{=} (i\not{\partial}_x + m)_{ab}\left[D(x-y) - \underset{\downarrow}{D(y-x)}\right]$$

$$\overset{(x-y)^2<0}{=} (i\partial_x + m)_{ab}\left[\underbrace{D(x-y) - \underline{D(x-y)}}_{=0}\right]$$

$$= 0$$

3.5. Discrete Symmetries of the Dirac Theory

Review of the Lorentz group



Time reversal

$$T: (t, \vec{x}) \to (-t, \vec{x})$$

Parity

$$P: (t, \vec{x}) \to (t, -\vec{x})$$

Parity

1| Unitary rep. on Fock space:

$$U(P)\, a^s_{\vec p}\, U^{-1}(P) = \overset{+1}{\eta_a}\, a^s_{-\vec p}$$

$$U(P)\, b^s_{\vec p}\, U^{-1}(P) = \underset{-1}{\eta_b}\, b^s_{-\vec p}$$

2|

$$\boxed{U(P)\,\psi(\overset{x}{\underset{\sim}{t,\vec x}})\,U^{-1}(P) = \underbrace{\gamma^0}_{P_{\frac{1}{2}}}\psi(t,-\vec x)}$$

3| Examples:

$$U(P)\,\bar\psi\psi\, U^{-1}(P) \overset{\varphi}{=} +\bar\psi\psi(t,-\vec x)$$

$$U(P)\,\bar\psi\gamma^5\psi\, U^{-1}(P) \overset{0}{=} -\bar\psi\psi(t,-\vec x)$$

Time Reversal

1|
- $U(T)\,\psi(t,\vec x)\,U^{-1}(T) = T_{\frac{1}{2}}\psi(-t,\vec x)$
- $U(T)\,a^s_{\vec p}\,U^{-1}(T) = a^{\bar s}_{-\vec p}$ ← flip spins
- $[U(T), H] = 0$
- $U^{-1}(T) = U^{+}(T)$

(Wigner's theorem)

2| Problem

$$\psi(t,\vec x) = e^{iHt}\psi(\vec x)\,e^{-iHt}$$

$$\overset{U\cdot U^{-1}}{\Rightarrow} \underbrace{U\psi(t,\vec x)U^{-1}} = e^{iHt}\underbrace{U\psi(\vec x)U^{-1}}e^{-iHt}$$

$$\overset{|0\rangle}{\Rightarrow} T_{\frac{1}{2}}\underbrace{\psi(-t,\vec x)}_{\downarrow}|0\rangle = e^{iHt}T_{\frac{1}{2}}\psi(\vec x)|0\rangle$$

$$\Rightarrow T_{\frac{1}{2}}e^{-iHt}\psi(\vec x)|0\rangle = \text{---}\,''\,\text{---}$$

$$\rightarrow \underbrace{e^{-2iHt}}_{t\,\underset{0}{\vee}} T_{\frac{1}{2}}\psi(\vec x)|0\rangle = \underbrace{T_{\frac{1}{2}}\psi(\vec x)|0\rangle}_{\text{no } t}$$

3| Solution:

$U(T)$ must be antiunitary / antilinear.

$$\boxed{U(T)\,\underset{\overset{\uparrow}{\in \mathbb{C}}}{c} = c^* \, U(T)}$$

A	U
A∘K	U∘K

4| Transformation of Spin

Spin flipped under TR

$$\overline{a^1_{\vec p}} \to a^2_{\vec p}$$

$$\overline{a^2_{\vec p}} \to - a^1_{\vec p}$$

5| Definition

- $U(T)$: antiunitary
- $U(T)\, a^s_{\vec p}\, U^{-1}(T) = \overline{a^s_{-\vec p}}$

$$b \qquad\qquad b$$

$$\Rightarrow U(T)\, \psi(t,x)\, U^{-1}(T) \doteq \underbrace{\overbrace{(\gamma^1\gamma^3)}^{\begin{pmatrix} i\sigma^y & \\ & i\sigma^y \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}}}_{T_{1/2}} \psi(-t,\vec x)$$

2| Problem

$$\psi(t,\vec x) = e^{iHt}\, \psi(\vec x)\, e^{-iHt}$$

$$\overset{U\cdot,\ \cdot U^{-1}}{\Rightarrow}\ \underline{U\,\psi(t,\vec x)\,U^{-1}} = e^{-iHt}\, \underbrace{U\,\psi(\vec x)\,U^{-1}}\, e^{+iHt}$$

$$\overset{|0\rangle}{\Rightarrow}\ T_{\frac12}\, \underbrace{\psi(-t,\vec x)}\, |0\rangle = e^{-iHt}\, T_{\frac12}\, \psi(\vec x)\, |0\rangle$$

$$\Rightarrow T_{\frac12}\, e^{-iHt}\, \psi(\vec x)\, |0\rangle = \text{— " —}$$

$$\Rightarrow \underbrace{e^{-2iHt}}_{t\ \underset{0}{V}}\, T_{\frac12}\, \psi(\vec x)\, |0\rangle = \underbrace{T_{\frac12}\, \psi(\vec x)\, |0\rangle}_{\text{no } t}$$

$0Ht$

no t

6] Examples:

$$U(T)\underset{\bar\Psi\gamma^\mu\psi}{j^\mu}(t,x)\,U^{-1}(T) = \begin{cases} +j^\mu(-t,\vec{x}) & \mu=0 \\ -j^\mu(-t,\vec{x}) & \mu=1,2,3 \end{cases}$$

Charge Conjugation

1] Discrete, non-spacetime symmetry.

$$\boxed{\begin{array}{c} U(C)\,a^s_{\vec{p}}\,U^{-1}(C) = b^s_{\vec{p}} \\ b \qquad\qquad a \end{array}}$$

2] —

3] $U(C)\,\psi\,U^{-1}(C)$

$$\overset{\circ}{=} -i\gamma^2\left(\psi^+\right)^T$$

$$= -i\left(\bar\Psi\gamma^0\gamma^2\right)^T$$

$$U(C)\,\bar\Psi\,U^{-1}(C) \overset{\circ}{=} -i\left(\gamma^0\gamma^2\psi\right)^T$$

5] Examples:

$$U(C)\,\bar\Psi\psi\,U^{-1}(C) \overset{\circ}{=} \bar\Psi\psi$$

$$U(C)\,\underbrace{\bar\Psi\gamma^\mu\psi}_{j^\mu}\,U^{-1}(C) \overset{\circ}{=} -\bar\Psi\gamma^\mu\psi$$

Note 3.3

- relativistic QFT: invariant $SO^+(1,3)$
- $(i\gamma^\mu\partial_\mu - m)\psi = 0$ is $\{C,P,T\}$ invariant
- The (quantized) Dirac theory is $\{C,P,T\}$ invariant:

  $[H, U(X)] = 0 \qquad X = C,P,T$
- Weak interactions violate C and P but preserve CP and T (Wu exp)
- Rare processes (neutral Kaon decay) violate CP and T
- CPT Theorem:
  - $SO^+(1,3)$ invariance
  - Causality
  - Locality
  - Stable vacuum

  $\Longrightarrow$ CPT is symmetry.

4. Interacting Fields and Feynman Diagrams

4.1. Preliminaries

- $$H_{int} = \int d^3x\, \mathcal{H}_{int}(\phi(x)) = -\int d^3x\, \mathcal{L}_{int}(\phi)$$
- Causality → Interactions are local

* Examples:

1) ϕ^4-Theory

$$\boxed{\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \underbrace{\overset{\text{dimensionless coupling constant}}{\frac{\lambda}{4!}}\phi^4(x)}_{\text{Interaction}}}$$

EOM
$$\longrightarrow \quad (\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3$$

⟶ Cannot be solved by Fourier modes

2) Yukawa theory

$$\boxed{\mathcal{L}_{Yukawa} = \bar{\psi}(i\not{\partial} - m)\psi + \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \underbrace{- g\,\phi\cdot\bar{\psi}\psi}_{\text{Interaction}}}$$

3) QED

$$\mathcal{L}_{QED} = \underbrace{\bar{\psi}(i\not{\partial} - m)\psi}_{\partial_\mu\gamma^\mu} - \underbrace{\frac{1}{4}(F_{\mu\nu})^2} - \underbrace{e\,\bar{\psi}\gamma^\mu\psi A_\mu}_{\text{Interaction}}$$

$$\boxed{\begin{aligned} &e = -|e| < 0 \text{ . Electron charge} \\ &D_\mu = \partial_\mu + ieA_\mu \end{aligned}}$$

covariant derivative

$$= \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}(F_{\mu\nu})^2$$

Gauge theory,

- $A'_\mu = A_\mu - \frac{1}{e}\partial_\mu \alpha(x)$
- $\psi'(x) = e^{i\alpha(x)}\psi(x)$

EOM
$\longrightarrow$

- $i(\not\partial - m)\psi(x) = 0$
- $\partial_\mu F^{\mu\nu} = e\, j^\nu \quad \bar{\psi}\gamma^\mu\psi$

- No known exactly solvable <u>interacting</u> QFTs in $D > 1+1$ → $\boxed{\text{CFT (conformal field theories}}$

→ <u>Perturbation Theory</u>

(→ Numerics (Lattice gauge theories)

4.2. Perturbation Expansion of Correlation Functions

<u>1</u> <u>Goal</u>: $\langle \underline{\Omega} | T\{\underline{\phi}(x)\phi(y)\} | \Omega \rangle$

- $|0\rangle$: Ground state of free theory
- $|\Omega\rangle$: —"— interacting —"—

<u>2</u> <u>Remember</u>.

$$\langle 0 | T\, \phi(x)\phi(y) | 0 \rangle = D_F(x-y)$$

<u>3</u> <u>Now</u>:

$$H_{\phi^4} = H_0 + \underbrace{\int d^3x \frac{\lambda}{4!}\phi^4(\vec{x})}_{H_{int} \,=\, \text{Perturbation}}$$

<u>4</u> <u>Todo</u>.

$$\text{Express} \begin{Bmatrix} \phi(x) \\ |\Omega\rangle \end{Bmatrix} \text{ in terms } \begin{Bmatrix} \text{free field } \phi_I(x) \\ \text{free vacuum } |0\rangle \end{Bmatrix}$$

5] $\tilde{x}$ Reference time t_0

$$\phi(t_0,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_{\vec{p}}\, e^{i\vec{p}\vec{x}} + \boxed{a^\dagger e^{-i\vec{p}\vec{x}}\right)}$$

6] **Definitions:**

Heisenberg picture:

$$\phi(x) = \phi(t,\vec{x}) = e^{iH(t-t_0)}\,\phi(t_0,x)\,e^{-iH(t-t_0)}$$

Interaction picture:

$$\phi_I(t,\vec{x}) = e^{iH_0(t-t_0)}\,\phi(t_0,x)\,e^{-iH_0(t-t_0)}$$

$$\Rightarrow \quad \phi_I(t,\vec{x}) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_{\vec{p}}\, e^{-ipx} + a^\dagger_{\vec{p}}\, e^{ipx}\right)$$

7]

$$\phi(t,\vec{x}) = U^\dagger(t,t_0)\,\phi_I(t,\vec{x})\,U(t,t_0)$$

$$U(t,t_0) = e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)}$$

$$\boxed{\begin{aligned} &\cdot\ i\partial_t U(t,t_0) \doteq H_I(t)\,U(t,t_0) \\ &\cdot\ U(t_0,t_0) = \mathbb{1} \end{aligned}}$$

with $H_I(t) = e^{iH_0(t-t_0)}\,H_{int}\,e^{-iH_0(t-t_0)}$

$$= \int d^3x\,\frac{\lambda}{4!}\,\phi_I^4(t,\vec{x})$$

8] Solution: Dyson series.

$$\boxed{\begin{aligned} U(t,t_0) &= \mathbb{1} + (-i)\int_{t_0}^{t} dt_1\, H_I(t_1) \\ &\quad + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1\int_{t_0}^{t} dt_2\, T\left\{H_I(t_1)H_I(t_2)\right\} \\ &\quad + \ldots\ldots \\ &= T\exp\left[-i\int_{t_0}^{t} d\tau\, H_I(\tau)\right] \end{aligned}}$$

9) Properties:

- $U(t,t') \overset{t>t'}{\doteq} e^{iH_0(t-t_0)} e^{-iH(t-t')} e^{-iH_0(t'-t_0)}$ (*)
- $U^{-1}(t,t') = U^{+}(t,t')$
- $U(t_1,t_2)U(t_2,t_3) \overset{t_1>t_2>t_3}{\doteq} U(t_1,t_3)$

10) Grand state $|\Omega\rangle$?

$\lambda \ll 1 \longrightarrow \langle\Omega|0\rangle \neq 0$

$$e^{-iHT}|0\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle$$

$$= e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle$$

$$+ \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle$$

$E_n > E_0$, $n \neq 0$

$$\boxed{|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} \boxed{e^{-iHT}}|0\rangle \overset{(*)}{=} \lim_{T\to} \left(e^{-iE_0(t_0+T)}\langle\Omega|0\rangle\right)^{-1} \boxed{U(t_0,-T)}|0\rangle}$$

$$\langle\Omega| = \lim_{T\to\infty(1-i\varepsilon)} \langle 0|U(T,t_0)\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}$$

11] ※ Two point correlator

$x^0 > y^0 > t_0$

$\langle\Omega|\cancel{T}(\phi(x)\phi(y))|\Omega\rangle$

$$= \lim_{T\to\infty(1-i\varepsilon)} N_T^{-1}\langle 0|U(T,x^0)\,\phi_I(x)\,U(x^0,y^0)\,\phi_I(y)\,U(y^0,-T)|0\rangle$$

$\langle\Omega|\Omega\rangle = 1$

$$N_T \overset{\downarrow}{=} \langle 0|U(T,t_0)\,U(t_0,-T)|0\rangle$$

$$= \langle 0|\,U(T,-T)|0\rangle$$

$$\boxed{\langle\Omega|T\,\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T}dt\,H_I(t)\right]\right\}|0\rangle}{\langle 0|T\left\{\exp\left[-i\int_{-T}^{T}dt\,H_I(t)\right]\right\}|0\rangle}}$$

4.3. Wick's Theorem

Goal. Evaluate

$$\langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle = \sum \langle 0|T\{\phi_I(x_1)\,\phi_I(x_2)\,\phi_I\dots\}|0\rangle$$

1]
$$\phi_I(x) = \underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\, a_p e^{-ipx}}_{\phi_I^+(x)} + \underbrace{\int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\, a_p^\dagger e^{+ipx}}_{\phi_I^-(x)}$$

$\phi^+|0\rangle = 0$

$\langle 0|\phi^- = 0$

2] Observation. $\boxed{x^0 > y^0}$ $\boxed{x^0 < y^0}$

$$T(\phi_I(x)\phi_I(y)) = \underset{y\quad x}{\phi^+_x\phi^+_y} + \underset{y\quad x}{\phi^+_x\phi^-_y} + \underset{x\quad x}{\phi^-_x\phi^+_y} + \underset{y\quad x}{\phi^-_x\phi^-_y}$$

$$\downarrow$$

$$= -''- \quad \phi^-_y\phi^+_x \quad -''- \quad -''- \quad + [\phi^+_x, \phi^-_y]$$

$$\phi^-\ \phi^+_y \qquad + [\phi^+, \phi^-$$

3] Definition

- Contraction

$$\overbrace{\phi(x)\phi(y)} := \begin{cases}[\phi^+(x), \phi^-(y)] & x^0 > y^0 \\ [\phi^+(y), \phi^-(x)] & x^0 < y^0\end{cases}$$

$$\overset{\circ}{=} D_F(x-y)$$

- Normal order:

$$:a_1^{(\dagger)}\dots a_n^{(\dagger)}: \; := \text{creation op} \times \text{annihilation op}.$$

$$T\{\phi(x)\phi(y)\} = :\phi(x)\phi(y) + \overbrace{\phi(x)\phi(y)}:$$

$$\to \langle 0|T\phi(x)\phi(y)|0\rangle$$

$$= D_F(x-y)$$

4) Wick's theorem:

$$T\{\phi(x_1)\ldots\phi(x_n)\} = :\phi(x_1)\ldots\phi(x_n) + \text{all possible contractions}:$$

where $$:A\wick{\c1\phi_i B \c1\phi_j} C: := D_F(x_i - x_j)\,:ABC:$$

5)

$$\langle 0|T(\phi_1\ldots\phi_n)|0\rangle = \text{all full contractions}$$

$$\langle 0|T(\phi_1\phi_2\phi_3\phi_4)|0\rangle = \wick{\c1\phi_1 \c2\phi \c2\phi \c1\phi} \quad \wick{\c1\phi \c1\phi \c2\phi \c2\phi} \quad \wick{\c1\phi \c2\phi \c1\phi \c2\phi}$$

Recap

① Interactions:

$$\mathcal{L}_{\phi^4} = \underbrace{\boxed{\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi}}_{\text{free KG field}} \underbrace{\boxed{-\frac{\lambda}{4!}\phi^4}}_{\text{Interaction}}$$

$$\Rightarrow H_{int} = \int d^3x \, \frac{\lambda}{4!} \phi^4(x)$$

② Goal: Correlators

$$\langle\Omega| \mathcal{T} \phi(x)\phi(y) |\Omega\rangle$$

Vacuum of interacting theory

Heisenberg operators of interacting theory

③ Todo:

$\phi(x) = \ldots$ free fields $\phi_I(x)$? ✓

$|\Omega\rangle = \ldots$ free vacuum $|0\rangle$? ✓

④ Definitions:

Free field:

free KG Hamiltonian

$$\phi_I(t,\vec{x}) = e^{iH_0(t-t_0)} \phi(t_0,\vec{x}) e^{-iH_0(t-t_0)}$$

interaction picture

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_{\vec{p}} e^{-ipx} + a_{\vec{p}}^{\dagger} e^{ipx} \right)\Big|_{p^0=E_p}$$

Interacting field:

interacting Hamiltonian: $H_0 + H_{int}$

$$\phi(t,\vec{x}) = e^{iH(t-t_0)} \phi(t_0,\vec{x}) e^{-iH(t-t_0)}$$

$$= U^{\dagger}(t,t_0)\, \Phi_I(t,\vec{x})\, U(t,t_0)$$

⑤ Time evolution operator.

$$U(t,t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$$

$$\Leftrightarrow \boxed{\begin{aligned} i\partial_t U &= H_I(t)\, U \\ U(t_0,t_0) &= \mathbb{1} \end{aligned}}$$

"interaction in interaction picture"

$$H_I(t) = \int d^3x \, \frac{\lambda}{4!} \phi_I^4(t,\vec{x})$$

→ Solution: Dyson series

$$U(t,t_0) = \mathcal{T} \exp\left[-i \int_{t_0}^{t} d\tau \, H_I(\tau) \right]$$

time ordering operator

⑥ Ground state.

$T \to \infty(1-i\varepsilon)$ "Trick"

$$\Downarrow$$

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \Big(e^{-iE_0(t_0+T)} \underbrace{\langle\Omega|0\rangle}_{\neq 0} \Big)^{-1} \times U(t_0,-T)|0\rangle$$

⑦ Group property.

$$U(t_1,t_2)\,U(t_2,t_3) = U(t_1,t_3)$$

⑧ Main result:

$$\langle \Omega | \underline{T}\, \phi(x)\phi(y) | \Omega \rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0 | T\left\{ \phi_I(x)\phi_I(y)\, \exp\left[-i\int_{-T}^{T} dt\; H_I(t)\right]\right\} |0\rangle}{\langle 0 | T\left\{ \exp\left[-i\int_{-T}^{T} dt\; \underset{\uparrow}{H_I}(t)\right]\right\} |0\rangle}$$

\ depends on Φ_I

⑨ Perturbative expansion of exp[..] in orders of λ

Efficient evaluation ? ⟶ Wick's theorem

$$\langle \Omega | T \phi(x)\phi(y) | \Omega \rangle = \sum_{n} \overbrace{\langle 0 | T\{\phi_I(x_1) \quad \phi_I(x_n)\} |0\rangle}$$

similar terms in denominator !

⑩ Definitions.

Contraction:

$$\overbrace{\phi(x)\phi(y)} := \begin{cases} [\phi^+(x), \phi^-(y)] & x^0 > y^0 \\ [\phi^+(y), \phi^-(x)] & x^0 < y^0 \end{cases} \doteq D_F(x-y)$$

drop the I

interaction picture fields !

Feynman propagator

Normal order: $:a_1^{(\dagger)} \ldots a_n^{(\dagger)}: \;:= (\text{creation op.}) \times (\text{annihilation op})$

(Example. $:aa^\dagger: = a^\dagger a$)

$$\boxed{\phi_i \equiv \phi(x_i) \equiv \phi_I(x_i)}$$

Note: $:A\overbrace{\phi_i B \phi_j} C: \;= D_F(x_i - x_j) \cdot \underbrace{:ABC:}$

products of fields / modes

number

normal ordered operators

⑪ Wick's theorem:

including the "trivial contraction" $\phi_1 \ldots \phi_n$

$$\boxed{T\{\phi_1 \ldots \phi_n\} = :\text{all possible contractions}:}$$

Since $\langle 0 | :A: | 0\rangle = 0$ if $:A: \neq \mathbb{1}$ $\Rightarrow$ $\langle 0 | T\{\phi_1 \quad \phi_n\} | 0\rangle =$ all full contractions

6] Example:

$$T\{\phi_1\phi_2\phi_3\phi_4\} =$$

$$:\phi_1\phi_2\phi_3\phi_4: +$$

$$\overline{\phi_1\phi_2}\phi_3\phi_4 + \overline{\phi\,\phi\,\phi}\,\phi + \overline{\phi\,\phi\,\phi\,\phi}$$

$$+ \phi\,\overline{\phi\,\phi}\,\phi + \phi\,\overline{\phi\,\phi\,\phi} + \phi\,\phi\,\overline{\phi\,\phi}$$

$$+ \overline{\phi\,\phi}\,\overline{\phi\,\phi} + \overline{\phi\,\phi\,\phi\,\phi} + \overline{\phi\,\overline{\phi\,\phi}\,\phi} :$$

$$\int :\phi^\dagger\phi + \pi^\dagger\pi:$$

$$:e^{i\phi}:$$

$$\cancel{a^\dagger}a = :\underline{aa^\dagger}: \overset{\text{CCR}}{=} :\underline{a^\dagger a + 1}:$$

$$\theta = 1$$

linear

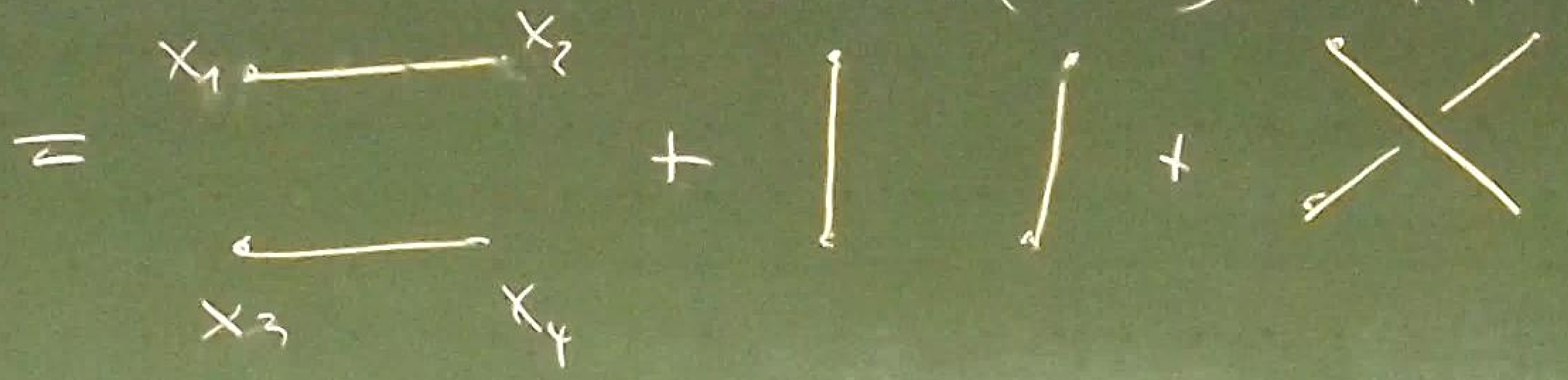
$$\overset{\checkmark}{=} :a^\dagger a: + :1: = \cancel{aa^\dagger} + 1$$

$$\langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle$$

$$= \overline{\phi_1\phi_2}\,\overline{\phi_3\phi_4} + \overline{\phi_1\phi_2\phi_3\phi_4} + \overline{\phi_1\overline{\phi_2\phi_3}\phi_4}$$

$$= D_F(x_1-x_2)\,D_F(x_3-x_4) + D_F(x_1-x_3)\,D_F(x_2-x_4)$$

$$+ D_F(x_1-x_4)\cdot D_F(x_2-x_3)$$



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44. Feynman Diagrams

1] $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$

$$\propto \langle 0|T\Big\{\underbrace{\phi(x)\cdot\phi(y)}_{\lambda^0} + \overbrace{\phi(x)\phi(y)\Big[-i\int dt\, H_I(t)\Big]}^{\lambda^1} + \overbrace{\ldots\ldots}^{\lambda^2}\Big\}|0\rangle$$

2] λ^0: $\langle 0|T\phi(x)\phi(y)|0\rangle$

$$= \overline{\phi\,\phi} = D_F(x-y)$$

3] λ^1:

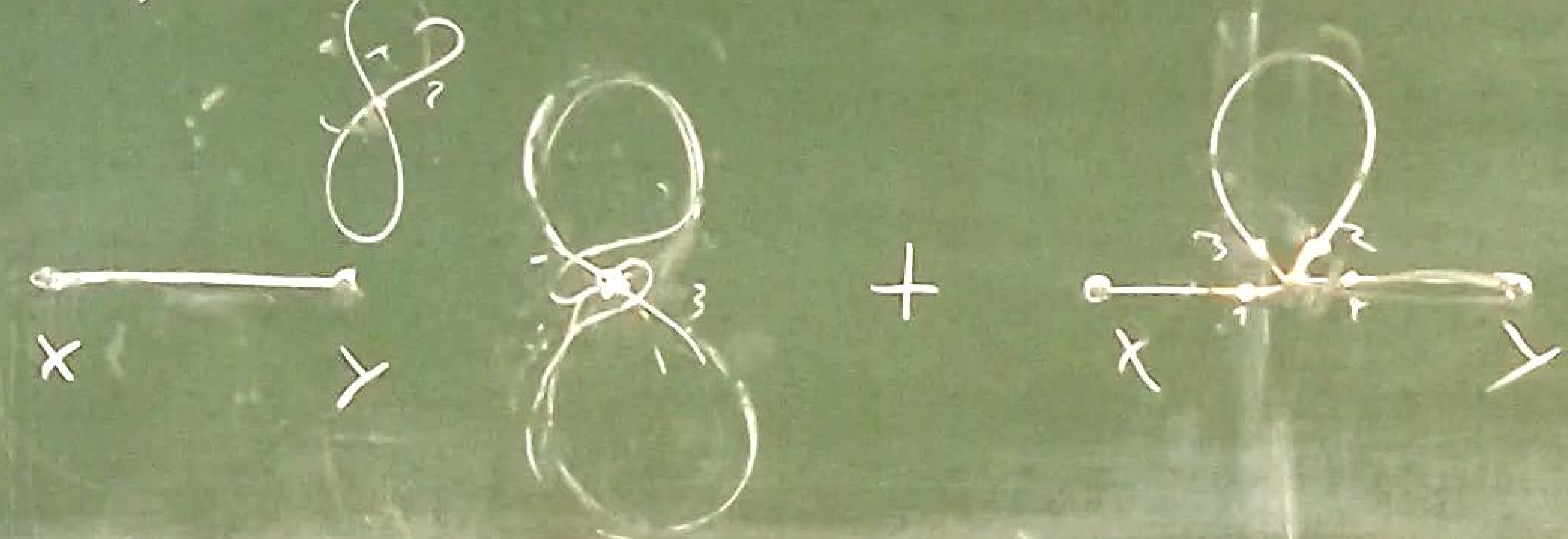
$$\langle 0|T\left\{\phi(x)\phi(y)\frac{-i\lambda}{4!}\int d^4z\,\phi(z)\phi(z)\phi(z)\phi(z)\right\}|0\rangle$$

W-T

$$= 3\,\frac{-i\lambda}{4!}\,D_F(x-y)\int d^4z\,D_F(z-z)\,D_F(z-z)$$

$$+\,12\cdot\frac{-i\lambda}{4!}\int d^4z\,D_F(x-z)\,D(y-z)\,D_F(z-$$

=



→ Interpretation:

Feynman diagram
- · edges = propagators ↔ D_F
- · Interal points = vertices ↔ $-i\lambda\int d^4z$
- external points = spacetime points ↔ x, y, . .

Analytic expression

4] Prefactors:

· F.D. ⇔ sum all identical terms.

· ϕ λ^n

→ $\frac{1}{n!}$, n integrals/vertices

→ n! equivalent permutation

→ ignore the $\frac{1}{n!}$

· 4 contractions at each vertex

→ 4! possibilities to interchange contractions

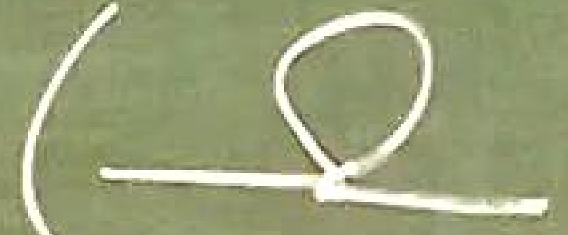
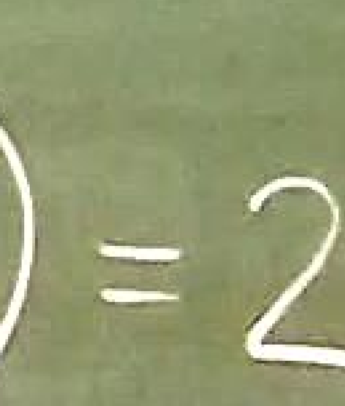
→ $\frac{1}{4!}$ cancels 4!

→ internal vertex → $-i\lambda\int d^4z$

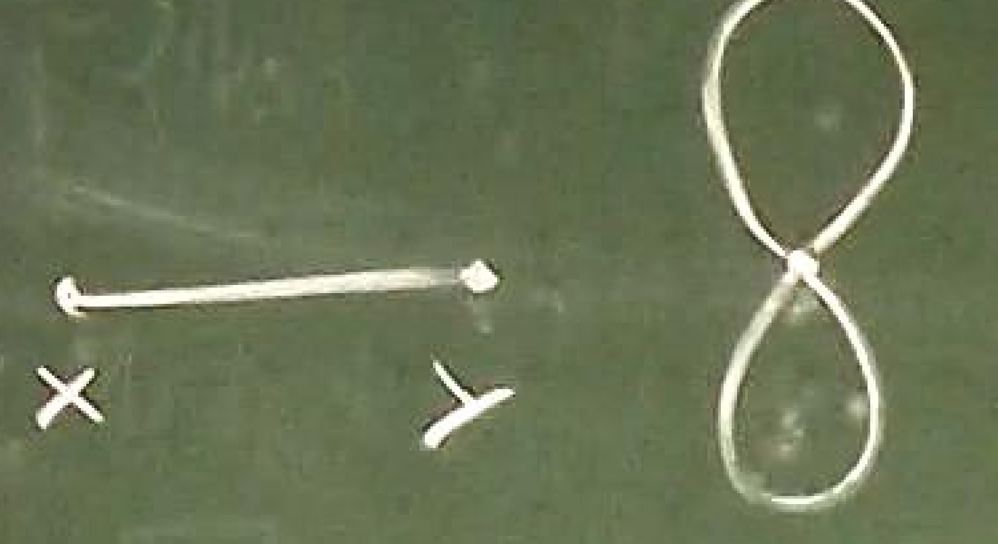
· Symmetries of diagrams reduce number of different contractions.

→ divide expression by symmetry factor S

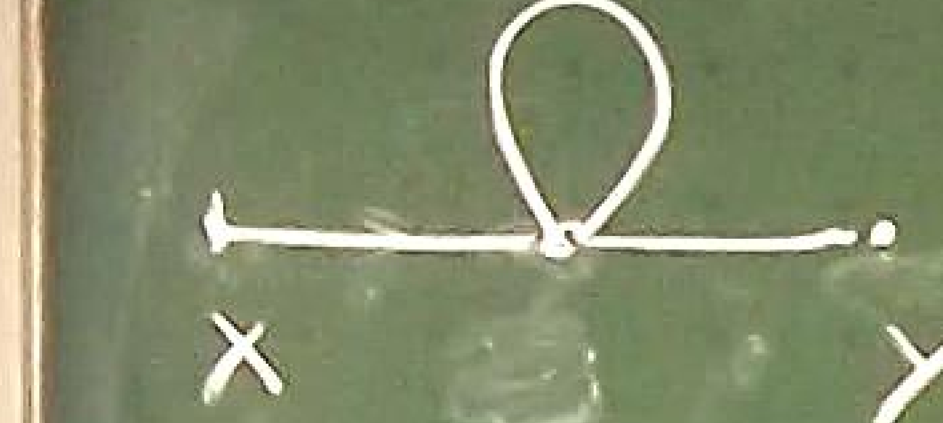
Examples:

S() = 2, S() = 2·2·2 = 8

4] Therefore:



$$= \frac{1}{8}\cdot D_F(x-y)(-i\lambda)\int d^4z\, D_F(z-z)\, D_F(z-z)$$



$$= \frac{1}{2}(-i\lambda)\int d^4z\, D_F(x-z)\, D_F(z-y)\, D_F(z-z)$$

$$\langle\Omega|T\,\phi(x)\phi(x)\phi(z)|\Omega\rangle = 0$$

5] Therefore

$$\langle 0|T\left\{\phi(x)\phi(y)e^{-i\int dt\, H_I(t)}\right\}|0\rangle$$

$$= \sum\left\{\begin{matrix}\text{Feynman diagrams with}\\ \text{two external points}\end{matrix}\right\}$$

with position/real-space **Feynman rules** for ϕ^4-theory:

1. x — y $= D_F(x-y)$

2.  $= (-i\lambda)\int d^4z_i$

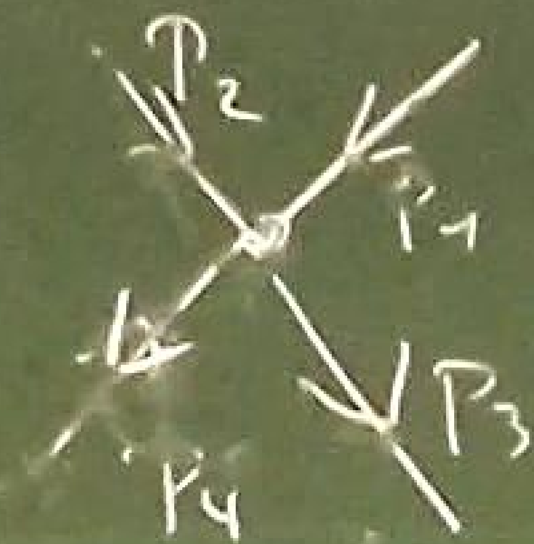
3. 

$= 1$

4. Divide by sym factor: $\frac{1}{S}$.

6] → Momentum space:

$$D_F(x-y) = \int\frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\varepsilon}\,e^{-ip(x-y)}$$



$$= -i\lambda\int d^4z$$

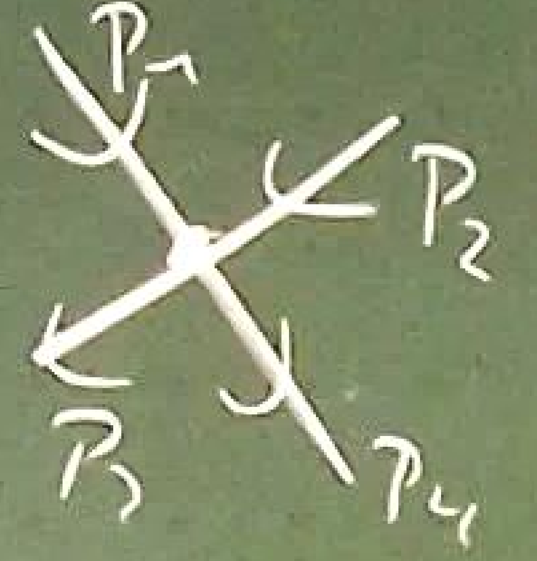
$$= (-i\lambda)(2\pi)^4\,\delta^{(4)}(p_1+p_2-p_3-p_4)$$

→ Momentum conservation at vertices

→ Momentum-space Feynman rules.

1. 

$$= \frac{i}{p^2-m^2+i\varepsilon}$$

2. 

$$= (-i\lambda)(2\pi)^4 \delta(p_1+p_2-p_3-p_4)$$

3. 

$$= e^{-ipx}$$

4. Integrate: $\prod_i \int \frac{d^4p_i}{(2\pi)^4}$

5. Divide by sym. fac. $\frac{1}{S}$

Recap

Hint in interaction picture

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|T\{\phi_I(x)\phi_I(y)\exp[-i\int_{-T}^{T}dt\,H_I(t)]\}|0\rangle}{\langle 0|T\{\exp[-i\int_{-T}^{T}dt\,H_I(t)]\}|0\rangle}$$

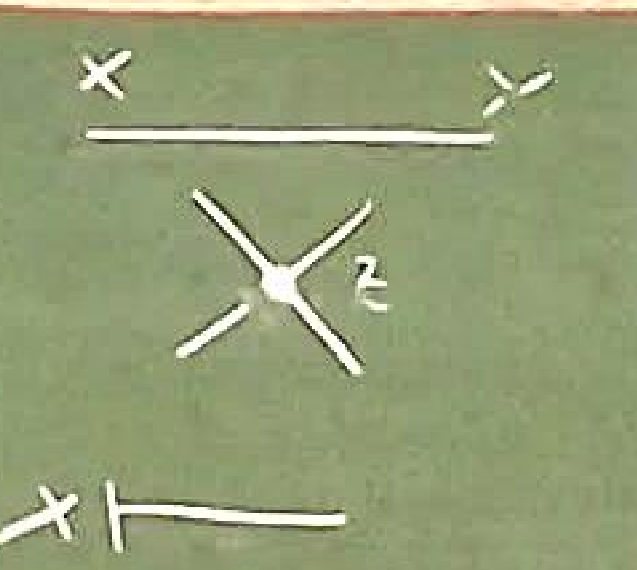
Wick's Theorem

Many equivalent terms ← { All full contractions ↓ Feynman propagators }

Feynman Diagrams

$$= \sum{}' \left\{ \begin{array}{l} \text{Feynman diagrams} \\ \text{with two external} \\ \text{points } x \text{ and } y \end{array} \right\}$$

Feynman diagram $\xrightarrow{\text{Feynman rules}}$ Analytic expression



① $= D_F(x-y)$

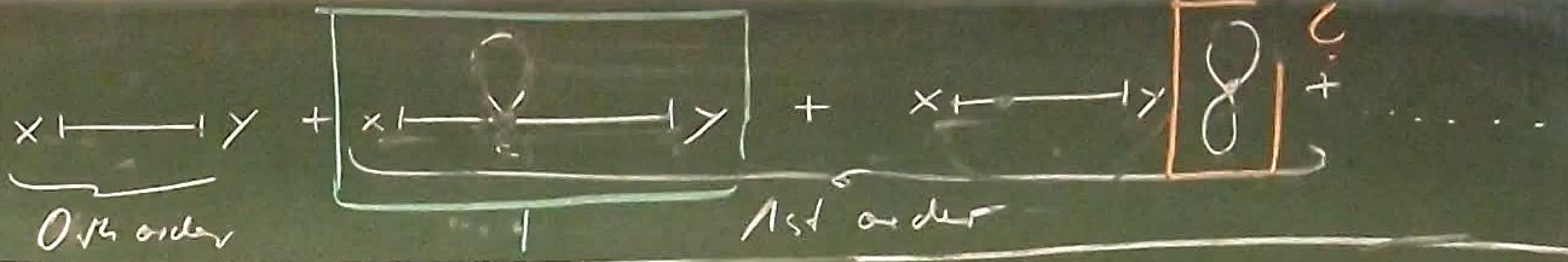
② $= (-i\lambda)\int d^4z$

③ $= 1$

④ Divide by Sym. factor $\frac{1}{S}\times$

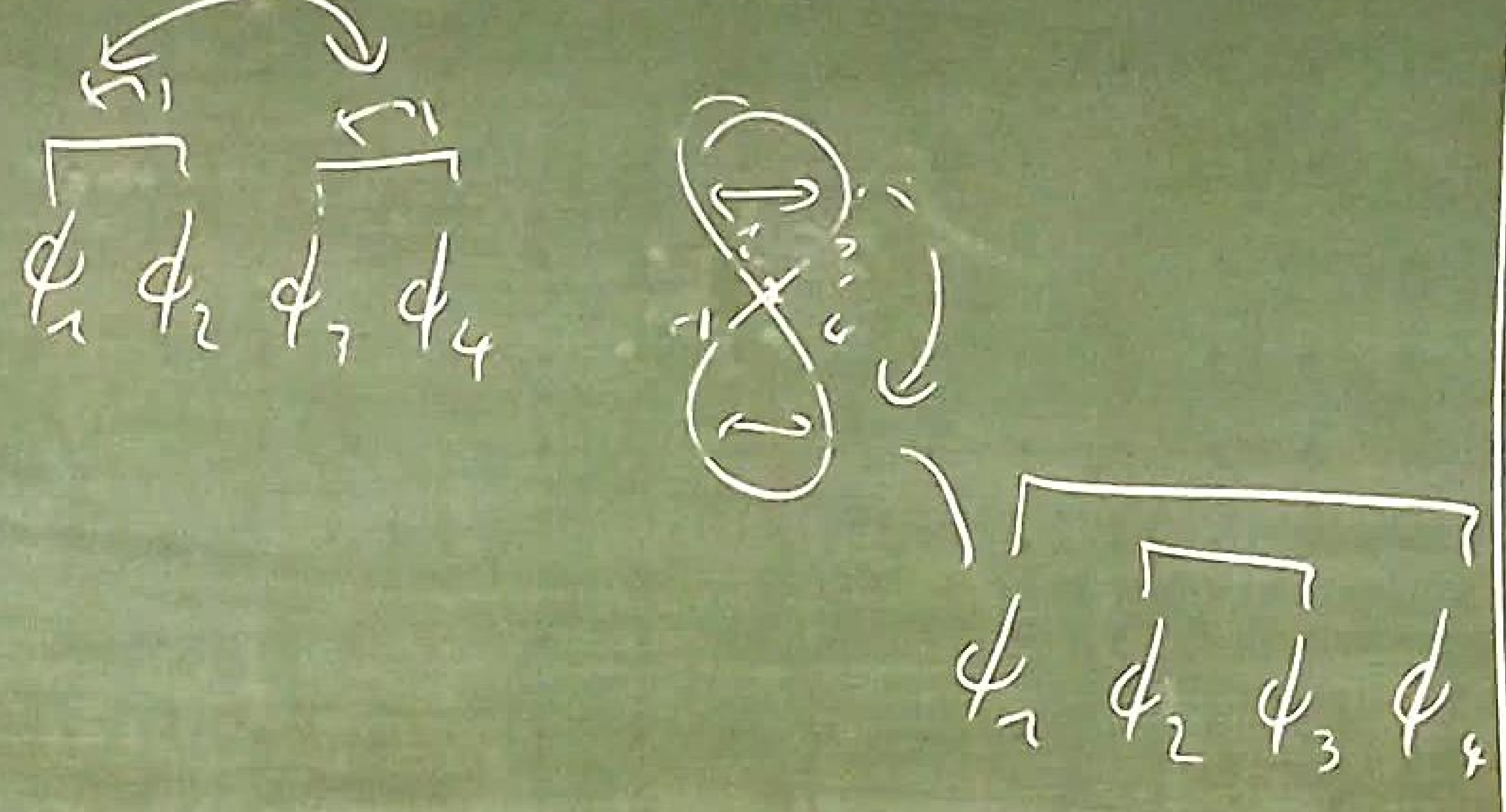
Example:

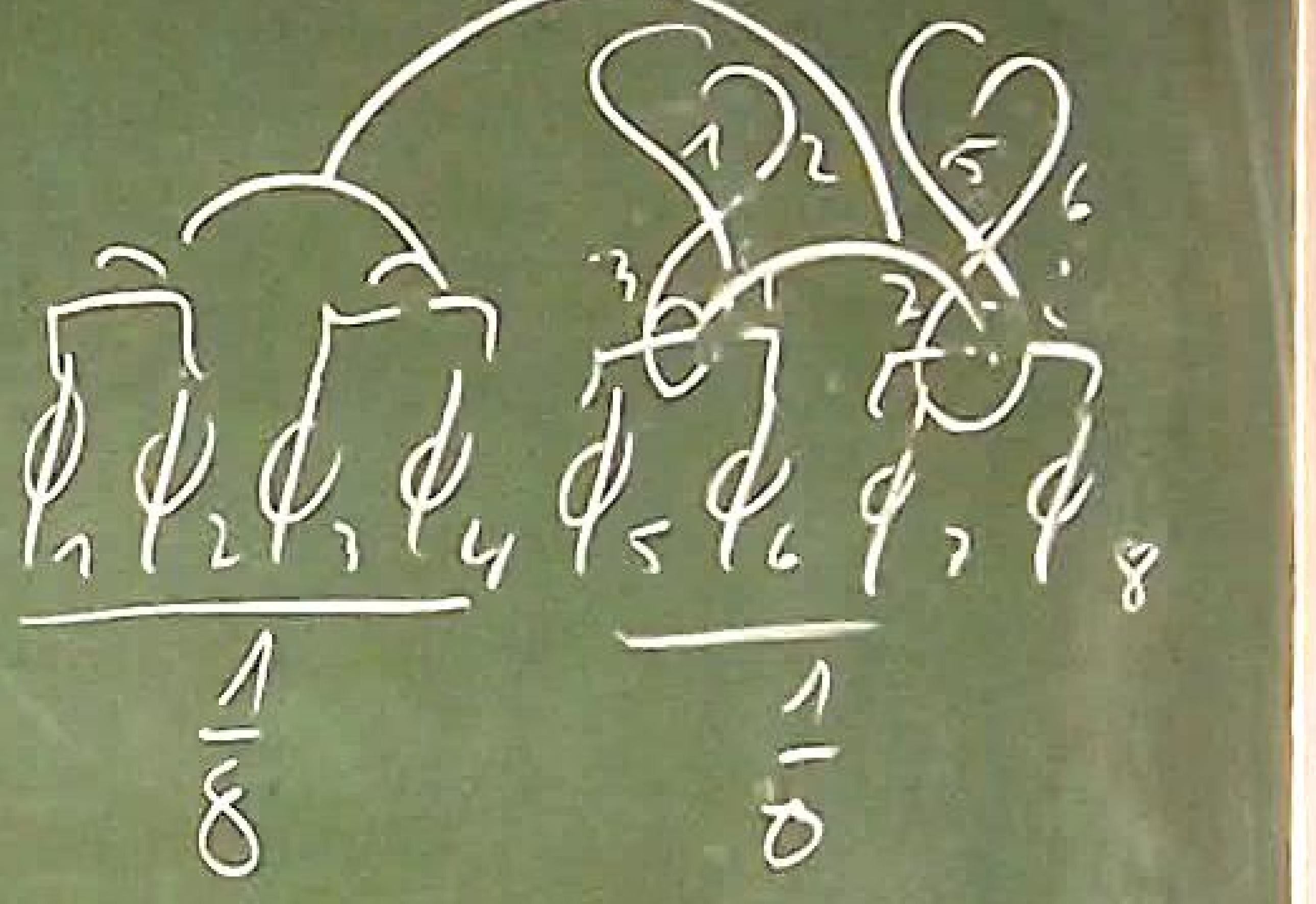
$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle =$



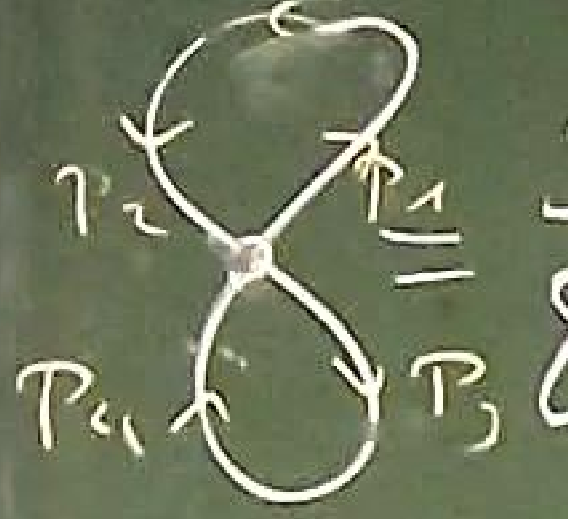
— " — ?

$$= \frac{1}{2}(-i\lambda)\int d^4z\; D_F(x-z)\,D_F(z-z)\,D_F(z-y)$$





7) Problem: Disconnected pieces of diagrams diverge!



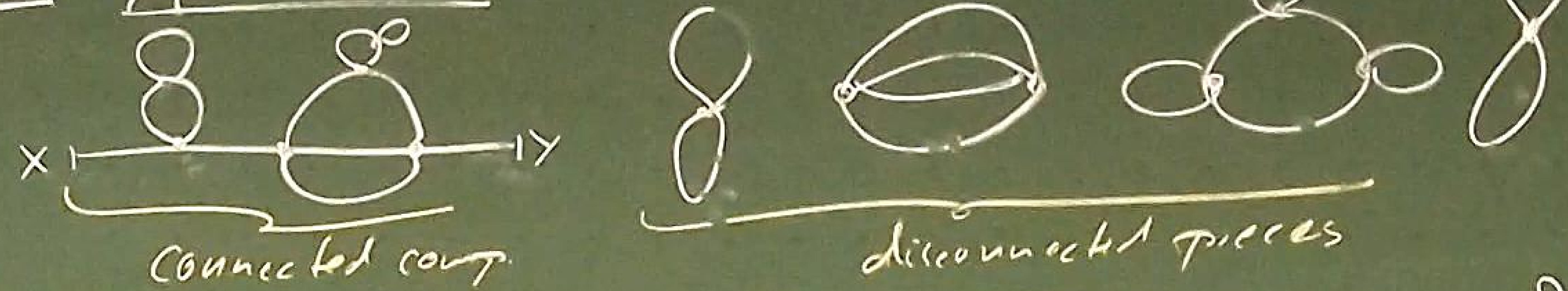
$$= \frac{1}{8}(-i\lambda)\int d^4z\, \underbrace{D_F(z-z)\,D_F(z-z)}_{\text{const.}}$$

$$\int d^4z \to \int_{-T}^{T} dt \int d^3x \to 2T\cdot(\text{volume of space})$$

$$\to \delta(\cancel{p_2}+\cancel{p_4}-\cancel{p_1}-\cancel{p_3}) = \delta(0)$$

8) Exponentiation of disconnected diagrams

i) Typical diagram:



ii)

$$V = \{V_1, V_2, \dots\} = \left\{\begin{array}{c}\text{Disconnected}\\ \text{Feynman diagrams}\\ \text{without external points}\end{array}\right\}$$

$$\mathcal{F}_{xy} = \left\{\begin{array}{c}\text{Connected Feynman}\\ \text{diagrams with external points } x, y\end{array}\right\}$$

$$\to \text{Feynman diagram } F = \{\overset{\cap}{F}^{xy}, \underbrace{V_1, V_1, ..V_1}_{n_1}, \underbrace{V_2, .. V_2}_{n_2}, V_3, \dots\}$$

$$F = F^{xy}\cdot\prod_i \frac{1}{n_i!}(V_i)^{n_i}$$

iv)

$$\langle 0|T\{\phi(x)\phi(y)\, e^{-i\int dt\, H_I(t)}\}|0\rangle$$

$$= \sum_{F\in\mathcal{F}^{xy}} \sum_{n_1, \dots n} \left[F\cdot\prod_i \frac{1}{n_i!}(V_i)^{n_i}\right]$$

$$= \left[\sum_{F\in\mathcal{F}^{xy}} F\right] \times \left[\sum_{n_1 \dots} \prod_i \frac{1}{n_i!}(V_i)^{n_i}\right]$$

$$= \text{—''—} \left[\prod_i \sum_{n_i} \frac{1}{n_i!}(V_i)^{n_i}\right]$$

$$= \text{—''—} \exp\left[\sum_i V_i\right]$$

$$\langle 0|T\{\phi(x)\phi(y)\, e^{-i\int dt\, H_I(t)}\}|0\rangle$$

$$= \sum(\tilde{\uparrow}^{xy}) \cdot e^{\Sigma(V)}$$

9] Denominator:

$$\langle 0|T\{e^{-i\int dt\, H_I(t)}\}|0\rangle = e^{\Sigma(V)}$$

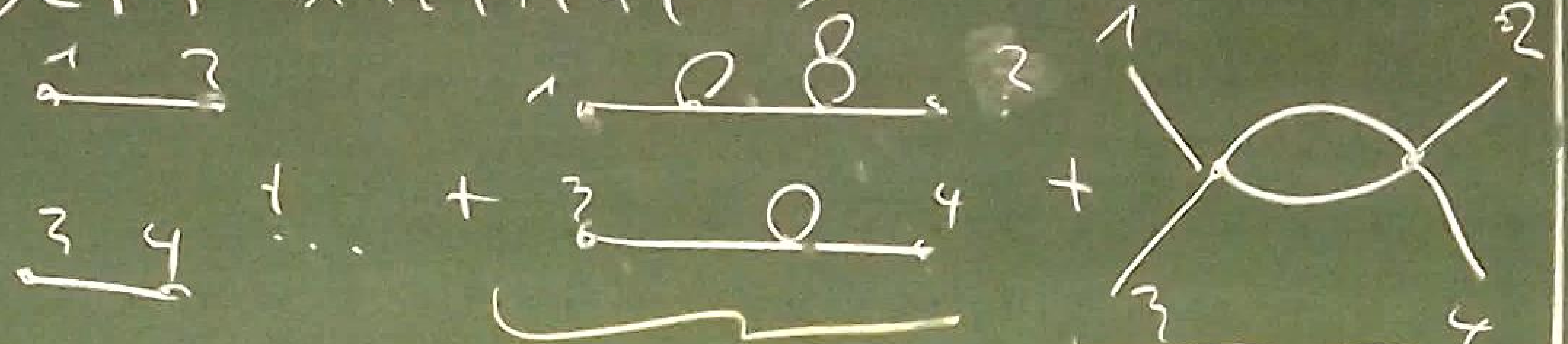
10] Two-point correlator:

$$\boxed{\langle\Omega|T\{\underset{\phi(x_1)\phi(x_2)\ldots\phi(x_4)}{\phi(x)\phi(y)}\}|\Omega\rangle = \sum\Big(\overset{x_1\ldots x_4}{\tilde{\uparrow}^{xy}}\Big) = \left\{\begin{array}{l}\text{Sum of all } \underline{\text{connected}} \\ \text{diagrams with } \overset{n}{\underline{\text{two}}} \text{ external} \\ \text{points}\end{array}\right\}};$$

Note 4.1

- $\langle\Omega|T\,\phi_1\phi_2\phi_3\phi_4|\Omega\rangle$

$=$



connected diag.
disconnected graph

connected dia.
connected graph

- Disconnected diagrams = "Vacuum bubbles"

- Interpretation:

$$\lim_{T\to\infty(1-i\epsilon)} \langle 0|T\,\phi(x)\phi(y)\exp\Big[-i\int_{-T}^{T} dt\, H_I(t)\Big]\}|0\rangle$$

$$= \underbrace{\langle\Omega|T\,\phi(x)\phi(y)|\Omega\rangle}_{||} \times \lim_{T\to\infty(1-i\epsilon)} \underline{|\langle 0|\Omega\rangle|^2} \boxed{e^{-iE_0 2T}}$$

$$= \sum(\mathcal{F}^{xy}) \cdot e^{\Sigma(V)}$$

$$\sum V_i = \Sigma(V) = -iE_0 2T$$

$$V_i = \tilde{V}_i \cdot (2T \cdot V)$$

Vacuum energy density

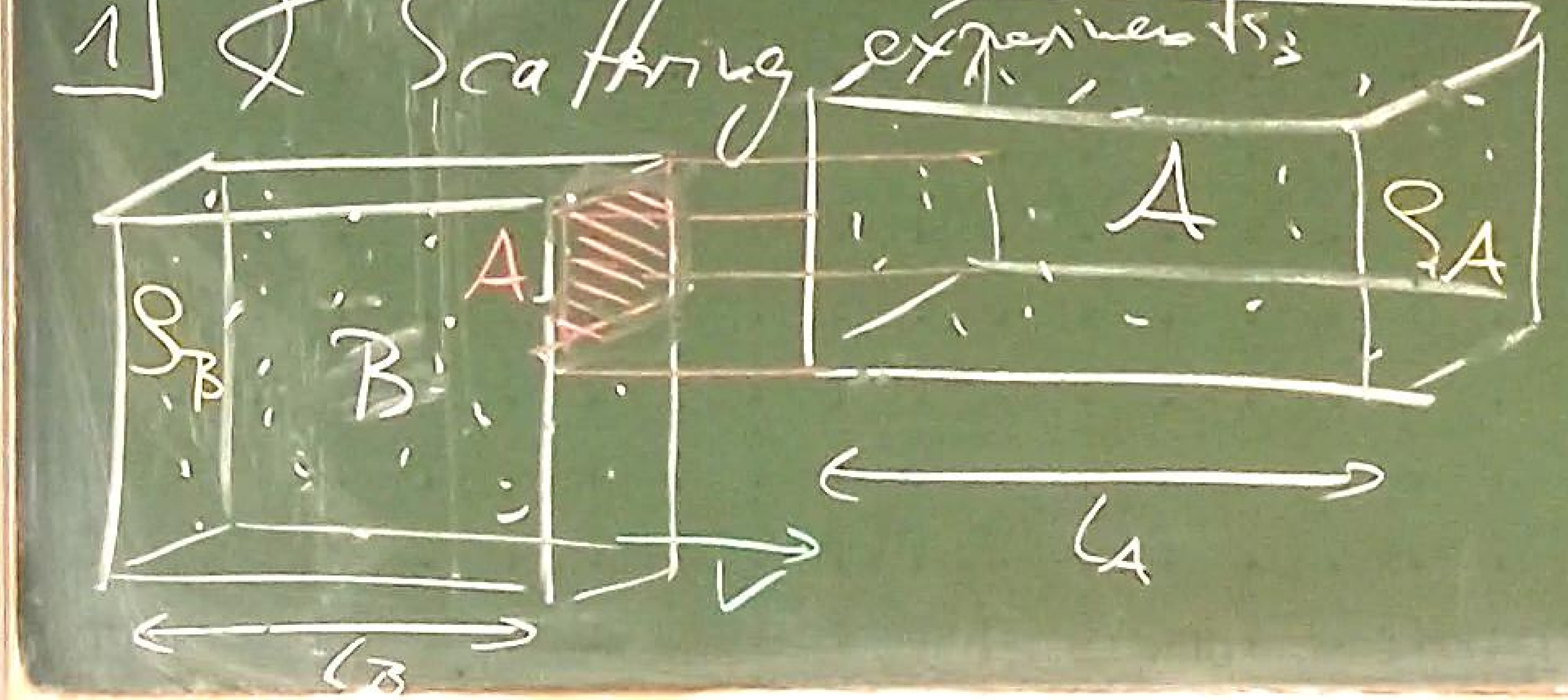
$$\Rightarrow \boxed{\boxed{\frac{E_0}{V}} = i\sum \tilde{V}_i}$$

→ Vacuum bubbles determine the vacuum energy density

4.5 Cross Sections and the S-Matrix

The Cross Section

1] $\mathcal{X}$ Scattering experiments:

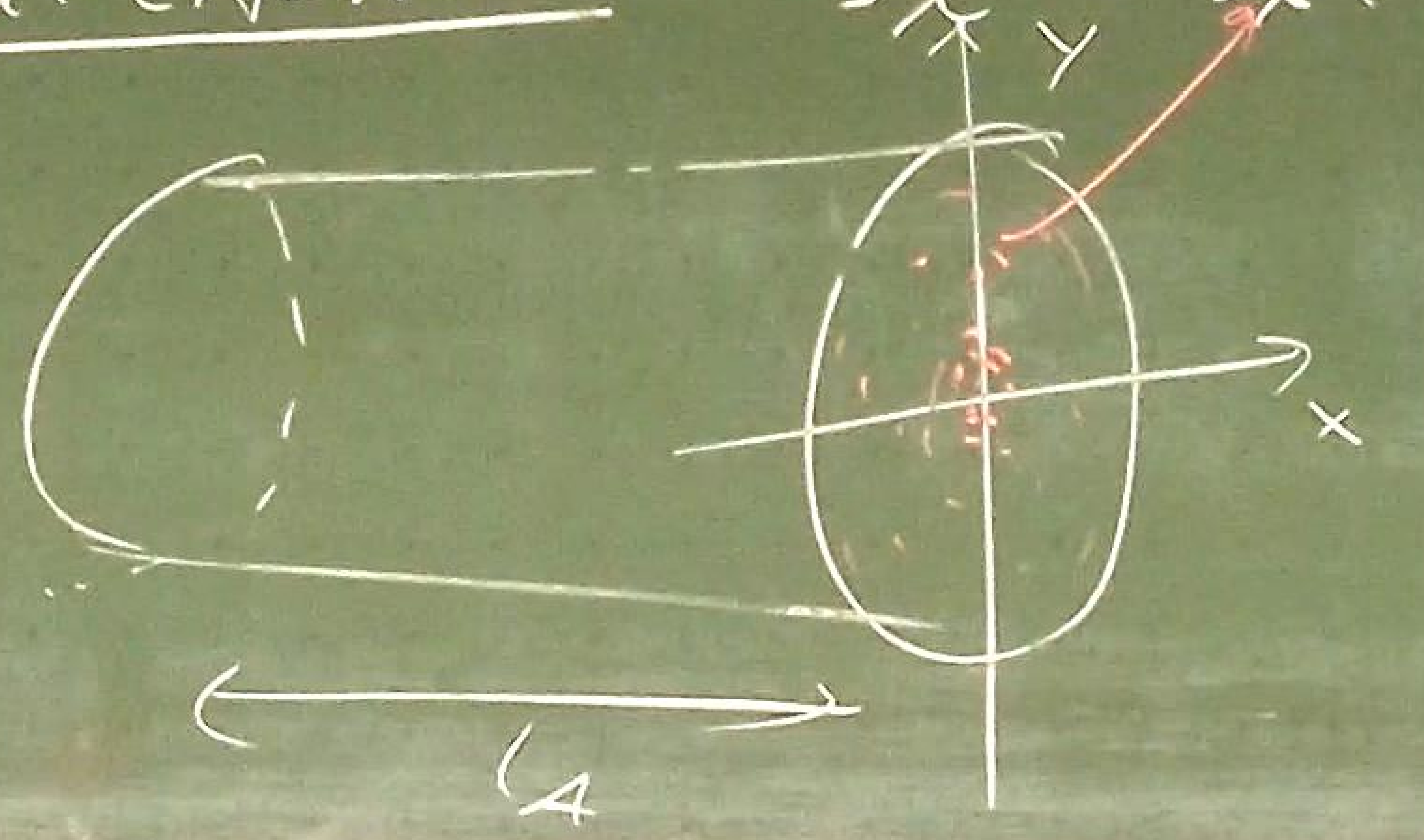


2] Cross section:

$$\sigma_X \equiv \frac{\#\text{scattering events (with outcome } X)}{\rho_A l_A \rho_B l_B A}$$

$$[\sigma_X] = L^2 = \text{Area}$$

3] Real experiments: $\rho_{\mathcal{X}} \to \rho_{\mathcal{X}}(x,y)$



$$\#\text{scattering events }(X) = \sigma_X \cdot l_A l_B \int dx\,dy\, \rho_A(x,y)\, \rho_B(x,y)$$

4] Many outcomes possible.

$$e^+e^- \to \left\{ \begin{array}{l} e^+e^- \\ \mu^+\mu^- \\ \mu^+\mu^-\gamma \\ \vdots \end{array} \right\} X$$

5] Differential cross section

- Scattering outcomes X of n final particles with moment $(\vec{p}_1, \dots, \vec{p}_n) \in V_p \subset \mathbb{R}^{3n}$

$$\sigma_{X|V_p} = \int_{V_p} d^3p_1 \dots d^3p_n \underbrace{\frac{d\sigma}{d^3p_1 \dots d^3p_n}}_{\text{Differential cross section}}$$

→ Constrained by 4-momentum conservation. $\sum_i p_i = \text{const.}$

Special case : $n=2$

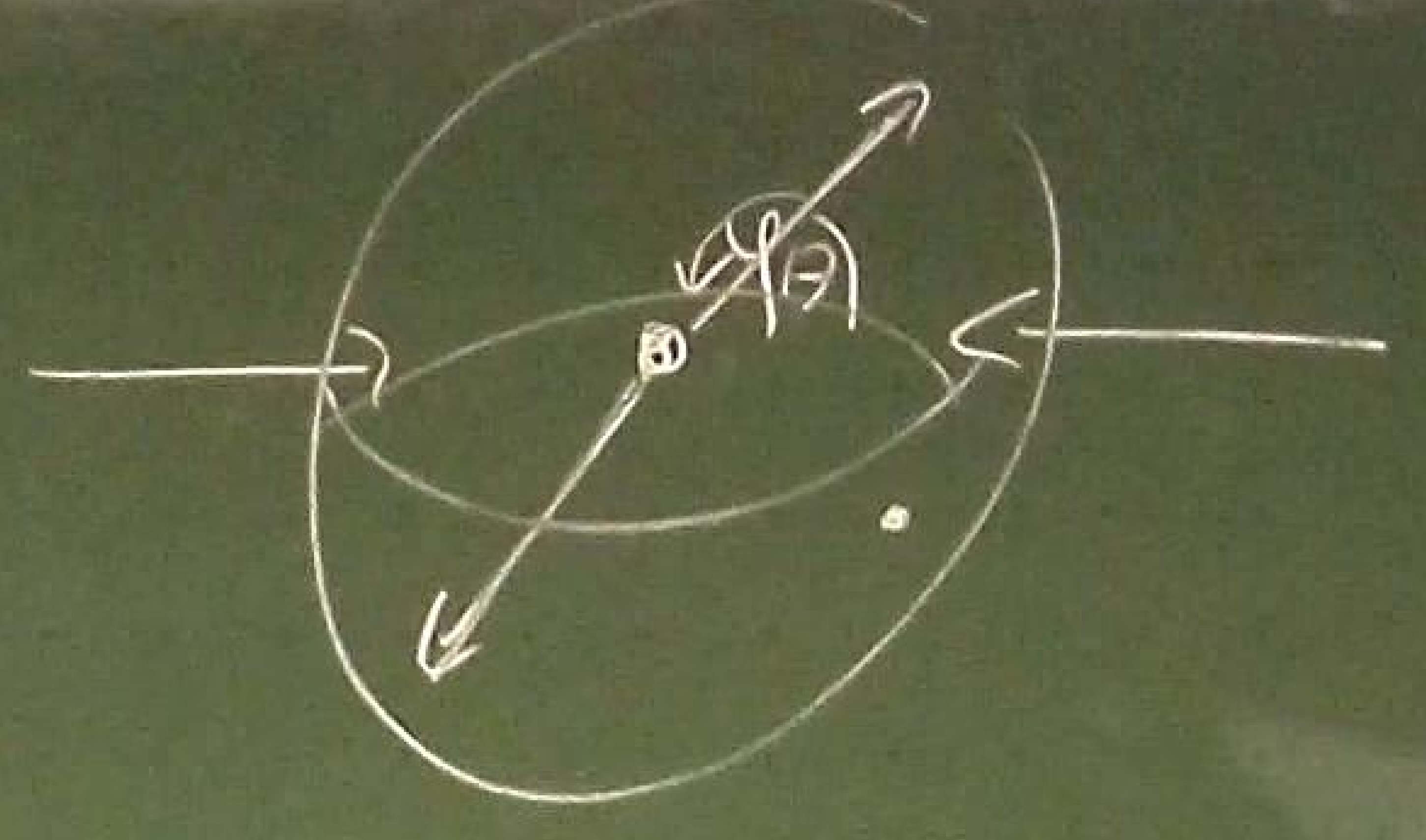
→ 6 dof $(\vec{p}_1, \vec{p}_2)$ + 4 constraints

→ 2 dof

→ Scattering directions (φ, ϑ) in the centre-of-mass frame

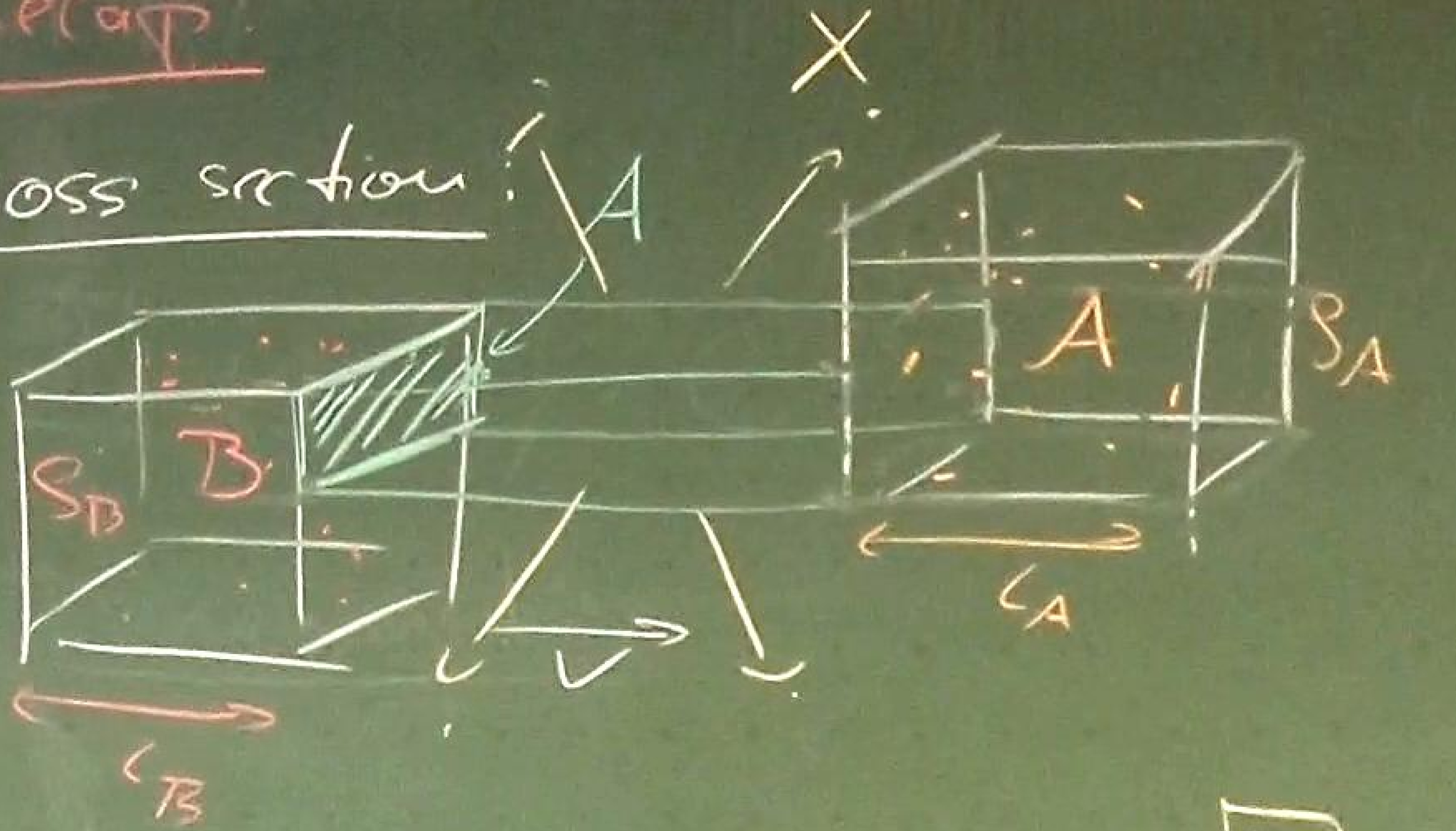
$$\frac{d\sigma}{d^3p_1 d^3p_2} \longrightarrow \boxed{\frac{d\sigma}{d\Omega}}$$

$d\Omega = \sin\vartheta \, d\vartheta \, d\varphi$



Recap:

Cross section:



$$\sigma_X = \frac{\#\text{ scattering events with outcome }\boxed{X}}{\underbrace{|A\rho_A\ell_A|}_{N_A}\underbrace{|\rho_B\ell_B A|}_{N_B}} \cdot A$$

The S-Matrix

1| ☆ One-particle wavepacket

$$|\phi\rangle = \int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\phi(\vec{k})|\vec{k}\rangle$$

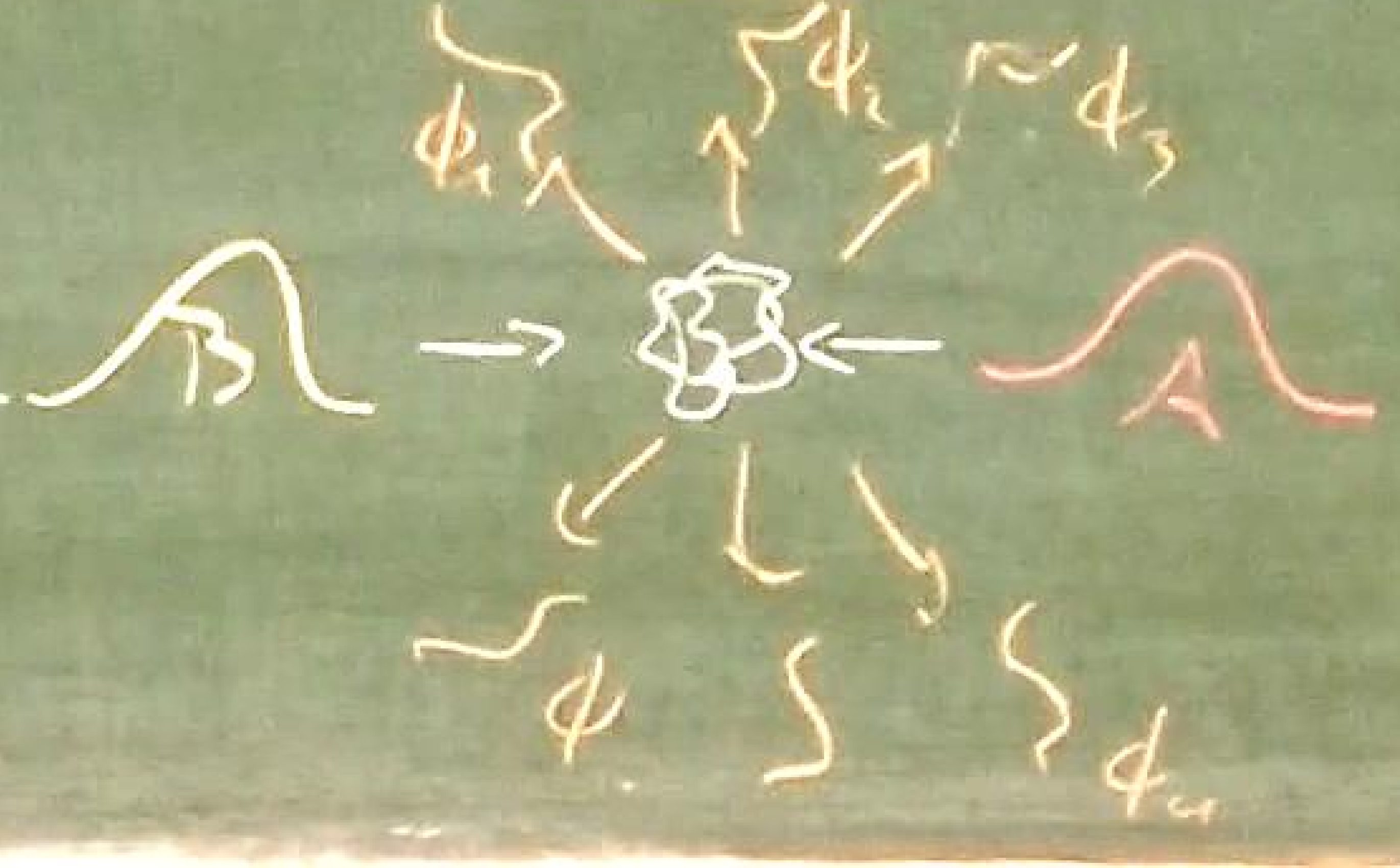
$$\langle\phi|\phi\rangle = \int\frac{d^3k}{(2\pi)^3}|\phi(k)|^2 = 1$$

$|\vec{k}\rangle$: one-particle state of interacting theory

$$\boxed{\lambda=0 \rightarrow |\vec{k}\rangle_0 = \sqrt{2E_k}\,a_k^\dagger|0\rangle}$$

2| Probability.

$$P = \left|{}_{out}\langle\phi_1\ \phi_n|\phi_A\phi_B\rangle_{in}\right|^2 =$$

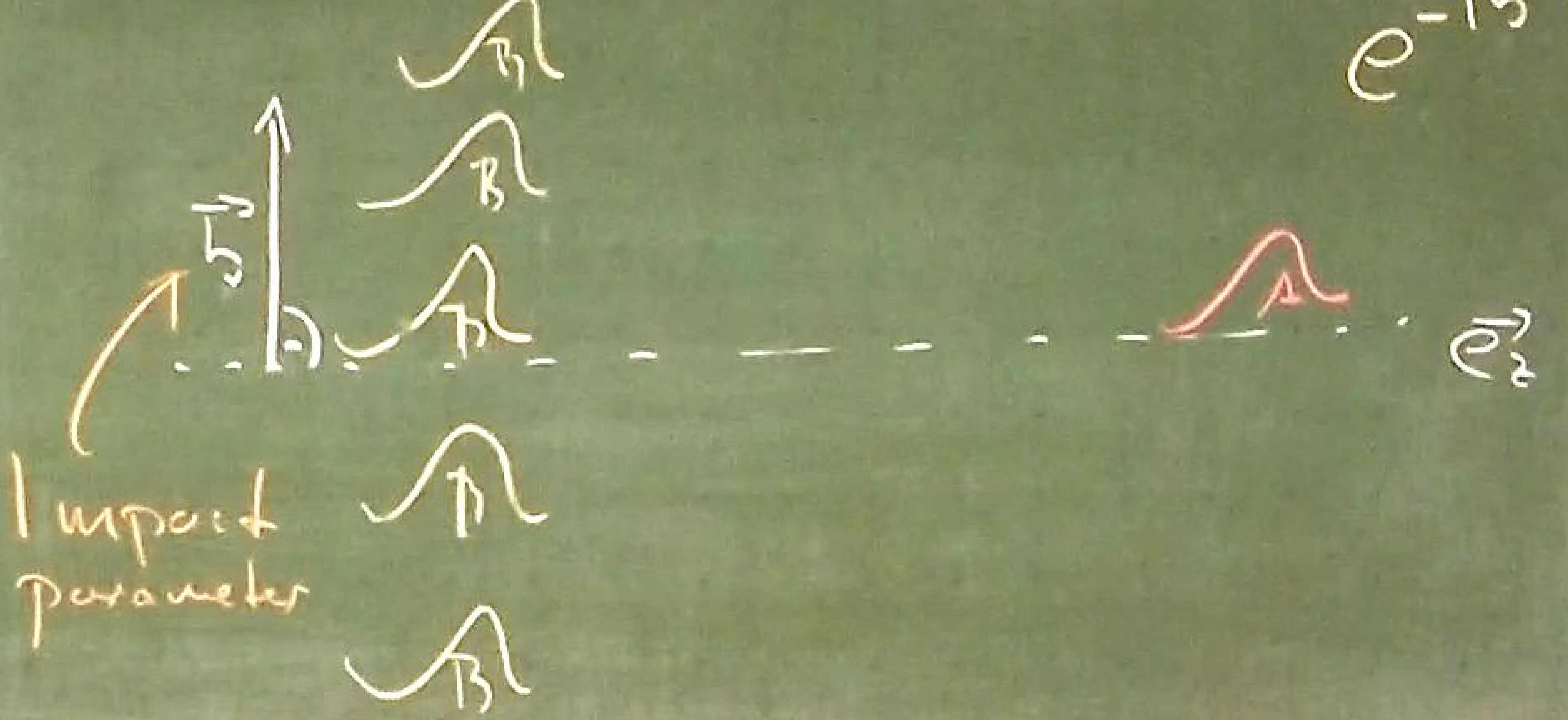


- $|\phi_A\phi_B\rangle_{in}$: in-state ($T \rightarrow -\infty$ two separate wavepackets)
- $|\phi_1\ldots\phi_n\rangle_{out}$: out-state ($T \rightarrow +\infty$ n separate wavepackets)

3| Fourier transformation

$$|\phi_A\phi_B\rangle_{in} = \int\frac{d^3k_A}{(2\pi)^3}\int\frac{d^3k_B}{(2\pi)^3}\frac{\phi_A(k_A)\phi_B(k_B)}{\sqrt{2E_{k_A}\,2E_{k_B}}}\overset{\uparrow}{}|k_Ak_B\rangle_{in}$$

$$e^{-i\vec{b}\cdot\vec{k}_B}$$

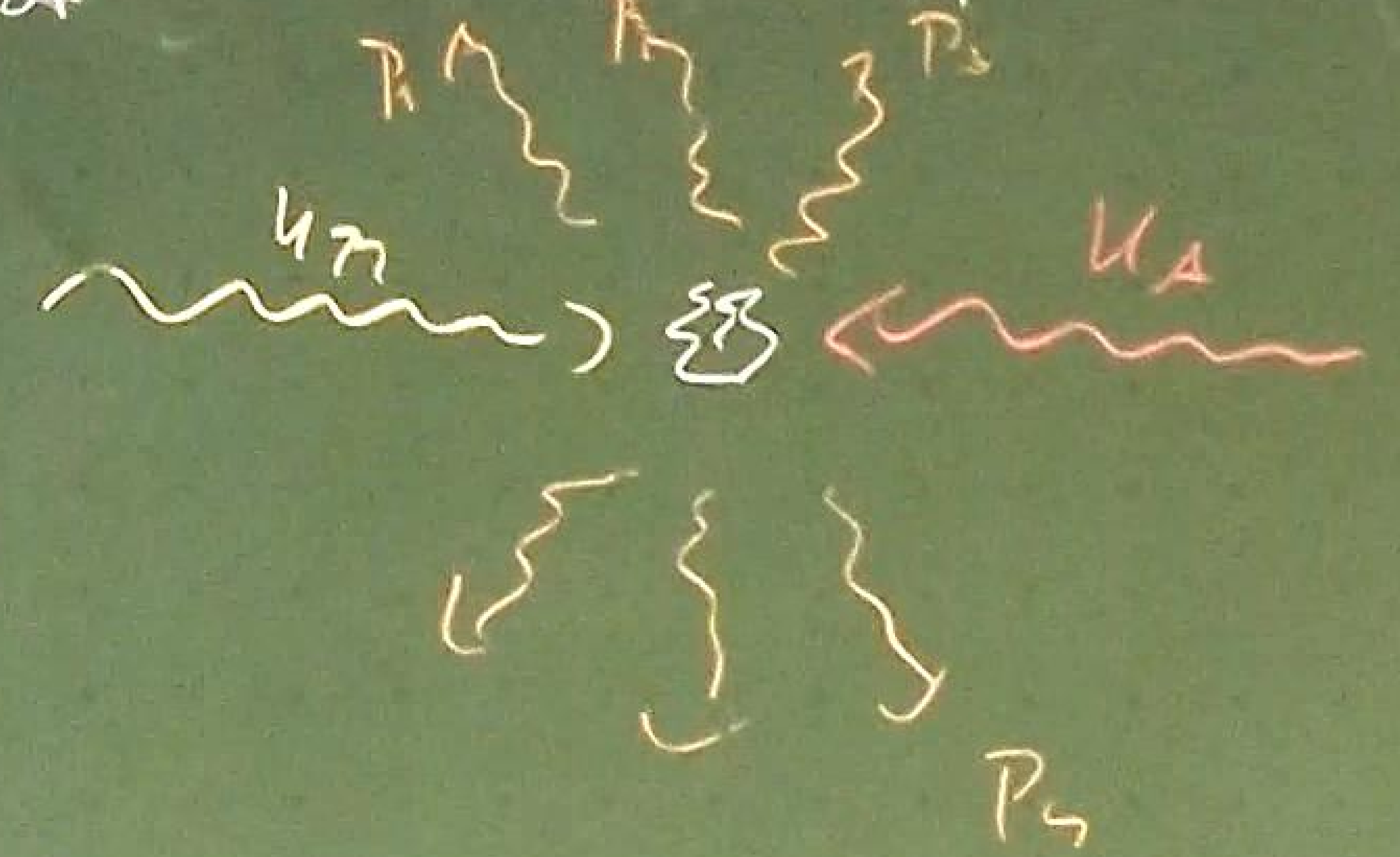


4|

$$|\phi_1 \ldots \phi_n\rangle_{out} \rightarrow |\vec{p}_1 \ldots \vec{p}_n\rangle_{out}$$

$$\Rightarrow \; {}_{out}\langle p_1 \ldots p_n | k_A k_B\rangle_{in}$$

=



5| S-matrix

$${}_{out}\langle \vec{p}_1 \ldots \vec{p}_n | \vec{k}_A \vec{k}_B\rangle_{in} := \lim_{T\to\infty} {}_T\langle \vec{p}_1 \ldots \vec{p}_n | \vec{k}_A \vec{k}_B\rangle_{-T}$$

$$= \lim_{T\to\infty} {}_{t_0=0}\langle p_1 \ldots p_n | e^{-i2TH} | k_A k_B\rangle_{t_0=0}$$

$$= {}_0\langle p_1 \ldots p_n | S | k_A k_B\rangle_0$$

Example: $S = \mathbb{1}$ for free theory

6| T-matrix:

$$\boxed{S = \underbrace{\mathbb{1}}_{\text{particles miss}} + \underbrace{iT}_{\text{non-trivial scattering}}}$$

Heisenberg state

$$P(t)|k\rangle_t = k\,|k\rangle_t$$

$$|k_t\rangle = e^{iHt}|k\rangle_0$$

$$|k_A k_B\rangle_{-T} = e^{-iHT}|k\rangle_{t_0=0}$$

$$|p_1 \ldots\rangle_T = e^{iHT}|p_1 \ldots p_n\rangle_{t_0=0}$$

7| → 4-momentum conservation

$$p^0 = E_p = \sqrt{\vec{p}^2 + m^2}$$

$$\langle \vec{p}_1 \ldots \vec{p}_n | iT | \vec{k}_A \vec{k}_B\rangle$$

$$= (2\pi)^4 \delta^{(4)}\Big(k_A + k_B - \sum_f p_f\Big)$$

$$\cdot \underbrace{i\mathcal{M}\big(k_A k_B \to \{p_f\}\big)}_{\text{Invariant matrix element}}$$

Two questions:

1. $\mathcal{M} = ?$
2. $\boxed{\sigma = \sigma(\mathcal{M})}$

8] ☆ Probability to scatter $d\mathcal{V}_p = \prod_f d^3p_f$

$$dP(AB_{\vec b} \to 1\ldots n) = \underbrace{\left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_{p_f}}\right)}_{\text{normalization}} \left|{}_{out}\langle \vec p_1 \ldots \vec p_n | \phi_A \phi_B(\vec b)\rangle_{in}\right|^2$$

9] ☆ Single target A and <u>many</u> incident particles $B_{\vec b}$

$$d(\#\,\text{scattering events}) = \int d^2b\, \underset{\substack{\uparrow\\ \text{Area density of B-particles}}}{n_B}\, dP(AB_{\vec b} \to 1\ldots n)$$

$$d\sigma = \frac{d(\#\ldots)}{\underbrace{n_B}_{\rho_B l_B}\cdot\underbrace{1}_{\rho_A l_A \cdot A}} = \int d^2b\, dP(AB_{\vec b} \to 1\ldots n) \;\textcircled{=}$$

$$\textcircled{=}\left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_{p_f}}\right)\int d^2b \prod_{i=A,B}\left(\int \frac{d^3k_i}{(2\pi)^3}\frac{\phi_i(\vec k_i)}{\sqrt{2E_{k_i}}}\int \frac{d^3q_i}{(2\pi)^3}\frac{\phi_i^*(\vec q_i)}{\sqrt{2E_{q_i}}}\right)$$

$$\times \underbrace{e^{i\vec b(\vec q_B - \vec k_B)}}_{(2\pi)^2\delta^{(2)}(k_B^\perp - q_B^\perp)}\; \underbrace{{}_{out}\langle\{p_f\}|\{k_i\}\rangle_{in}}_{iM(\{k_i\}\to\{p_f\})\cdot(2\pi)^4\delta^{(4)}(\sum k_i - \sum p_f)} \cdot \underbrace{{}_{out}\langle\{p_f\}|\{q_i\}\rangle_{in}^*}_{-iM^*(\{k_i\}\to\{p_f\})\,(2\pi)^4\delta^{(4)}(\sum q_i - \sum p_f)}$$

$$\boxed{q_i^\perp = (q_i^x, q_i^y)}$$

→ Evaluate 6 q_i-integrals.

i) $q_B^\perp = k_B^\perp$

ii) → $q_A^\perp = k_A^\perp$

$$k_A^{\overset{?}{x}} + \cancel{k_B^{\overset{z}{x}}} = q_A^{\overset{z}{x}} + \cancel{q_B^{z}}$$

$$E_A^k + E_B^k = E_A^q + E_B^q$$

iii) ∢ $q_A^2\, q_B^2$ - integrals:

$\sqrt{\vec{p}_A^{\,2}+m^2}$

$$\int dq_A^2 \underbrace{dq_B^2\ \delta(q_A^2+q_B^2-\sum p_f^2)}\ \delta(E_A+E_B-\sum E_f)$$

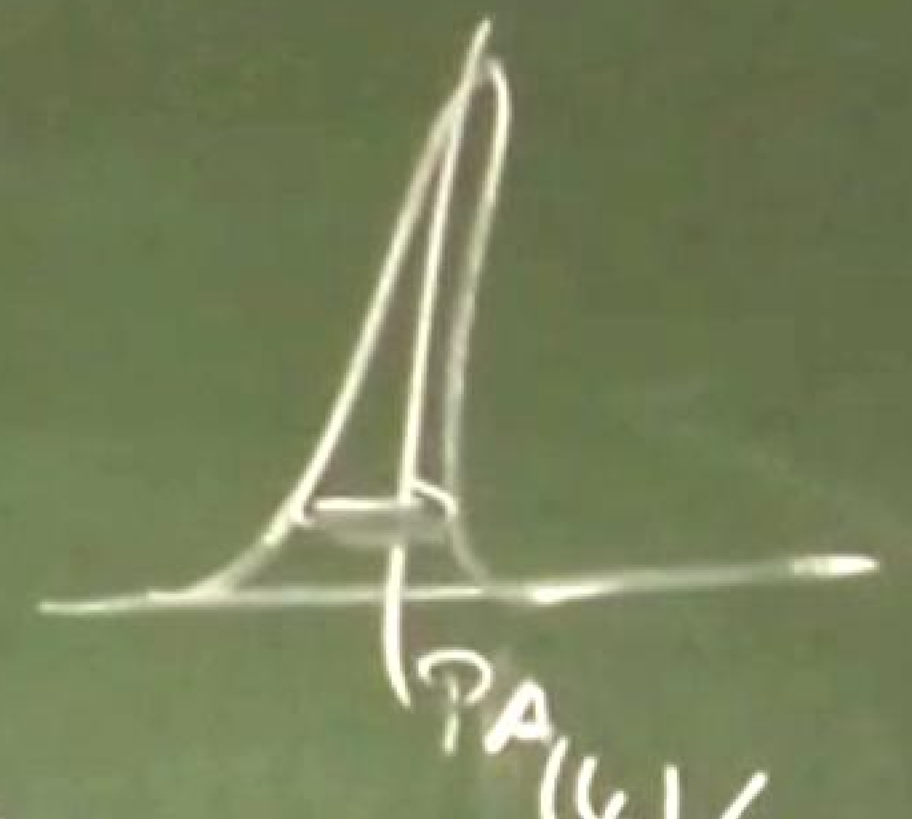
$q_B^2=\sum p_f^2-q_A^2$

$$=\int dq_A^2\ \delta\Big(\underbrace{\sqrt{q_A^2+m_A^2}+\sqrt{q_B^2+m_B^2}-\sum E_f}_{g(q_A^2)}\Big)\Bigg|_{q_{B}^2=\sum p_f^2-q_A^2}$$

$$=\frac{1}{|g'(q_A^2)|}\Bigg|_{g(q_A^2)=0}\overset{\circ}{=}\frac{1}{\left|\underbrace{\frac{q_A^2}{E_A}}_{v_A}-\underbrace{\frac{q_B^2}{E_B}}_{v_B}\right|}=\frac{1}{|v_A-v_B|}$$

$$\boxed{v_{grp}=\frac{\partial E(\vec{q})}{\partial\vec{q}}=\frac{\vec{q}}{E_{\vec{q}}}}$$

10) $\phi_i(\vec{u}_i)$ peaked around $\vec{p}_A,\vec{p}_B$



$$d\sigma\approx\left(\prod\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_{p_f}}\right)\frac{|\mathcal{M}(p_Ap_B\to\{p_f\})|^2}{2E_{p_A}2E_{p_B}|v_A-v_B|}$$

$$\underbrace{\int d^3u_A\ldots}\int d^3u_B\ldots\ \underbrace{|\phi_A(u_A)|^2}\,|\phi_B(u_B)|^2(2\pi)^4\delta^{(4)}(\underbrace{u_A+u_B}_{p_A+p_B}-\sum p_f)$$

1 $\quad$ 1

11)

$$\boxed{\begin{aligned}d\sigma&=\frac{1}{2E_{p_A}2E_{p_B}|v_A-v_B|}\left(\prod_f\frac{dp_f}{(2\pi)^3}\frac{1}{E_{p_f}}\right)|\mathcal{M}(p_Ap_B\to p_f)|^2\\&\times(2\pi)^4\delta(p_A+p_B-\sum p_f)\end{aligned}}$$

Special cases:

(2) Two final particles (p_1, p_2)

↪ center of mass frame $\vec{p}_A + \vec{p}_B = 0 = \vec{p}_1 + \vec{p}_2$

$$\boxed{\frac{d\sigma}{d\Omega} \overset{\circ}{=} \frac{1}{2E_{p_A}E_{p_B}|v_A - v_B|} \frac{|\vec{p}_1|}{(2\pi)^2 4 E_{cm}} |\mathcal{M}(p_A p_B \to p_1 p_2)|^2}$$

center of mass energy

(3) $m_A = m_B = m_1 = m_2$

$$\boxed{\frac{d\sigma}{d\Omega} \overset{\circ}{=} \frac{|\mathcal{M}(p_A p_B \to p_1 p_2)|^2}{64\pi^2 E_{cm}^2}}$$

$$E_{cm} = [E_{p_A} + E_{p_B}]_{cm} = \sqrt{(p_A + p_B)^2}$$

Recap

S-matrix:

$$\left\langle \vec{p}_1 \ldots \vec{p}_n \middle| \vec{k}_A \vec{k}_B \right\rangle_{in} = \lim_{T\to\infty} {}_{t_0}\left\langle \vec{p}_1 \ldots \vec{p}_n \middle| e^{-iH(2T)} \middle| \vec{k}_A \vec{k}_B \right\rangle_{t_0}$$

out: $T \to +\infty$ in: $T \to -\infty$

$$\equiv {}_{t_0}\left\langle \vec{p}_1 \ldots \vec{p}_n \middle| S \middle| \vec{k}_A \vec{k}_B \right\rangle_{t_0}$$

T-matrix:

$$S = \mathbb{1} + \underbrace{iT}_{\text{non-trivial scattering}}$$

Invariant matrix element:

$$\left\langle \vec{p}_1 \ldots \vec{p}_p \middle| iT \middle| \vec{k}_A \vec{k}_B \right\rangle = \underbrace{(2\pi)^4 \delta^{(4)}\left(k_A + k_B - \sum p_f\right)}_{\text{4-momentum conservation}} \cdot i \underset{?}{\boxed{\mathcal{M}\left(k_A k_B \to \{p_f\}\right)}}$$

Scattering cross section:

$$\boxed{d\sigma = \underbrace{\frac{1}{2E_{\vec{p}_A} 2E_{\vec{p}_B} |v_A - v_B|}}_{\text{not LI!}} \underbrace{\left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_{\vec{p}_f}}\right)}_{\text{LI}} \underbrace{\left|\mathcal{M}\left(p_A p_B \to \{p_f\}\right)\right|^2}_{\text{LI} \quad (\to \text{Weinberg})} \underbrace{(2\pi)^4 \delta^{(4)}\left(p_A + p_B - \sum p_f\right)}_{\text{LI}}}$$

2 outgoing particles, $m_A = m_B = m_1 = m_2$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\left|\mathcal{M}\left(p_A p_B \to p_1 p_2\right)\right|^2}{64\pi^2 E_{cm}^2}}$$

$$\left[E_{\vec{p}_A} + E_{\vec{p}_B}\right]_{cm} = \sqrt{(p_A + p_B)^2}$$

$$\vec{p}_A + \vec{p}_B = 0$$

4.6 Computing S-matrix elements from Feynman diagrams

Motivation

We want

1) $\langle \vec{p}_1 .. \vec{p}_n | S | \vec{p}_A \vec{p}_B \rangle = \lim_{T\to\infty} \langle \vec{p}_1 .. \vec{p}_n | e^{-iH2T} | \vec{p}_A \vec{p}_B \rangle$

Eigenstates of H_0

2) Problem

- $|p_A p_B\rangle_0 = \sqrt{2E_A}\sqrt{2E_B}\, a^\dagger_{p_A} a^\dagger_{p_B} |0\rangle$
- $|p_A p_B\rangle = ?$ $\qquad |p\rangle$: Eigenstate of H

3) Remember:

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle$$

4) Assume it holds

$$|p_A p_B\rangle = \lim_{T\to\infty(1-i\epsilon)} (??)\, e^{-iHT}|p_A p_B\rangle_0$$

5) If this holds:

$$\lim_{T\to\infty} e^{-iH2(T-i\epsilon)}$$

$$\langle \vec{p}_1 .. \vec{p}_n | S | \vec{p}_A \vec{p}_B \rangle \propto \langle \vec{p}_1 .. | \left(e^{-iHT(\cancel{1}-i\epsilon)}\right)^\dagger e^{-iH2T} e^{-iH(\cancel{T}-i\epsilon)} |p_A p_B\rangle_0$$

$$\propto \lim_{T\to\infty(1-i\epsilon)} {}_0\langle p_1 .. | e^{-iH2T} | p_A p_B\rangle_0$$

$$\propto \lim_{T\to\infty(1-i\epsilon)} {}_0\langle p_1 .. | T\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right] |p_A p_B\rangle_0$$

$$U(T,-T) = T\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right]$$

$$= e^{iH_0(T-t_0)}\, e^{-iH2T}\, e^{-iH_0(-T-t_0)}$$

6) Correct result:

$$\boxed{\langle \vec{p}_1 .. \vec{p}_n | iT | \vec{p}_A \vec{p}_B \rangle = \lim_{T\to\infty(1-i\epsilon)} \left\{ {}_0\langle \vec{p}_1 .. \vec{p}_n | T\exp\left[-i\int_{-T}^{T} dt\, H_I(t)\right] |p_A p_B\rangle_0 \right\}_{fc+a}}$$

fc+a "fully connected + amputated"

(LSZ reduction formula

Proof: P&S 7.2)

Interpretation + Application

$\cancel{\text{3}}$ ϕ^4-theory

1) λ^0-order

$${}_0\langle \vec{p}_1 \vec{p}_2 | \vec{p}_A \vec{p}_B \rangle_0 = \sqrt{2E_{p_1} 2E_{p_2} 2E_{p_A} 2E_{p_B}}$$

$$\langle 0 | a_{p_1} a_{p_2} \, a^\dagger_{p_A} a^\dagger_{p_B} | 0 \rangle$$

$$\overset{0}{=} 2E_{p_A} 2E_{p_B} (2\pi)^6$$

$$\left\{ \delta^{(3)}(\vec{p}_A - \vec{p}_1)\, \delta^{(3)}(\vec{p}_B - \vec{p}_2) + \delta^{(3)}(\vec{p}_A - \vec{p}_2)\, \delta^{(3)}(\vec{p}_B - \vec{p}_1) \right\}$$

(*)

=

→ States do not change

→ Contributes $\mathbb{1}$ in $S = \mathbb{1} + iT$

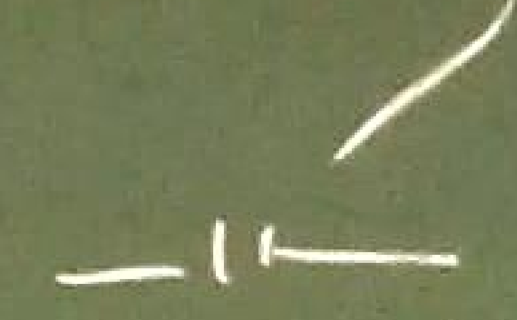
2) λ^1 order:

i) $${}_0\langle \vec{p}_1 \vec{p}_2 | -i\frac{\lambda}{4!} \int d^4x \, \underbrace{T(\phi_I^4(x))}_{} | p_A p_B \rangle_0$$

Wick's theorem

$: \phi_I^4(x) + \text{contractions} :$

=



ii) Careful:

$$\phi_I^+(x) | p \rangle_0 = \int \ldots\ldots \, a_k \, a_p^\dagger |0\rangle \overset{0}{=} e^{-ipx} |0\rangle$$

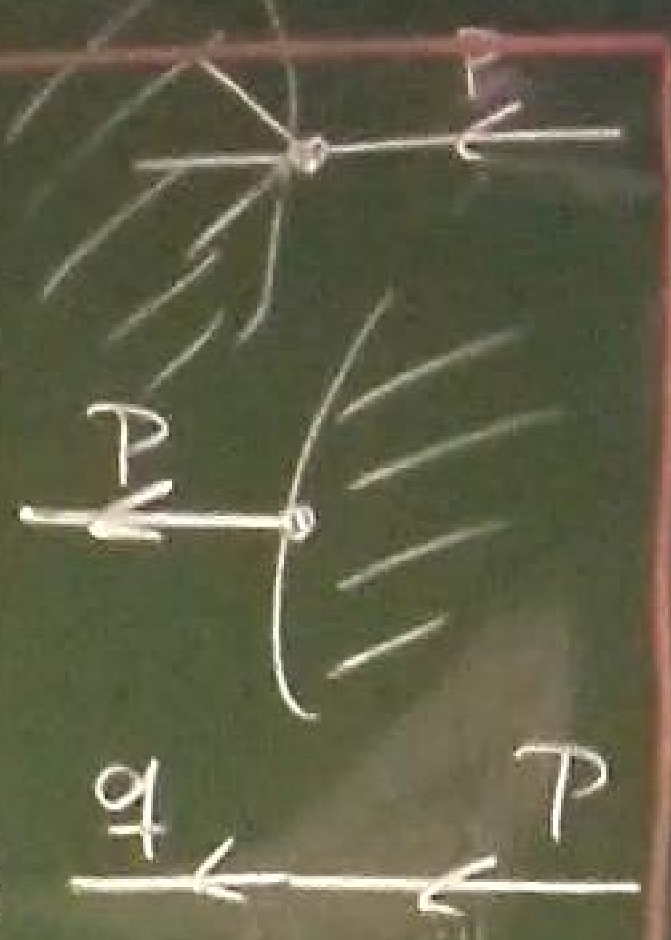
$${}_0\langle p | \phi_I^-(x) = \ldots\ldots\ldots = \langle 0 | e^{+ipx}$$

iii) Definitions:

$$\overbrace{\phi_I(x) | \vec{p}}\rangle \equiv e^{-ipx} |0\rangle \hat{=}$$

$$\overbrace{\langle \vec{p} | \phi_I(x)} \equiv \langle 0 | e^{+ipx} \hat{=}$$

$$\overbrace{\langle \vec{p} | \vec{q}} \rangle \equiv 2E_p (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \hat{=}$$



iv)

$${}_0\langle p_1 \ldots | T\{\phi_a \ldots \phi\} | p_A \ldots \rangle_0$$

$$= \left\{ \begin{array}{l} \text{Sum of all } \underline{\text{full}} \text{ contractions} \\ \text{of fields } \underline{\text{and}} \text{ external-state momenta} \end{array} \right\}$$

Example:

$${}_0\langle \vec{p}_1 \vec{p}_2 | \vec{p}_A \vec{p}_B \rangle_c = \langle p_1 p_2 | p_A p_B \rangle$$

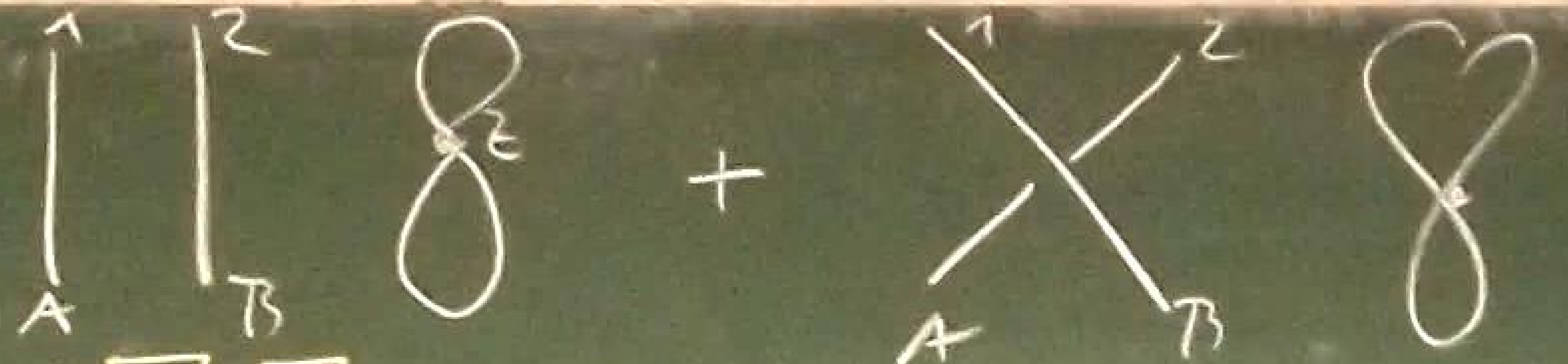
$$+ \langle p_1 p_2 | p_A p_B \rangle$$

$$= (*)$$

V)

$$-i\frac{\lambda}{4!}\int d^4x\, {}_0\langle p_1 p_2 | T\, \phi_I^4(x) | p_A p_B \rangle_0$$

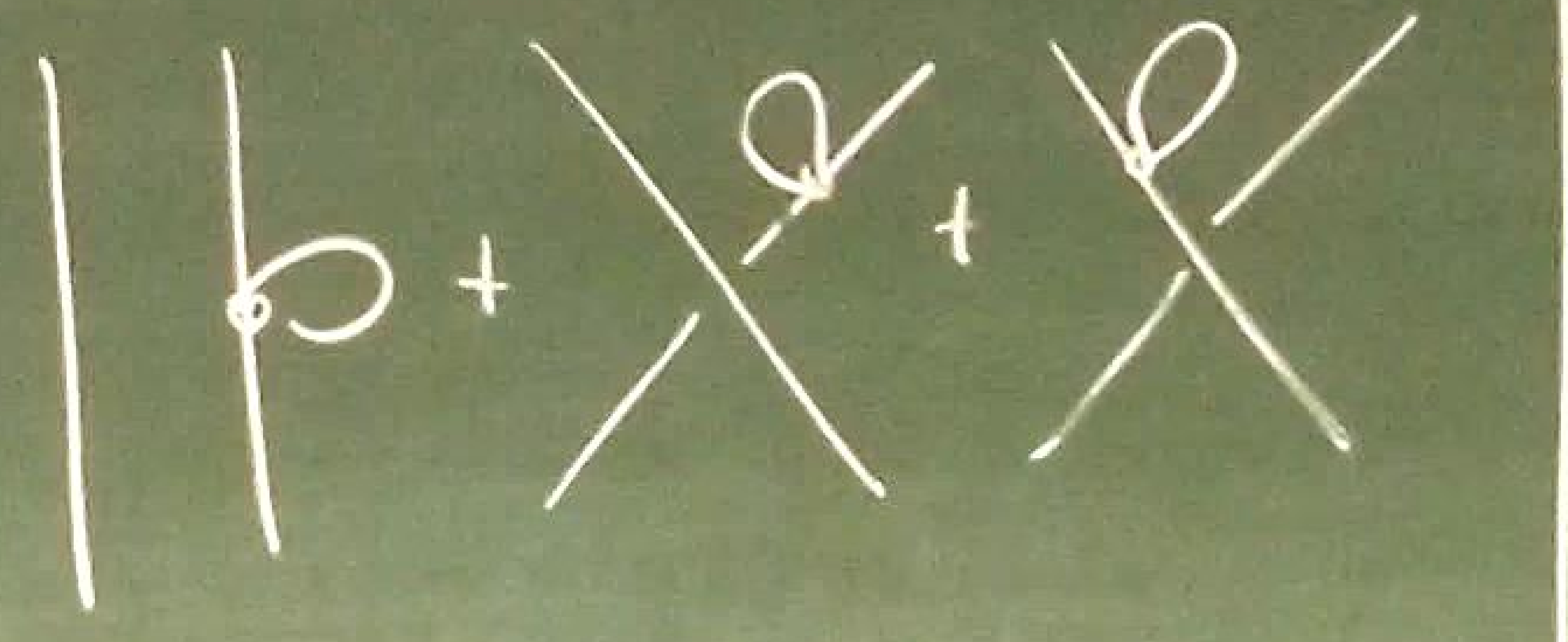
$=$



$$\langle p p | \phi\phi\phi\phi | p p \rangle$$

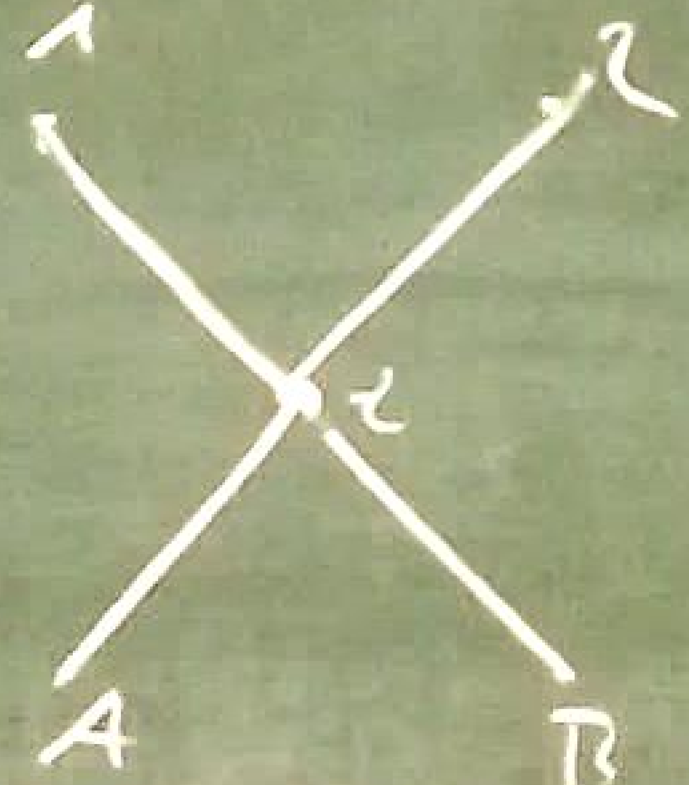
$+$

$$\langle p_1 p_2 | \phi\phi\phi\phi | p_A p_B \rangle$$



$+$

$$\langle p_1 p_2 | \phi\phi\phi\phi | p_A p_B \rangle$$



fully connected

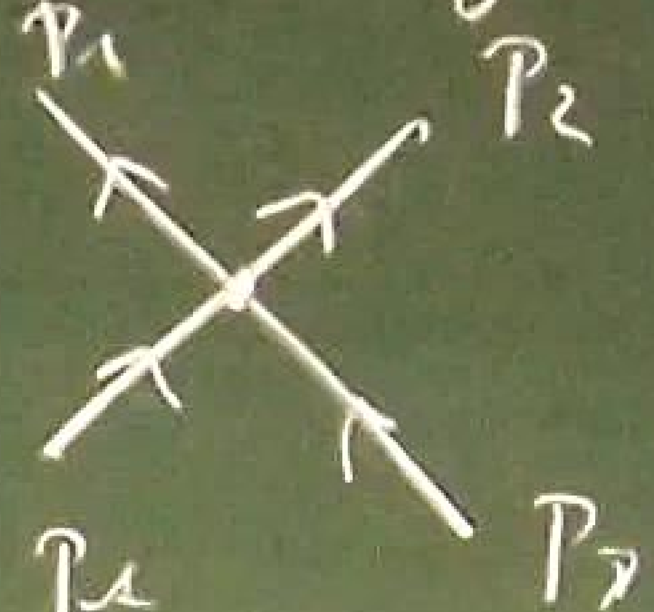
→ Terms with $\phi\phi\phi\phi$ and $\phi\phi\phi\phi$ do not contribute to T

→ only <u>fully connected</u> diagrams contribute

VI)

1st order

$$\langle p_1 p_2 | iT | p_A p_B \rangle =$$



$$= \cancel{4!}\left(-i\frac{\lambda}{\cancel{4!}}\right)\int d^4x\; e^{-i(p_A+p_B-p_1-p_2)x}$$

$$= (-i\lambda)(2\pi)^4\,\delta^{(4)}(p_A+p_B-p_1-p_2)$$

$$\overset{\text{def.}}{=} i\,\boxed{\mathcal{M}}\cdot(2\pi)^4\,\delta^{(4)}(p_A+p_B-p_1-p_2)$$

$$\rightarrow \mathcal{M}(p_A p_B \rightarrow p_1 p_2) \doteq -\lambda + \mathcal{O}(\lambda^2)$$

$$\rightarrow \sigma_{total} = \frac{\lambda^2}{32\pi E_{cm}^2}$$

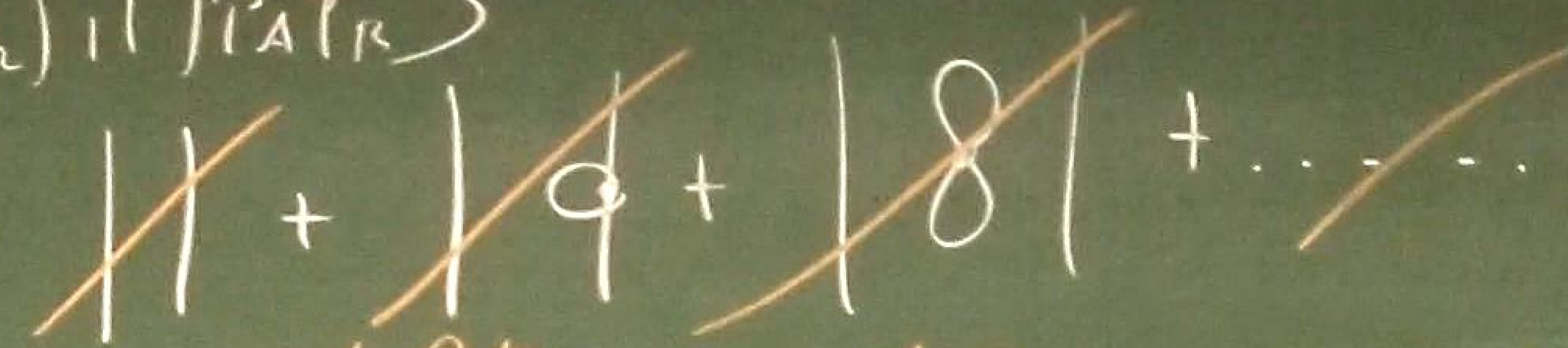
$\uparrow$

$$\int d\Omega \frac{d\sigma}{d\Omega} = 4\pi$$

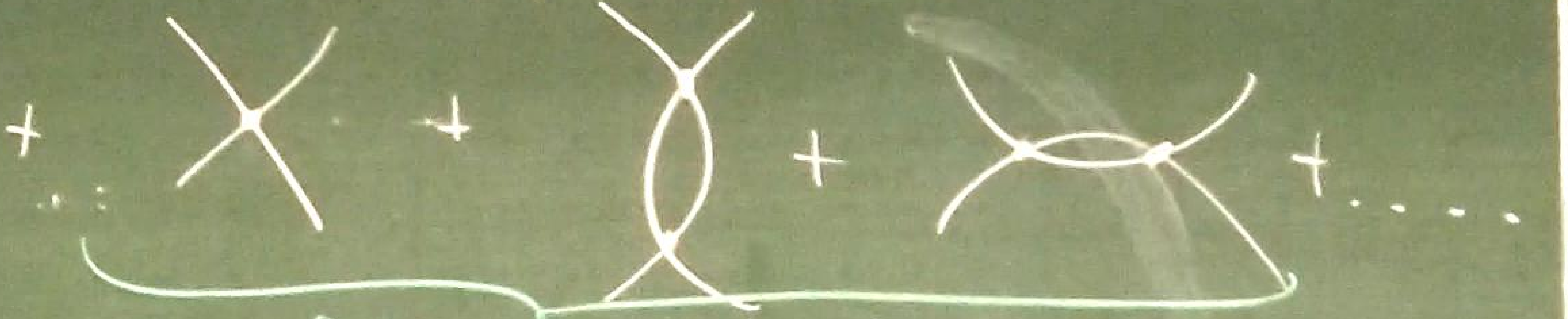
3) Higer-order contribution

$$\langle p_1 p_2 | iT | p_A p_B \rangle$$

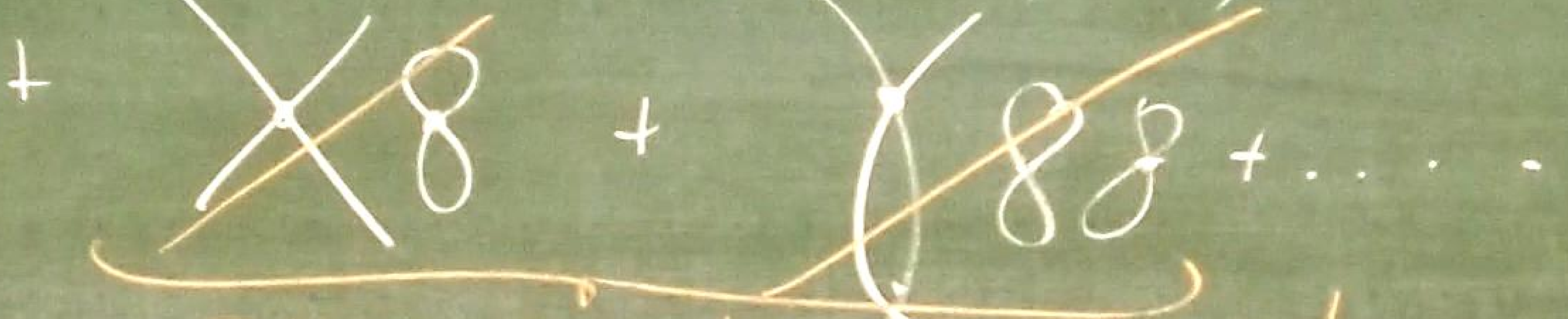
$=$



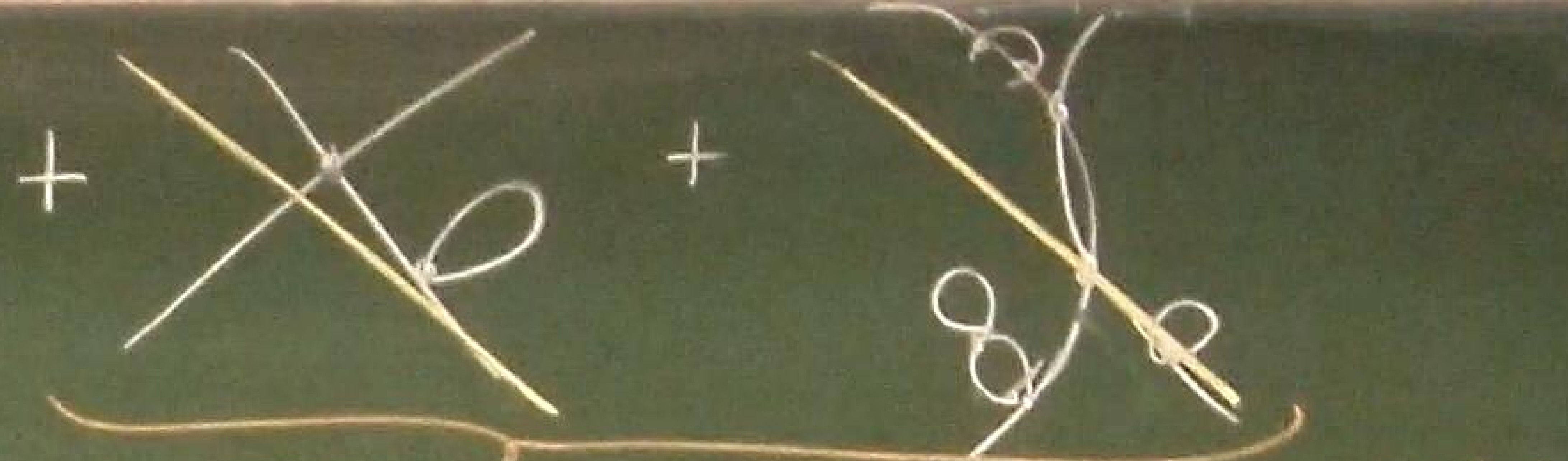
not fully connected



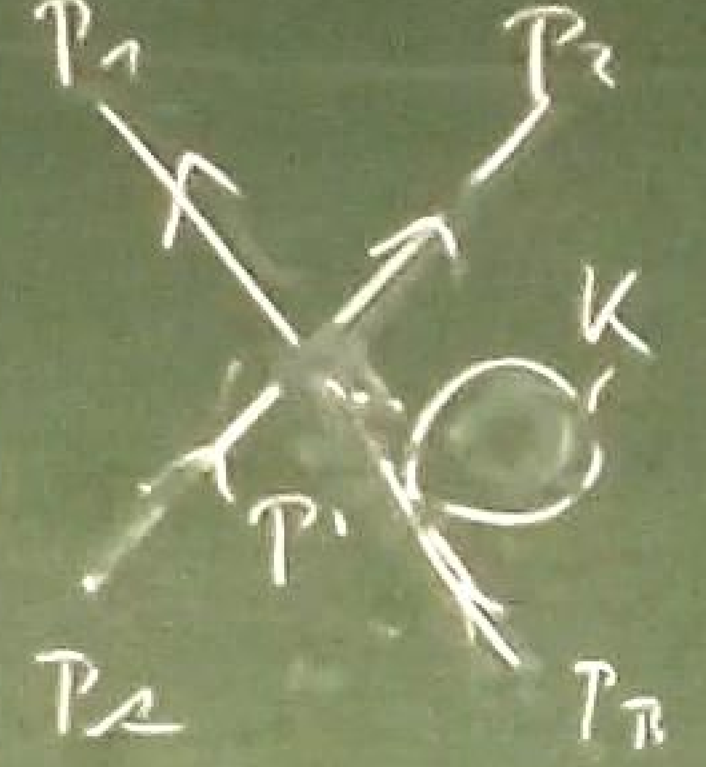
fully connected (amputated)



Bubbles exponentiate and drop out



Fully connected <u>but not</u> amputated



$$\hat{=} \frac{1}{2} \int \frac{d^4p'}{(2\pi)^4} \frac{i}{p'^2 - m^2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2}$$

$$\cdot (-i\lambda)(2\pi)^4 \delta(p_A + p' - p_1 - p_2)$$

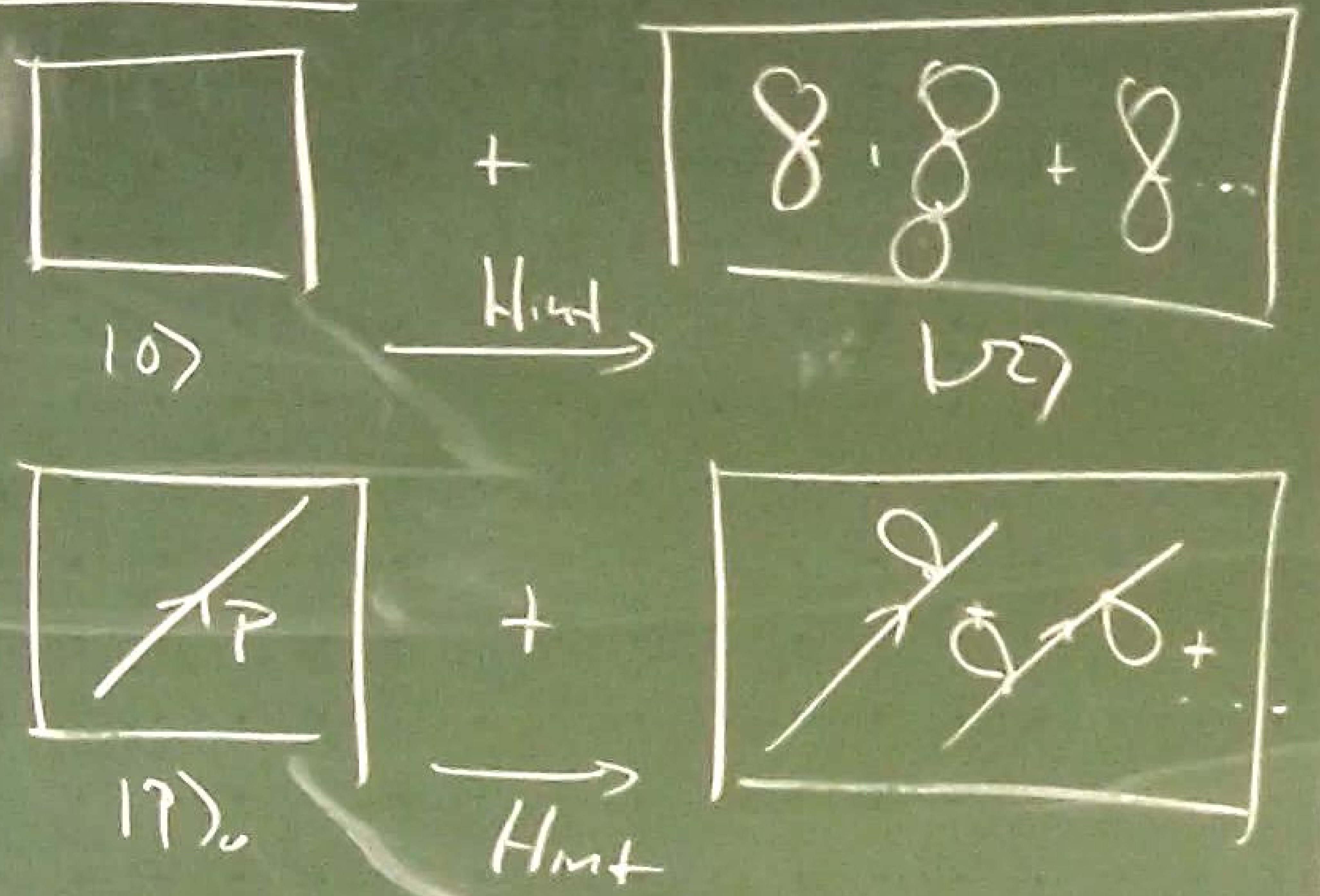
$$\cdot (-i\lambda)(2\pi)^4 \delta(p_B - p')$$

$$\sim \frac{-i}{p_B^2 - m^2 + i\epsilon} = \frac{1}{0} = \infty$$

$p_B^2 = m^2$ $\quad E_p = p^0 = \sqrt{\vec{p}^2 + m^2}$

$\longrightarrow$ Eq (4.132) makes only sense without these diagrams!

Interpretation:

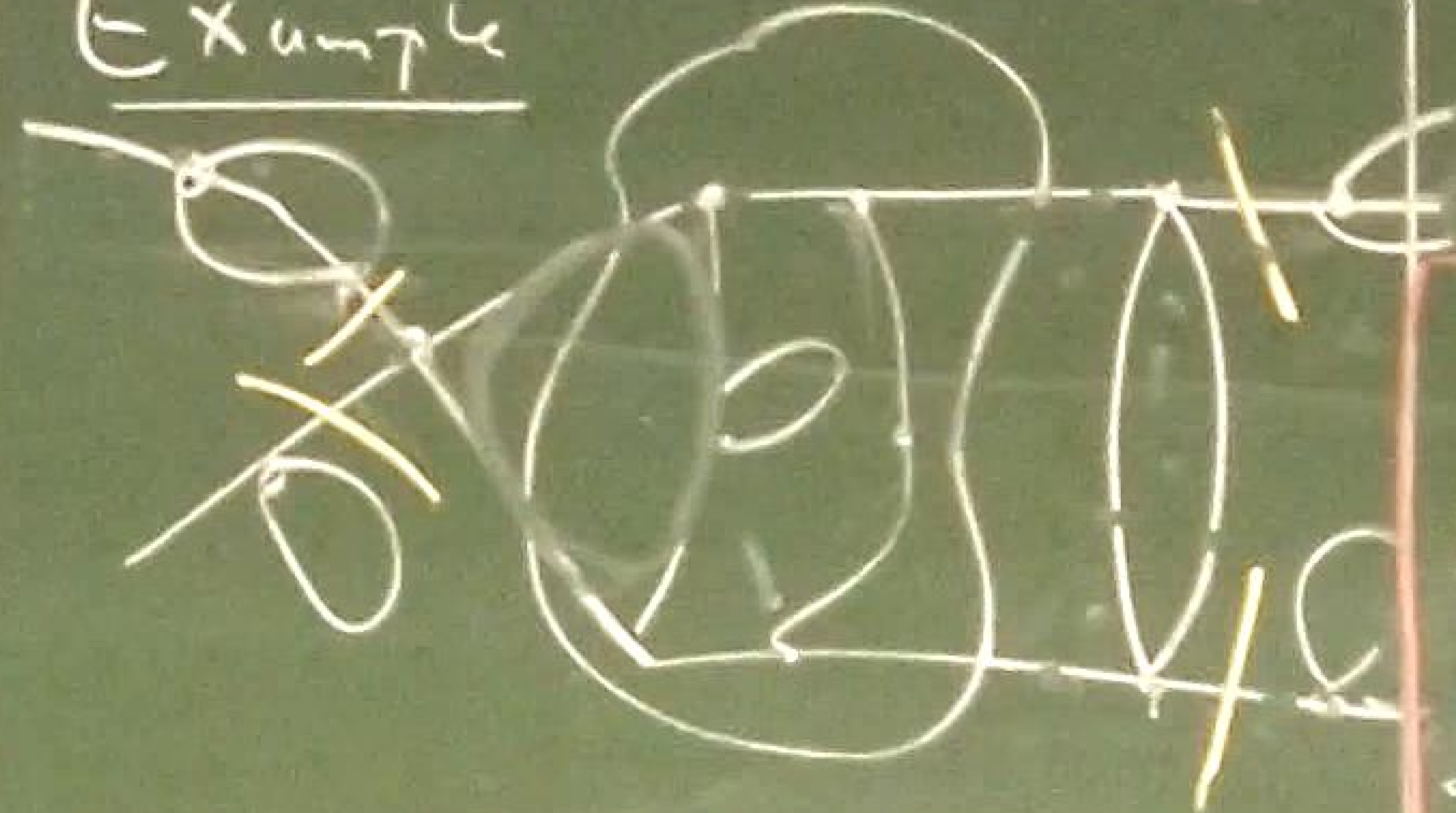


$\rightarrow$ Not related to scattering!

$\rightarrow$ Drop non-amputated diagrams

4] Amputation of diagram:

Example



Momentum space Feynman rules

1. Edges $\xrightarrow{p} = \frac{i}{p^2-m^2+i\epsilon}$
2. Vertex p_1, p_2, p_3, p_4 $= (-i\lambda)(2\pi)^4 \delta(p_1+p_2-p_3-p_4)$
3. External lines $\xleftarrow{q} = 1$
4. Integrate over all internal momenta $\prod_i \int \frac{d^4 p_i}{(2\pi)^4}$
5. Divide sym factor $\frac{1}{s}$

5]

$$\boxed{4.132} = iM (2\pi)^4 \delta^{(4)}(p_A+p_B-\sum p_f)$$

$$= \left\{ \begin{array}{l} \text{Sum of all } \underline{\text{fully connected}}, \\ \underline{\text{amputated}} \text{ Feynman diagrams} \\ \text{with } p_A, p_B \text{ incoming and } \{p_f\} \\ \text{outgoing} \end{array} \right\}$$

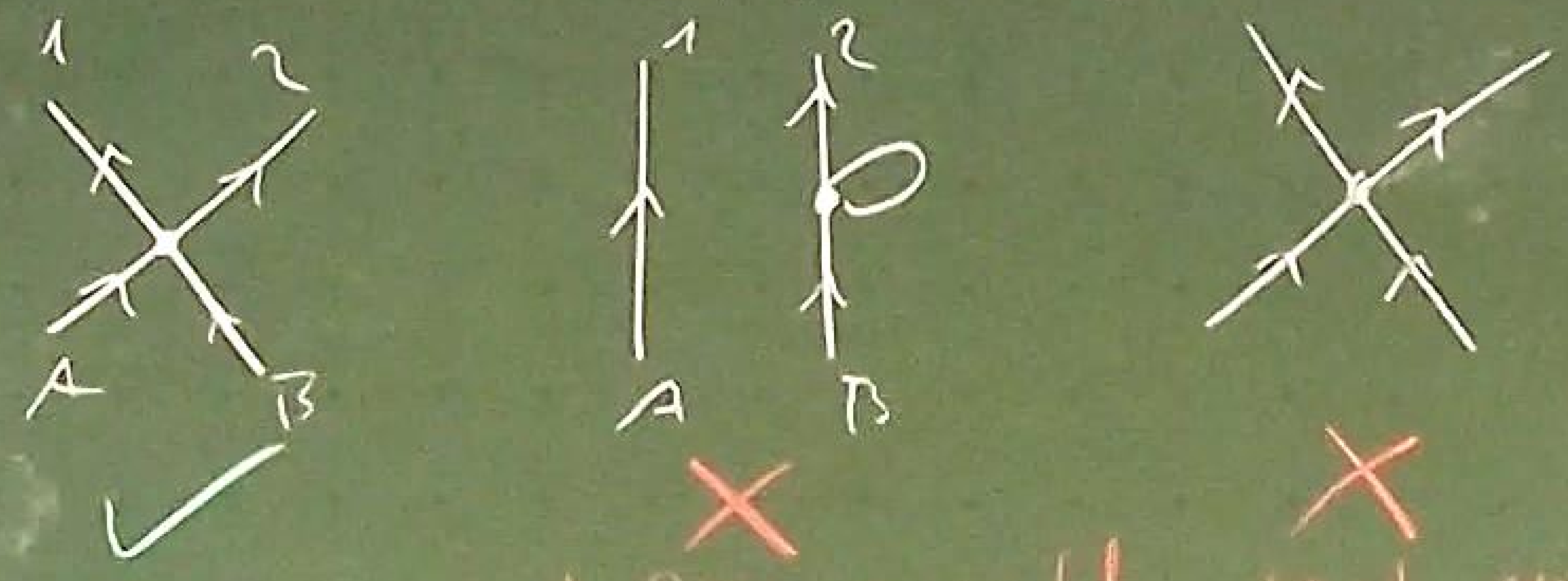
6] Position-space Feynman rules

1. For each edge $\overset{x}{\bullet}\!\!-\!\!-\!\!-\!\!\overset{y}{\bullet} = D_F(x-y)$
2. For each vertex $\times_z = (-i\lambda)\int d^4z$
3. For each external line $z \longleftarrow = e^{-ipz}$
4. Divide by sym factor: $\frac{1}{S}$

Recap:

$$\langle \vec{p}_1 \ldots \vec{p}_n | iT | \vec{p}_A \vec{p}_B \rangle = i\mathcal{M}(p_A p_B \to \{p_f\})\,(2\pi)^4\,\delta^{(4)}(p_A + p_B - \textstyle\sum p_f)$$

$= \sum \Big\{$ <u>fully connected</u> + <u>amputated</u> Feynman diagrams with $\vec{p}_A, \vec{p}_B$ incoming / $\{p_f\}$ outgoing $\Big\}$



not fully connected

not amputated

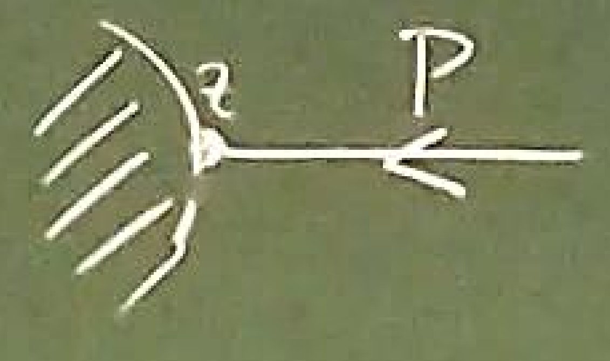
Feynman rules in

Position space

1. Edges $x \underline{\qquad} y = D_F(x-y)$

2. Internal vertices


$= (-i\lambda)\int d^4 z$

3. External lines


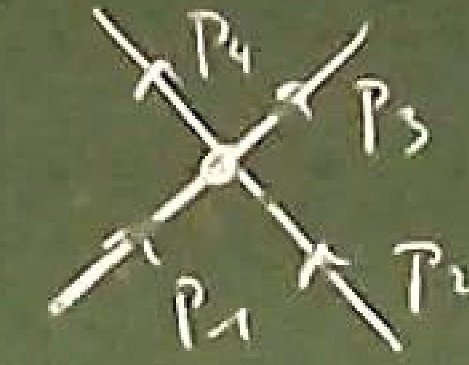
$= e^{-ipz}$

4. —

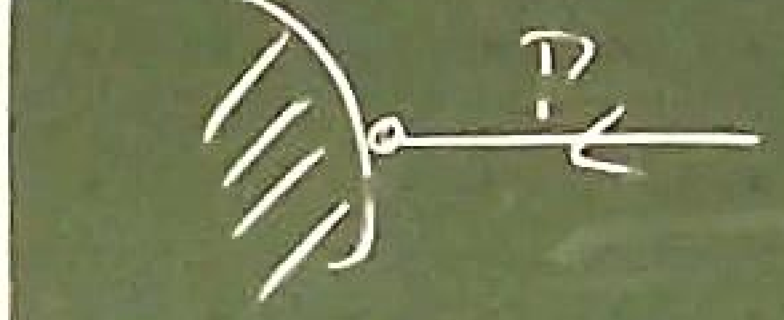
5. Divide by sym. factor $\frac{1}{S}$

momentum space

$\xrightarrow{p} = \frac{i}{p^2 - m^2 + i\varepsilon}$



$= (-i\lambda)(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4)$



$= 1$

Integrate internal momenta : $\prod_i \int \frac{d^4 p_i}{(2\pi)^4}$

— , —

4.7. Feynman Rules for QED

Setting the Stage

1) **Fields**

Fermions: $\Psi(x)$ (bispinor field)
Photons: $A_\mu(x)$ (vector field)

2) **Lagrangian**

$$\mathcal{L}_{QED} = \mathcal{L}_{Dirac} + \mathcal{L}_{Maxwell} + \mathcal{L}_{int}$$

$$= \bar\Psi(i\partial\!\!\!/ - m)\Psi - \frac{1}{4}\underbrace{F_{\mu\nu}}_{\partial_\mu A_\nu - \partial_\nu A_\mu}F^{\mu\nu} - \boxed{e}\,\underbrace{\bar\Psi\gamma^\mu\psi}_{j^\mu}A_\mu$$

mass of fermion

charge of fermion

$$= \bar\Psi(i\underbrace{D\!\!\!/}_{\text{covariant derivative}} - \boxed{m})\Psi - \frac{1}{4}F^2$$

$$D_\mu = \partial_\mu + ieA_\mu$$

3) **Hamiltonian**

$$H_{QED} = H_{Dirac} + H_{Maxwell} + \underbrace{H_{int}}_{e\int d^3x\,\bar\Psi\gamma^\mu\psi A_\mu}$$

4) **EOM**

$$(iD\!\!\!/ - m)\Psi = 0$$

$$\partial_\nu F^{\nu\mu} = j^\mu$$

Note 4.2 $\mathcal{L}_{QED}$ is invariant under local $U(1)$ gauge transformations,

$$\psi'(x) = e^{ie\alpha(x)}\psi(x)$$

$$A'(x) = A_\mu(x) - \partial_\mu\alpha(x)$$

for arbitrary $\alpha: \mathbb{R}^{1,3} \to \mathbb{R}$.

Note 43 Standard model:

$$``\mathcal{L}^{SM}_{QED}" = \sum_f{}' \Big\{ \overline{\Psi}_f (i\not{\partial} - m) \Psi_f - \underbrace{q_f \overline{\Psi}_f \gamma^\mu \psi}_{J^\mu_f} A_\mu \Big\} - \frac{1}{4} F^2$$

$f \in \{$Leptons, Quarks$\}$

$= \{e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau, u, d, c, s, t, b\}$

Notes on Fermion / Dirac sector

Remember: $$S^{ab}_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} e^{-ip(x-y)}$$

$$= \begin{cases} \langle 0| \psi_a(x) \overline{\psi}_b(y) |0\rangle & x^0 > y^0 \\ -\langle 0| \overline{\psi}_b(y) \psi_a(x) |0\rangle & y^0 > x^0 \end{cases}$$

$$= \langle 0| T\{\psi_a(x) \overline{\psi}_b(y)\} |0\rangle$$

→ need Wick's theorem

1] Time-ordering:

$\psi \in \{\psi, \overline{\psi}\}$

$$T\{\psi_{\sigma_1} \dots \psi_{\sigma_N}\} \equiv (-1)^{\#} \psi_1 \dots \psi_N$$

$(-1)^{\#}$: Signum of permutation σ $\qquad x^0_1 > \dots > x^0_N$

2] Normal order.

$X \in \{a^s_p, a^{s+}_p, b^s_p, b^{s+}_p\}$

$$:X_1 \dots X_N: \equiv (-1)^{\#} (\text{creation operators}) \times (\text{annihilation op.})$$

$\#$: number of operator interchanges

3] Contractions:

$$\overbrace{\psi_a(x)\psi_b(y)} \equiv T\{\psi_a(x)\psi_b(y)\} - :\psi_a(x)\psi_b(y):$$

$$\to \overbrace{\psi_a(x)\overline{\psi}_b(y)} \stackrel{\circ}{=} \begin{cases} \{\psi^+_a(x), \overline{\psi}^-_b(y)\} & x^0 > y^0 \\ -\{\overline{\psi}^+_b(y), \psi^-_a(x)\} & x^0 < y^0 \end{cases}$$

$$= S^{ab}_F(x-y)$$

$$\overbrace{\psi_a(x)\psi_b(y)} \stackrel{\circ}{=} 0$$

$$\overbrace{\overline{\psi}_a(x)\overline{\psi}_b(y)} \stackrel{\circ}{=} 0$$

4] **Contraction + Normal order.**

$$:A\,\overline{\psi_a(x)\,B\,\psi_b(y)}\,C:$$

$$\equiv(-1)^{\#}\,\overline{\psi_a(x)\psi_b(y)}\;:ABC:$$

# of operator interchanges

(ψ_a with A, ψ_b with AB)

5] **Wick's theorem**

$$\boxed{T\{\psi_a(x_1)\psi_b(x_2)\ldots\} = :\psi_a(x_1)\psi_b(x_2) + \text{all possible contractions}:}$$

Notes on Photon/Maxwell Sector

1] **Observation:** A^μ: four deg. of freedom

but photon has only 2 dof.

2] **Problem:** Gauge invariance

→ Unphysical dof

→ Fix gauge to quantize only phys dof

3] **Different solutions:**

- Coulomb gauge: $\nabla\vec{A}=0$
- Lorentz gauge $\partial_\mu A^\mu=0$ is Lorentz invariant $\Rightarrow$ Gupta-Bleuler quantization
- Faddeev-Popov procedure ↪ later

4] **Motivate:**

i] $\mathcal{L}$ $\partial_\mu A^\mu=0$ $\xrightarrow{\text{EOM}}$ $\underbrace{\partial^2}_{\Box} A^\mu=0$

ii] Expand field in classical solutions.

$$A_\mu(x)=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}\sum_{r=0}^{3}\Big[a^r_{\vec p}\,\underbrace{\boxed{\varepsilon^r_\mu(p)}}_{\text{polarization vectors.}}e^{-ipx}+a^{r\dagger}_{\vec p}\,\varepsilon^{r*}_\mu(p)e^{ipx}\Big]$$

$p^0=|\vec p|$

$$\boxed{p^2=0}$$

5] **Result:**

i) For physical (external) photons:

$$\boxed{\varepsilon^\mu(p)=\begin{pmatrix}0\\ \vec\varepsilon(p)\end{pmatrix},\qquad \underbrace{\vec\varepsilon(p)\cdot\vec p=0}_{\text{transverse pol.}}}$$

→ Two $r,s = 1,2$ independent bosonic modes for each momentum $\vec p$:

$$\boxed{[a_p^{r\mu}, a_q^{s\dagger\nu}] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec p - \vec q)}$$

$-g^{\mu\nu} = -1$ (+) (−) $+1$

$\mu,\nu = 0,1,2,3$

ii) Propagator

$$\boxed{\langle 0 | T\{A_\mu(x) A_\nu(y)\} | 0 \rangle \overset{*}{=} \int \frac{d^4q}{(2\pi)^4} \frac{-i g_{\mu\nu}}{q^2 + i\varepsilon} e^{-iq(x-y)}}$$

Feynman rules

1. Expectations:

a) Fermions: $a \longrightarrow b$ $\qquad S_F^{ba} = \psi_b \bar\psi_a$

Photons:  $\quad A_\mu(x) A_\nu(x)$

→ Two particle types (anti)fermions / photons

→ Fermions/Antifermions: $|\vec p, s\rangle_{a,b}$ spin ↑↓

Photon: $|\vec p, \xi\rangle$ polarization

b) Interaction: $(\bar\Psi_b \gamma^\mu_{ba} \psi_a) A$

→ Vertex: 

2) Momentum-space Feynman rules

- —● internal vertex
- —| external "vertex" $(\ldots)$
- — virtual cut

$\gamma^\mu \cdot \psi$ $\qquad \bar\Psi \gamma^\mu \psi A_\mu$

$(\ldots)(\ldots)^\mu$

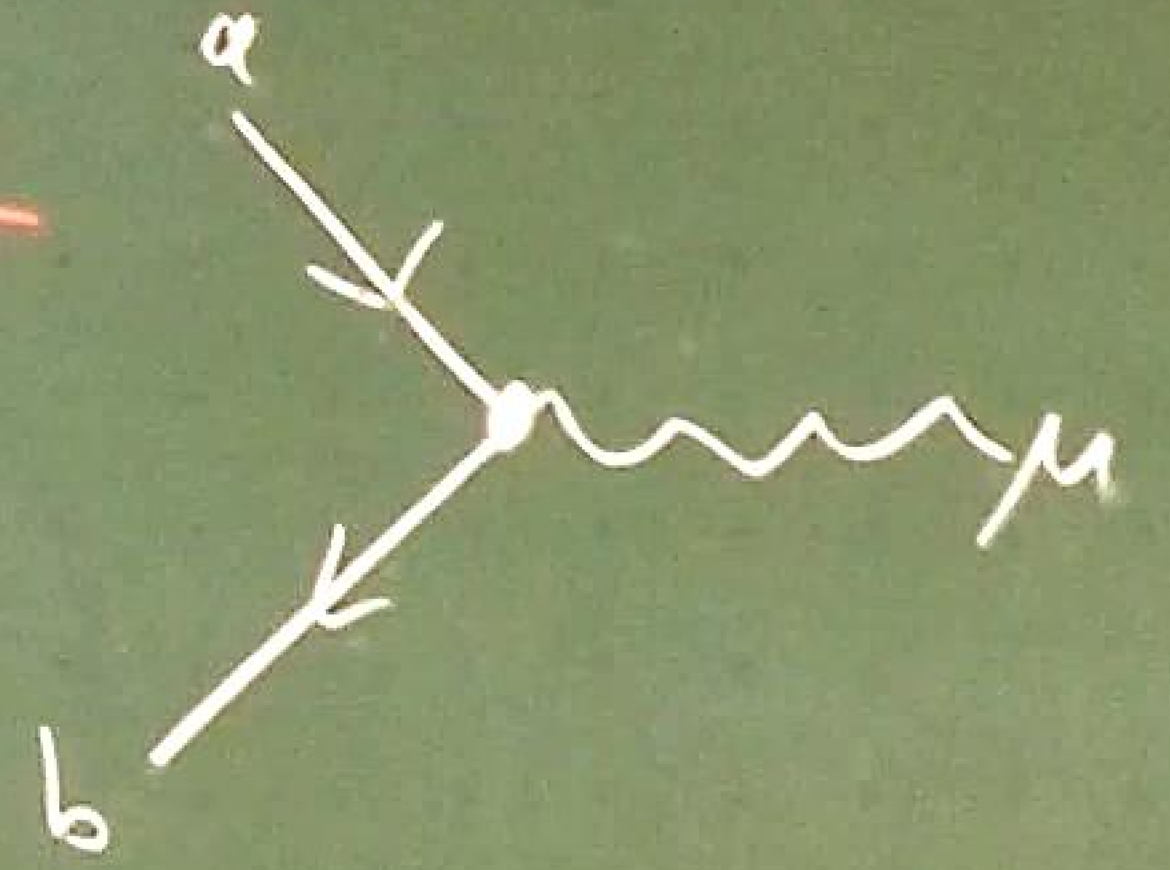
Propagators

Fermions: $a \xrightarrow{\;p\;} b = \dfrac{i(\not{p}+m)_{ba}}{p^2-m^2+i\varepsilon} \hat{=} \overbrace{\psi_b(x)\bar{\psi}_a(y)}$

$a \longrightarrow b$ simplified

Photons: $\mu \overset{\xrightarrow{q}}{\sim\sim\sim\sim} \nu = \dfrac{-ig_{\mu\nu}}{q^2+i\varepsilon} \hat{=} \overbrace{A_\mu(x)A_\nu(y)}$

Vertices



$= -ie\,\gamma^\mu_{ba} \hat{=} -ie\int d^4z\,\gamma^\mu_{ba}$

External legs:

Fermions:

$c \underset{\xleftarrow{p}}{\longleftarrow|}\, s = u_c^s(p) \hat{=} \overbrace{\psi_c\,|p,s\rangle}_a$

$s\, \underset{\xleftarrow{p}}{|\longleftarrow} c = \bar{u}_c^s(p) \hat{=} {}_a\overbrace{\langle p,s|\,\bar{\psi}_c}$

Antifermions:

$c \underset{\xleftarrow{p}}{\longrightarrow|}\, s = \bar{v}_c^s(p) \hat{=} \overbrace{\bar{\psi}_c\,|p,s\rangle}_b$

$s\, \underset{\xleftarrow{p}}{|\longrightarrow} c = v_c^s(p) \hat{=} {}_b\overbrace{\langle p,s|\,\psi_c}$

Photons:

$\mu \underset{\xleftarrow{q}}{\sim\sim\sim|}\, r = \varepsilon_\mu^r(q) \hat{=} \overbrace{A_\mu|q,r\rangle}$

$r\, \underset{\xleftarrow{q}}{|\sim\sim\sim} \mu = \varepsilon_\mu^{r*}(q) \hat{=} \overbrace{\langle q,r|A_\mu}$

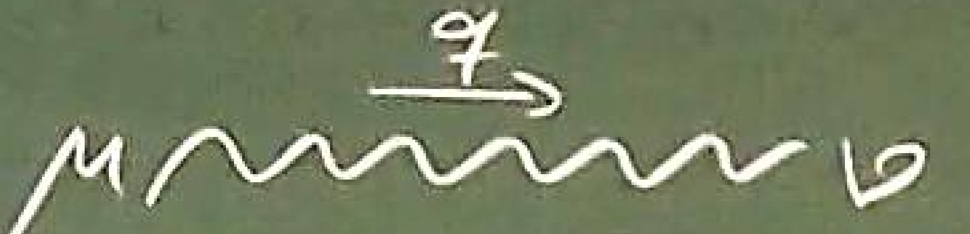
Evaluate:

1. Impose momentum conservation at internal vertices
2. Integrate over all undetermined momenta
3. Compute the sign of the diagram

Momentum space Feynman rules for scattering amplitudes in QED

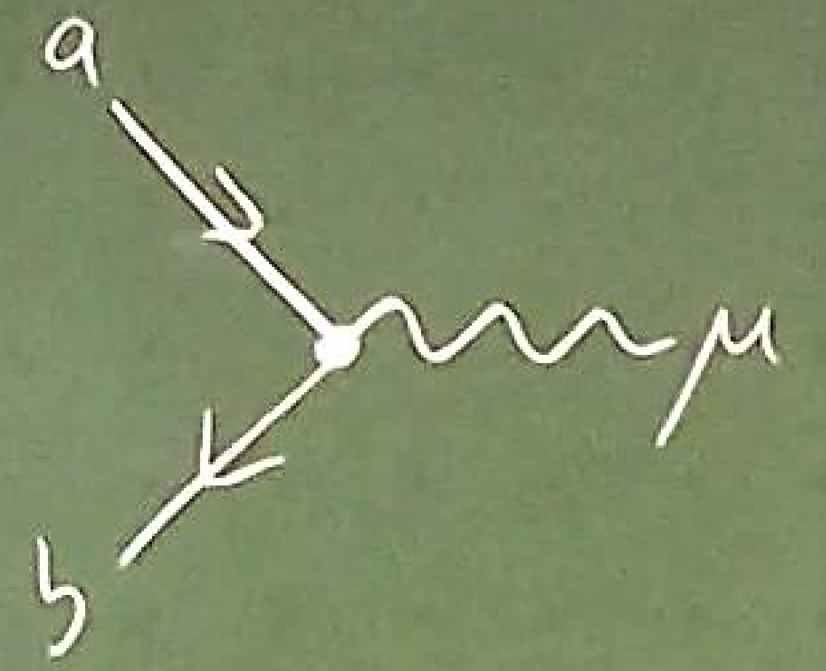
Propagators:

$= \frac{i(\not p + m)_{ba}}{p^2 - m^2 + i\varepsilon}$



$= \frac{-i g_{\mu\nu}}{q^2 + i\varepsilon}$

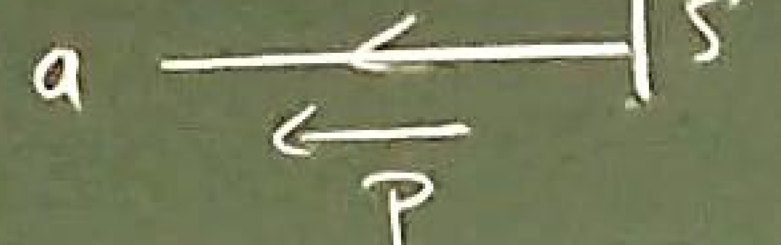
Vertices:



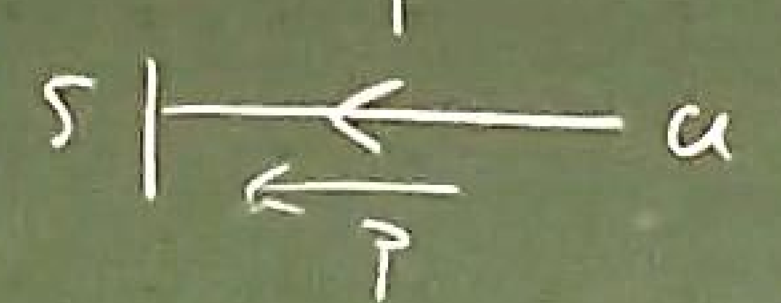
$= -ie\gamma^{\mu}_{ba}$

External legs:

Fermion



$= u^s_a(p)$ in



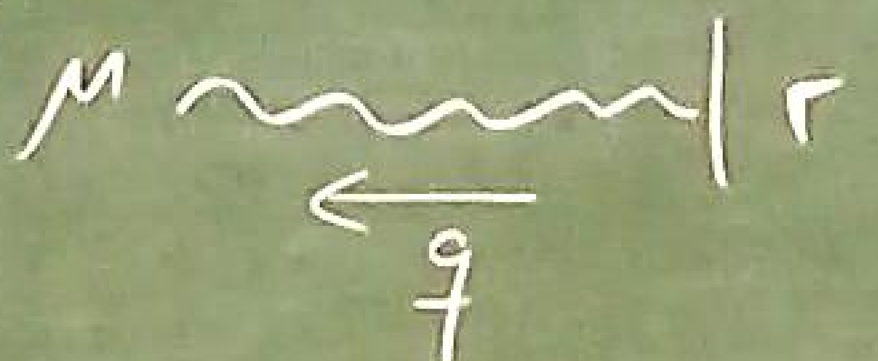
$= \bar{u}^s_a(p)$ out

Anti-fermion

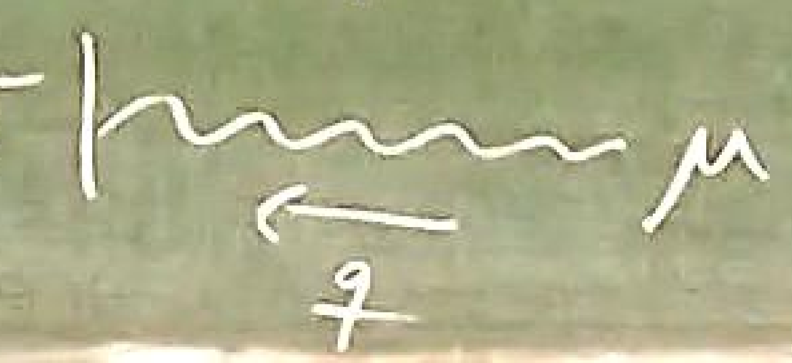
$= \bar{v}^s_a(p)$ in

$= v^s_a(p)$ out

Photon



$= \varepsilon^r_\mu(q)$ in



$= \varepsilon^{r*}_\mu(q)$ out

Evaluation:

1. Impose momentum cons. at vertices
2. Integrate undetermined momenta
3. Compute overall sign of the diagram

Minimal coupling

= only couple to first moments of charge distribution (= charges)

$$\mathcal{L} = \bar{\Psi}(i\not\partial - m)\Psi - \frac{1}{4}F^2$$

$$- e \underbrace{\bar{\Psi}\gamma^\mu\Psi}_{\text{charge current}} A_\mu \quad \Big\} \text{ minimal coupling}$$

$$\overset{\frac{eK}{8m}}{-a}\ \bar{\Psi}\sigma_{\mu\nu}\Psi F^{\mu\nu} \quad \Big\} \text{ Pauli coupling}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\nu$$

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

$$\Rightarrow (i\not{D} - m - a\sigma^{\mu\nu}F_{\mu\nu})\Psi = 0$$

$$\Rightarrow \vec{\mu}_{\text{eff}} = \frac{e}{2m}\underbrace{(2+K)}_{g}\vec{S}$$

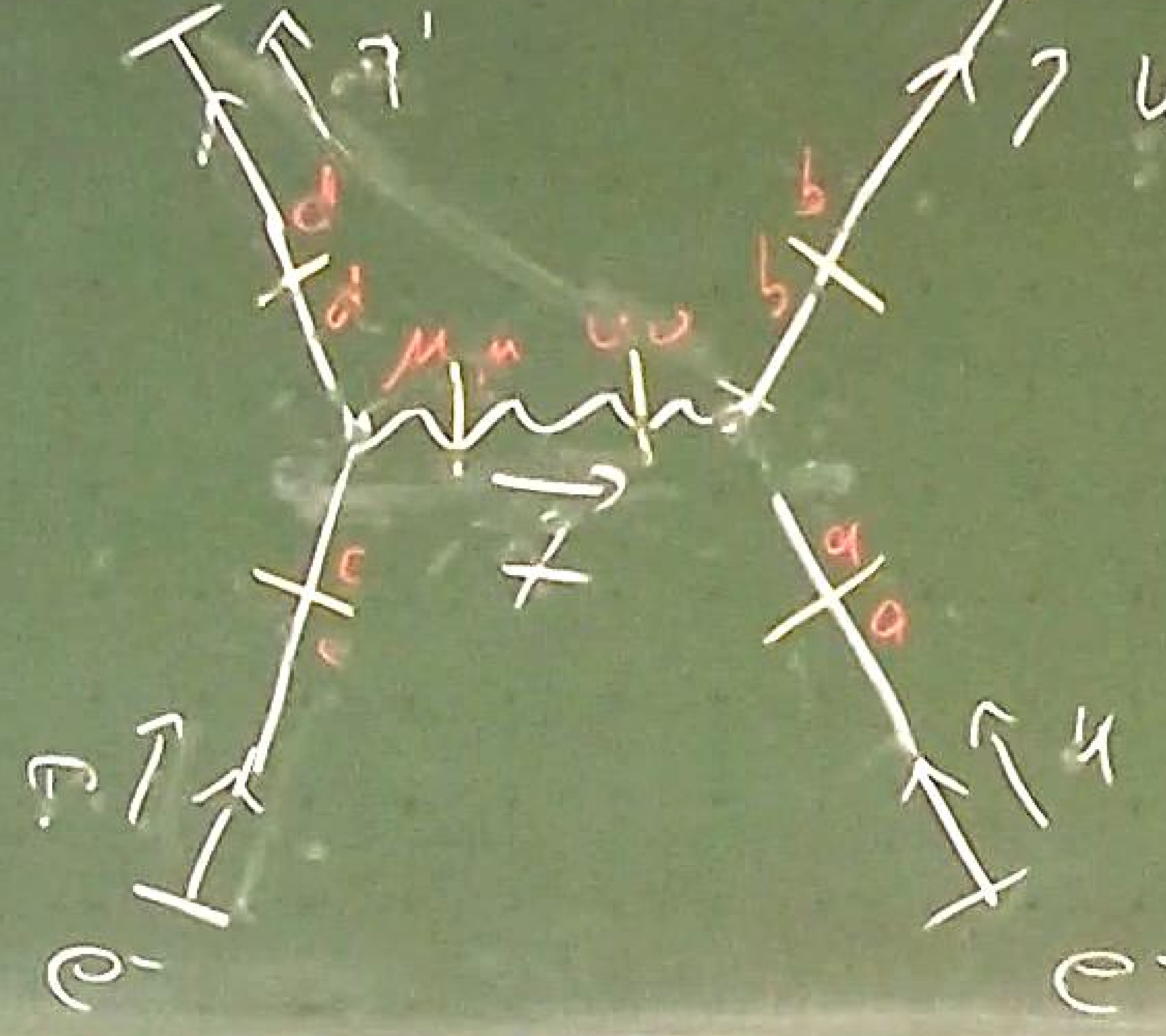
The Coulomb potential

1) Møller scattering:

$$e^- + e^- \longrightarrow e^- + e^-$$

i) Contribution to tree-level amplitude:

$$i\mathcal{M}(e^-(p)e^-(k) \longrightarrow e^-(p')e^-(k'))$$

$=$ 

$+ \ldots$

$$= \sigma \cdot \bar{u}_d(p')(-ie\gamma^\mu_{dc})u_c(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}_b(k')(-ie\gamma^\nu_{ba})u_a(k)$$

$$= \sigma \; \bar{u}(p')(-ie\gamma^\mu)u(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}(k')(-ie\gamma^\nu)u(k)$$

$$p - p' = q = k' - k$$

ii) Nonrelativistic limit. $|\vec{p}|^2 << m^2$

- $u(p) = \begin{pmatrix} \sqrt{p\sigma}\,\xi \\ \sqrt{p\bar{\sigma}}\,\xi \end{pmatrix} \approx \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$
- $\frac{1}{(p-p')^2} \approx \frac{-1}{|\vec{p}-\vec{p}'|^2}$

$$\rightarrow \bar{u}(p')\gamma^\mu u(p) \approx \begin{cases} 2m\,\xi^\dagger_{p'}\xi_p & \mu = 0 \\ 0 & \mu = 1,2,3 \end{cases}$$

$$\rightarrow i\mathcal{M} = \boxed{\sigma \frac{-ie^2}{|\vec{p}-\vec{p}'|^2}} (2m\,\xi^\dagger_{p'}\xi_p)(2m\,\xi^\dagger_{k'}\xi_k)$$

iii) Compare with non-relativistic scattering theory (first Born approxim.)

$$\langle p'|iT|p\rangle \overset{*}{=} \boxed{-i\hat{V}(\vec{q})}(2\pi)\,\delta(E_{\vec{p}'} - E_{\vec{p}})$$

$$\vec{q} = (\vec{p}' - \vec{p})$$

Fourier transform of scattering potential

$$\rightarrow \hat{V}(\vec{q}) = \sigma \frac{e^2}{|\vec{q}|^2} \Longrightarrow \boxed{V(\vec{r}) = \sigma \frac{e^2}{4\pi|\vec{r}|}} \; \alpha$$

iv| Sign:

$${}_{aa}\langle p', k' | \bar\psi \psi A \, \bar\psi \psi A | p, k \rangle_{aa}$$

$$= \langle 0 | a_{k'} a_{p'} \bar\psi \psi A \, \bar\psi \psi A \, a_p^\dagger a_k^\dagger | 0 \rangle$$

$$\rightarrow \quad 1 + 1 + 2 = 4 \rightarrow (-1)^\# = 1$$

$$\sigma = +1$$

v|

$$V_{e^-e^-}(r) = + \frac{e^2}{4\pi r}$$

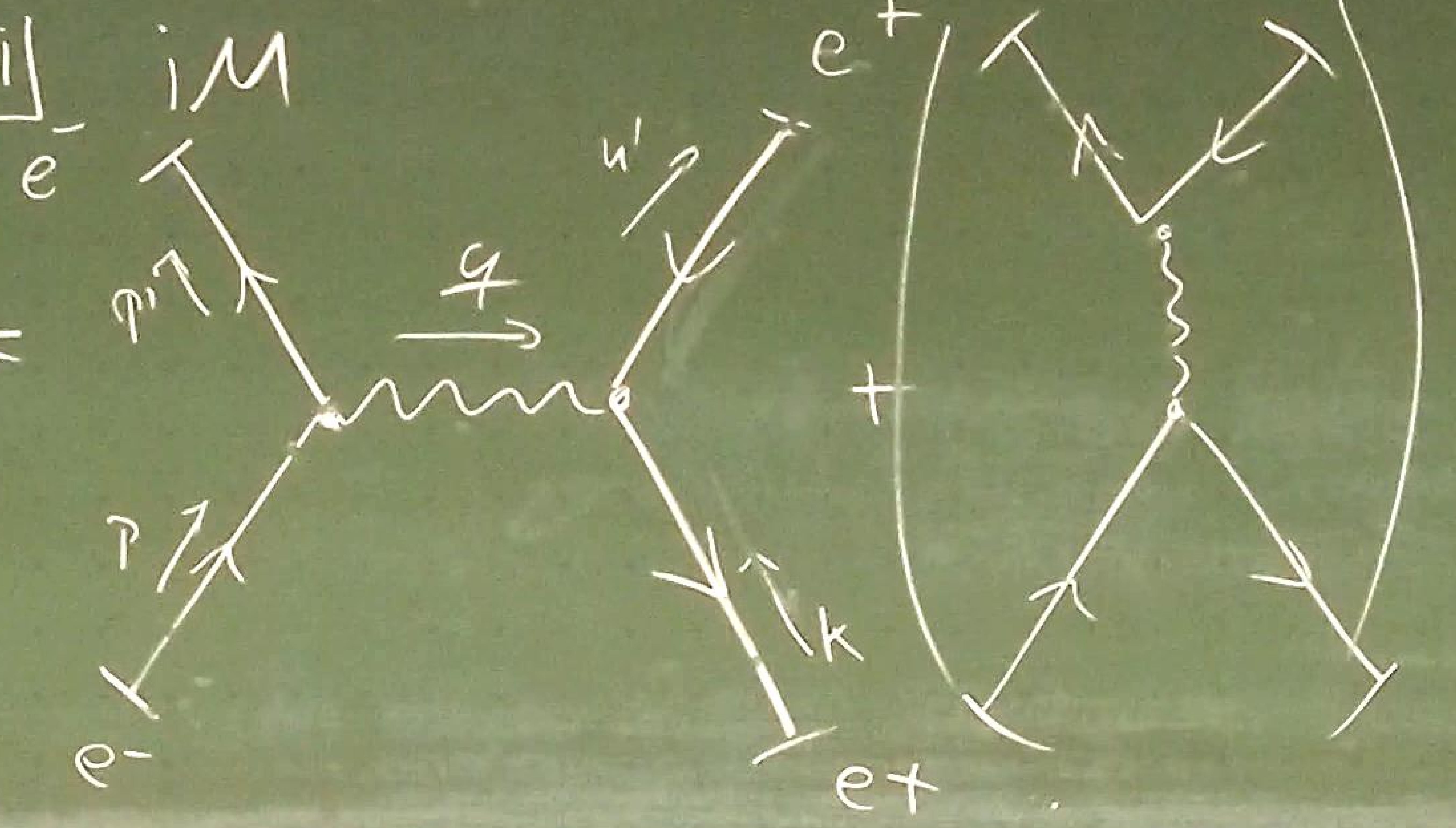
→ Equal charges repel each other ✓

2| Bhabha scattering

$$e^- + e^+ \longrightarrow e^- + e^+$$

i| iM

$=$



$$= \sigma \, \bar u(p')(-i\gamma^\mu) u(p) \left(\frac{-i g_{\mu\nu}}{q^2} \right) \bar v(k)(-i\gamma^\nu) v(k')$$

$$p - p' = q = k' - k$$

ii| Non-rel limit → Same resuld with $k \leftrightarrow k'$

but was is σ ?

iii| $\langle 0 | b_{k'} a_{p'} \bar\psi \psi A \, \bar\psi \psi A \, a_p^\dagger b_k^\dagger | 0 \rangle$

$$\rightarrow \quad 2 + 1 + 2 = 5$$

$$\rightarrow \quad (-1)^\# = -1 \Rightarrow \sigma = -1$$

iv| $\boxed{V_{e^+e^-}(r) = - \frac{e^2}{4\pi r}}$ → Attractive Coulomb potential

5. Elemandary Processes of QED

5.1. Cross section $e^+e^- \to \mu^+\mu^-$

scattering

1] Reaction:

$$e^- + e^+ \longrightarrow \mu^+ + \mu^-$$

$$m_e \ll m_m$$

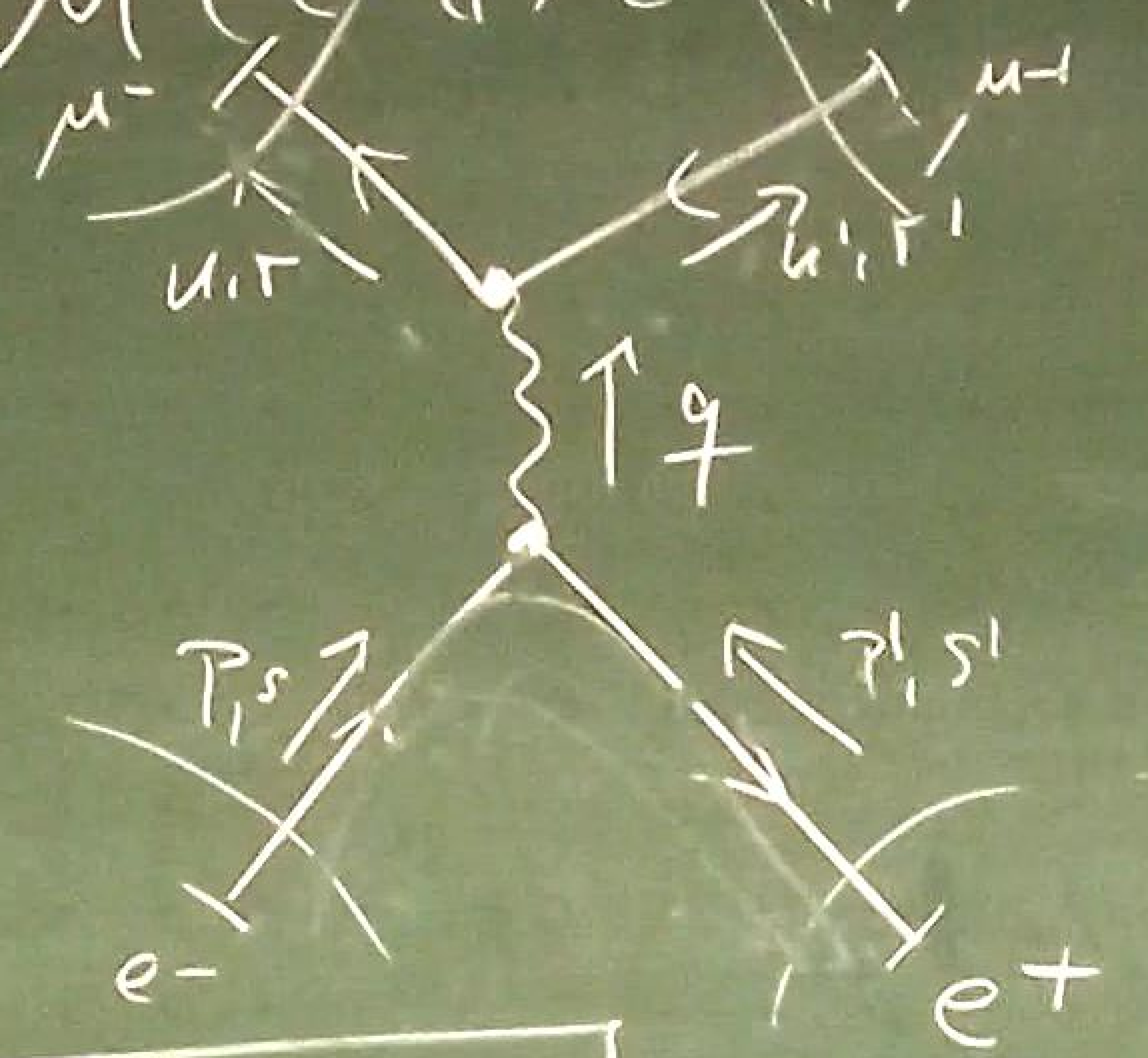
2] Note: $q_e = q_m = e = -|e|$

$$\mathcal{L}^{e,m} = \sum_{f=e,m} \bar{\psi}_f (i\partial\!\!\!/ - m_f)\psi_f - q_f \bar{\psi}_f \gamma^\mu \psi_f A_\mu - \frac{1}{4}F^2$$

3] Tree-level amplitude:

$$i\mathcal{M}(e^-(p)\,e^+(p') \longrightarrow \mu^-(k)\,\mu^+(k'))$$

$=$



$$= \underbrace{\bar{v}_e^{s'}(p')(-iq_e\gamma^\mu)u_e^s(p)}_{\text{electron}}\left(\frac{-ig_{\mu\nu}}{q^2}\right)\underbrace{\bar{u}_m^r(k)(-iq_m\gamma^\nu)v_m^{r'}(k')}_{\text{muon}}$$

$$= \frac{ie^2}{q^2}\left(\bar{v}(p')\gamma^\mu u(p)\right)\left(\bar{u}(k)\gamma_\mu v(k')\right)$$

$$p + p' = q = k + k'$$

4.] $d\sigma \propto |\mathcal{M}|^2 \to \mathcal{M}^*$

Use $(\bar{v}\gamma^\mu u)^* \overset{\circ}{=} \bar{u}\gamma^\mu v$

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\boxed{\left[(\bar{v}(p')\gamma^\mu u(p))(\bar{u}(p)\gamma^\nu v(p'))\right]} \cdot \left[(\bar{u}(k)\gamma_\mu v(k'))(\bar{v}(k')\gamma_\nu u(k))\right]$$

5.] Typical collider setup:

- e^+, e^- beams unpolarized
  - → average over spins of in-state
- Muon detector cannot resolve spin
  - → sum over spins of out-state

$$\to d\sigma \propto \frac{1}{4} \sum_{ss'} \sum_{rr'} |\mathcal{M}(ss' \to rr')|^2$$

6] Use spin sum relations.

$$\sum_s u^s \bar{u}^s = \not{p} + m$$
$$v \bar{v}^s = \not{p} - m$$

$$\sum_{ss'} (\bar{v}^{s'}_a(p') \gamma^\mu_{ab} u^s_b(p))(\bar{u}^s_c(p) \gamma^\nu_{cd} v^{s'}_d(p'))$$

$$\overset{\circ}{=} \mathrm{Tr}\left[(\not{p}' - m_e) \gamma^\mu (\not{p} + m_e) \gamma^\nu\right]$$

7] →

$$\frac{1}{4} \sum_{ss'rr'} |\mathcal{M}|^2 = \frac{e^4}{4q^4} \mathrm{Tr}\left[(\not{p}' - m_e) \gamma^\mu (\not{p} + m_e) \gamma^\nu\right]$$
$$\mathrm{Tr}\left[(\not{k} + m_\mu) \gamma_\mu (\not{k}' - m_\mu) \gamma_\nu\right]$$

8] Trace technology

Examples: $\mathrm{Tr}[\text{odd } \gamma^\mu] = 0$

- $\mathrm{Tr}[\gamma^\mu \gamma^\nu] = 4 g^{\mu\nu}$

$\vdots$

- $\gamma^\mu \gamma_\mu = 4 \mathbb{1}$

$\gamma^\mu \gamma^\nu \gamma_\mu = -2 \gamma^\nu$

9] $\longrightarrow$

- $\mathrm{Tr}\left[(\not p' - m_e)\gamma^\mu(\not p + m_e)\gamma^\nu\right]$

$$\overset{\circ}{=} 4\left[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(pp' + m_e^2)\right]$$

10] $m_e/m_e \approx 1/200 \longrightarrow m_e = 0$

$$\underbrace{\frac{1}{4}\sum_{ss'rr'}|M|^2}_{|\overline{M}|^2} \overset{\circ}{=} \frac{8e^4}{q^4}\left[(pk)(p'k') + (pk')(p'k) + m_\mu^2(pp')\right]$$

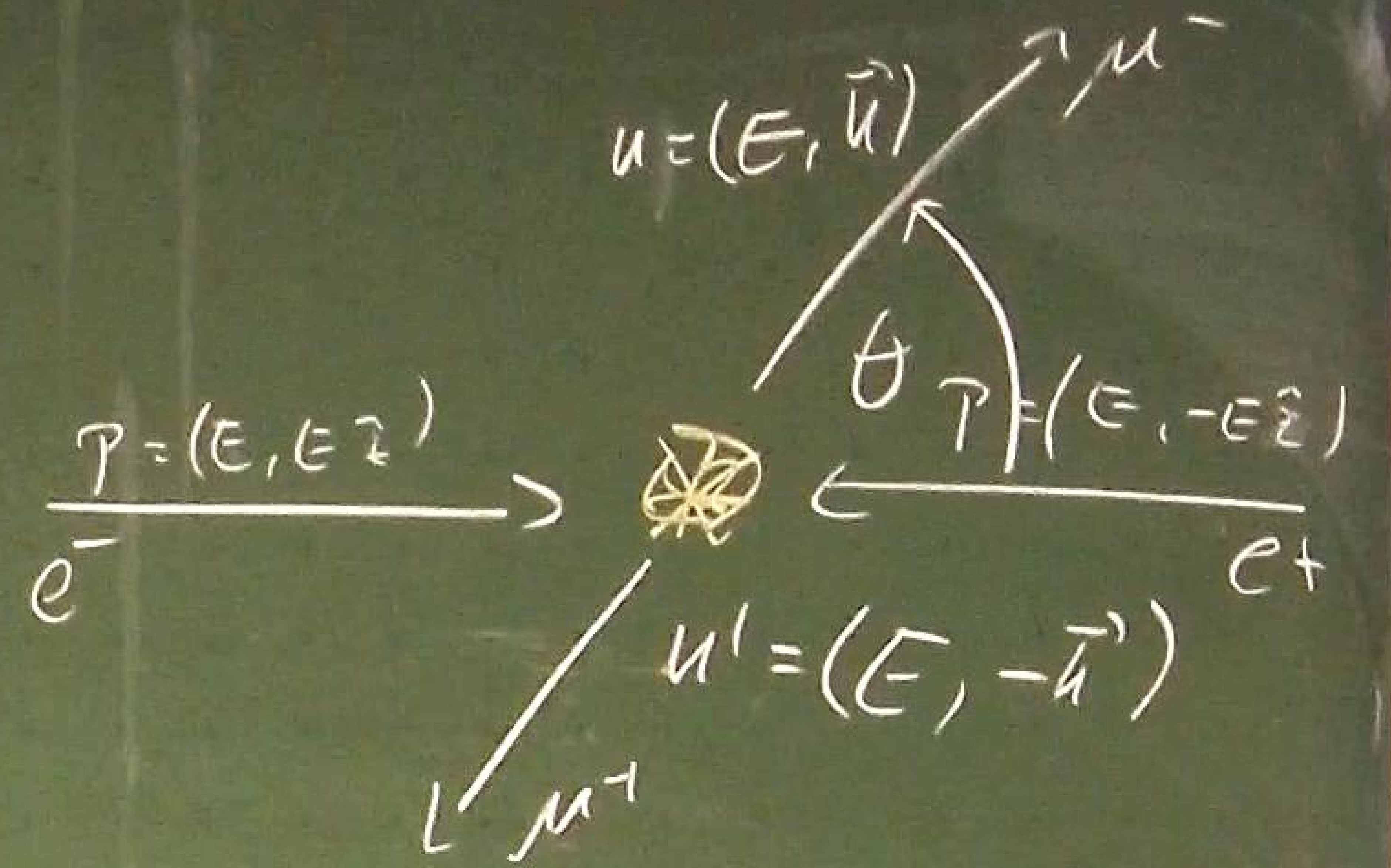
11] ⊗ Center of mass frame: $\vec p + \vec p\,' = 0 = \vec k + \vec k'$

- $p = (E, E\hat z)$, $p' = (E, -E\hat z)$
- $|\vec k| = \sqrt{E^2 - m_\mu^2}$
- $\vec k \cdot \hat z = |\vec k| \cos\theta$

$\Downarrow$

- $q^2 = (p + p')^2 = 4E^2$
- $pp' = 2E^2$
- $pk = p'k' = E^2 - E|\vec k|\cos\theta$

$pk' = p'k = E^2 + E|\vec k|\cos\theta$

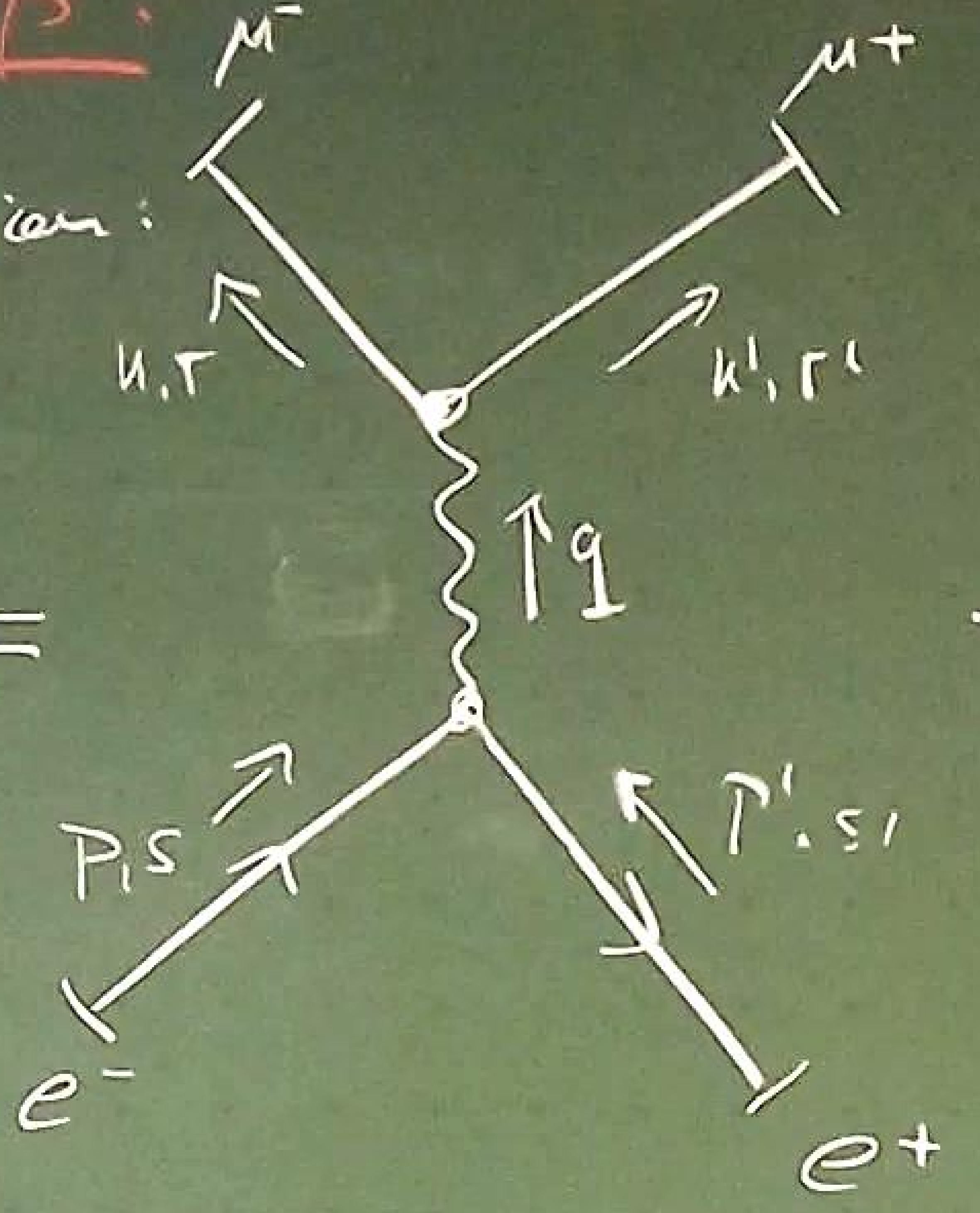


$$\Rightarrow \boxed{|\overline{M}|^2 = e^4\left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]}$$

Recap:

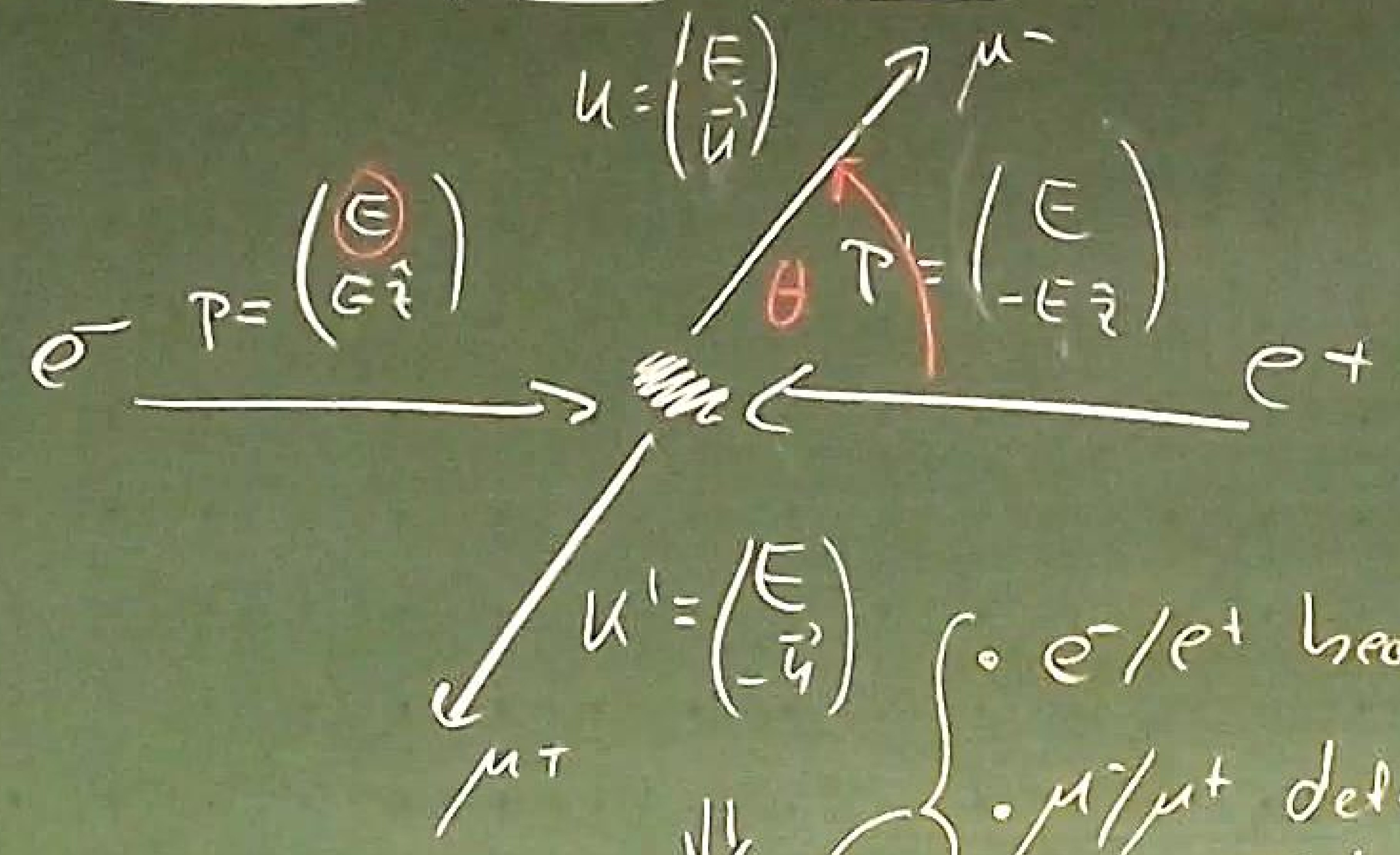
Reaction:

$$iM =$$



$$+ O(\alpha^3)$$

11] Center of mass frame:



- e^-/e^+ beam unpolarized
- μ^-/μ^+ detector cannot resolve spin

$\Downarrow$

$$\Longrightarrow \boxed{\begin{aligned} |\overline{M}|^2 &= \frac{1}{4} \sum_{s s' r r'} |M|^2 \\ &= e^4 \left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2\theta \right] \end{aligned}}$$

12] Differential scattering cross section

(special case for 2 outgoing particles, Eq. (4.122))

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{2E_p 2E_{p'} \underbrace{|v_p - v_{p'}|}_{2}} \frac{|\vec{k}|}{(2\pi)^2 4 \underbrace{E_{cm}}_{2E}} |\overline{M}|^2$$

$$= \frac{\alpha^2}{4E_{cm}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2\theta \right]$$

13]

$$\int d\Omega \dots = \boxed{\frac{4\pi\alpha^2}{3E_{cm}^2} \sqrt{1 - \frac{m_\mu^2}{E}} \left(1 + \frac{m_\mu^2}{2E^2}\right) = \sigma_{total}}$$

14| Discussion.

- $2m_\mu \lessgtr E = E_{cm}$
- Prediction of QED:
  non-trivial energy dependence
  of $|\overline{\mathcal{M}}|^2$
  → Experimentally verified
- Measure σ_{total} as function of E
  $\Longrightarrow$ muon mass m_μ
  ..

5.2. Summary of QED calculations

1. Draw diagrams
2. Use Feynman rules to compute $\mathcal{M}$
3. $|\overline{\mathcal{M}}|^2 = \sum_{spin} |\mathcal{M}|^2$
4. Evaluate trace
5. Fix a frame (eg center of mass)
   $P = P(E, \theta, \varphi)$
6. Plug $|\overline{\mathcal{M}}|^2$ in (4.120)
   integrate over momenta
   that are not measured.

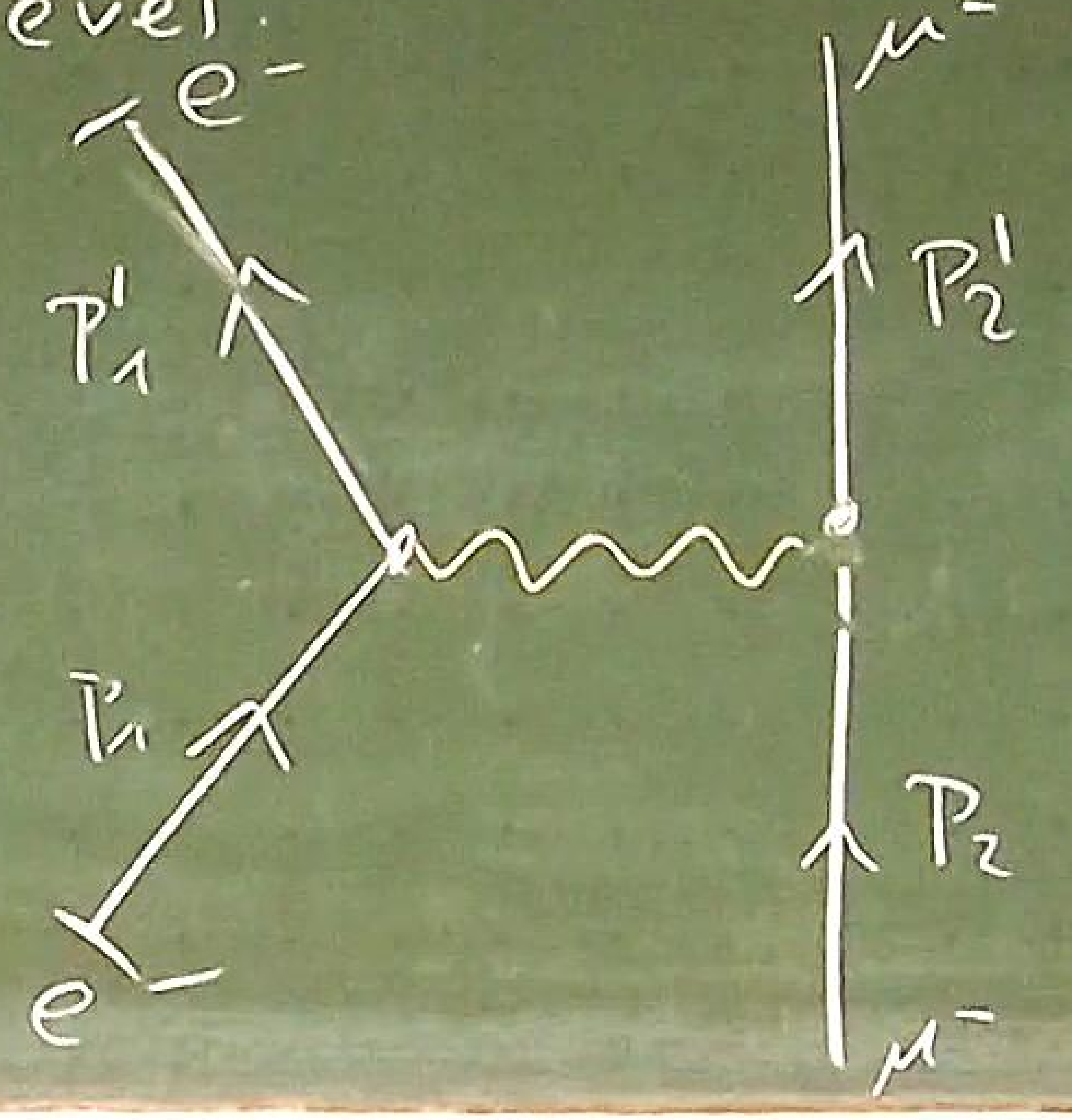
6. Radiative Corrections of QED

6.1. Overview

1| Process

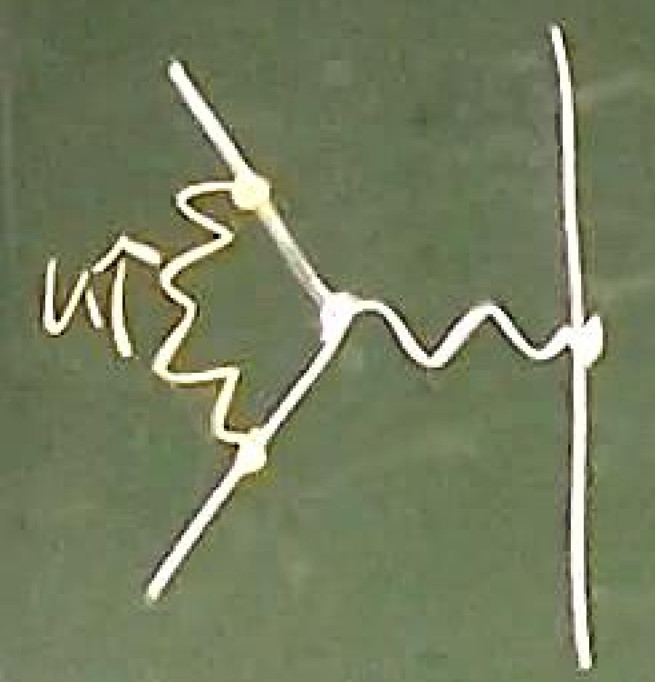
$$\lim_{m_\mu \to \infty} \left\{ e^- + \mu^- \longrightarrow e^- + \mu^- \right\}$$

2| Tree level.



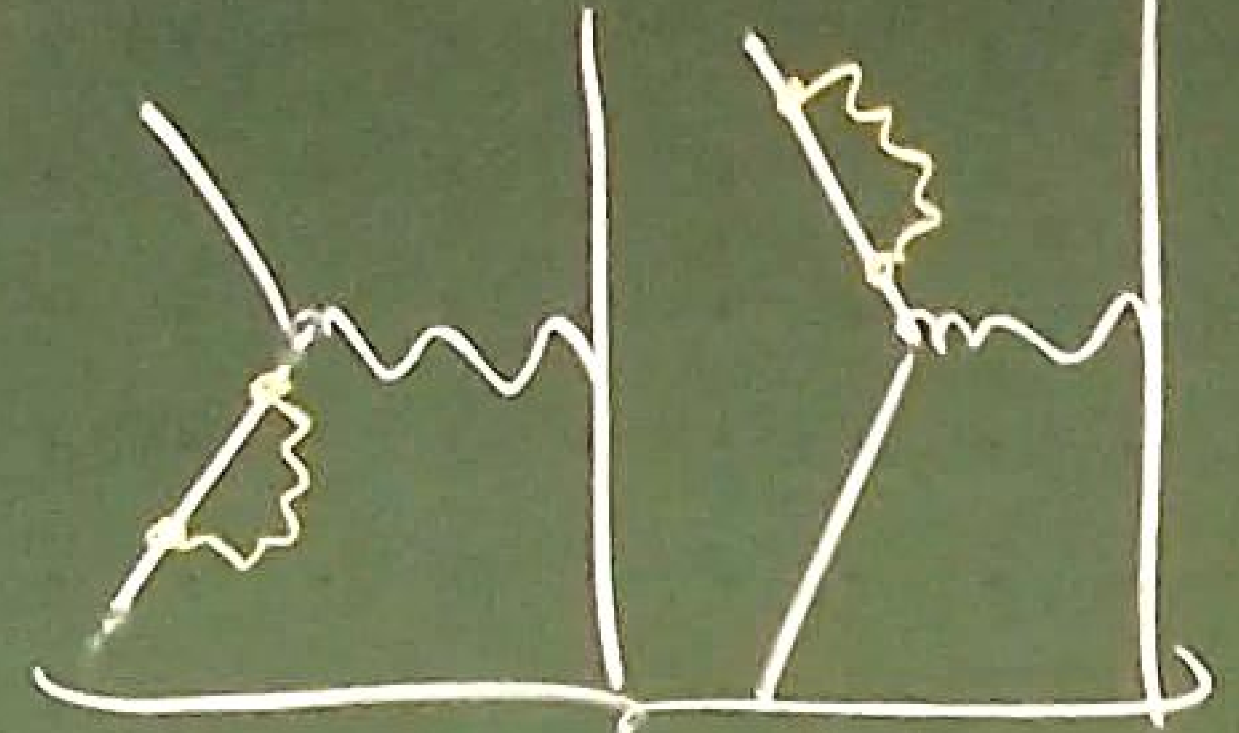
3| Radiative corrections:

- loops:



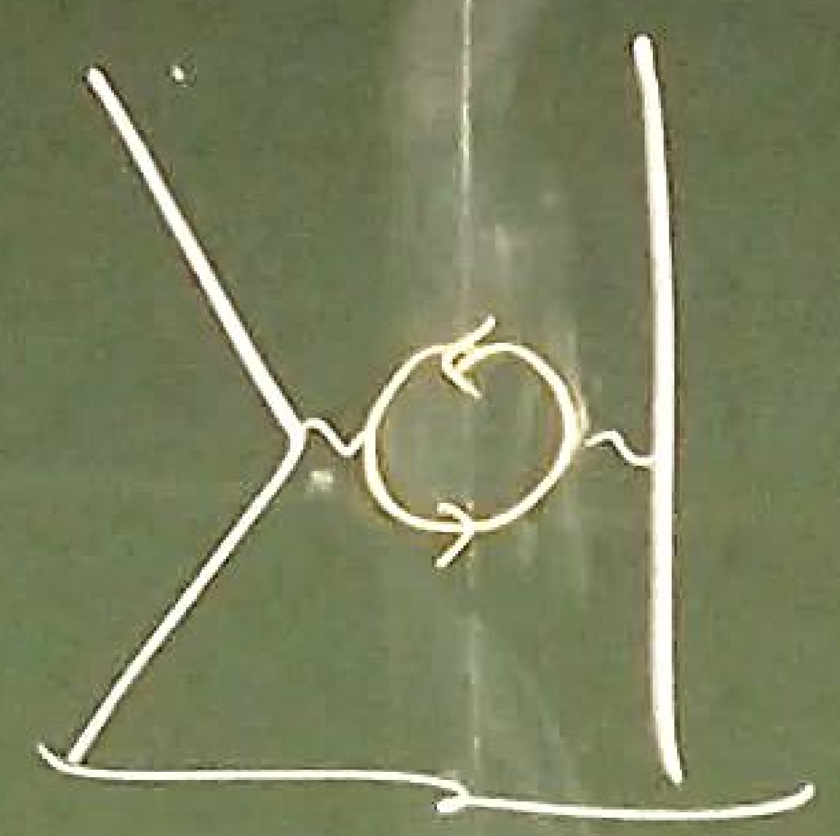
Vertex correction

- UV divergence
- IR divergence



External leg corrections

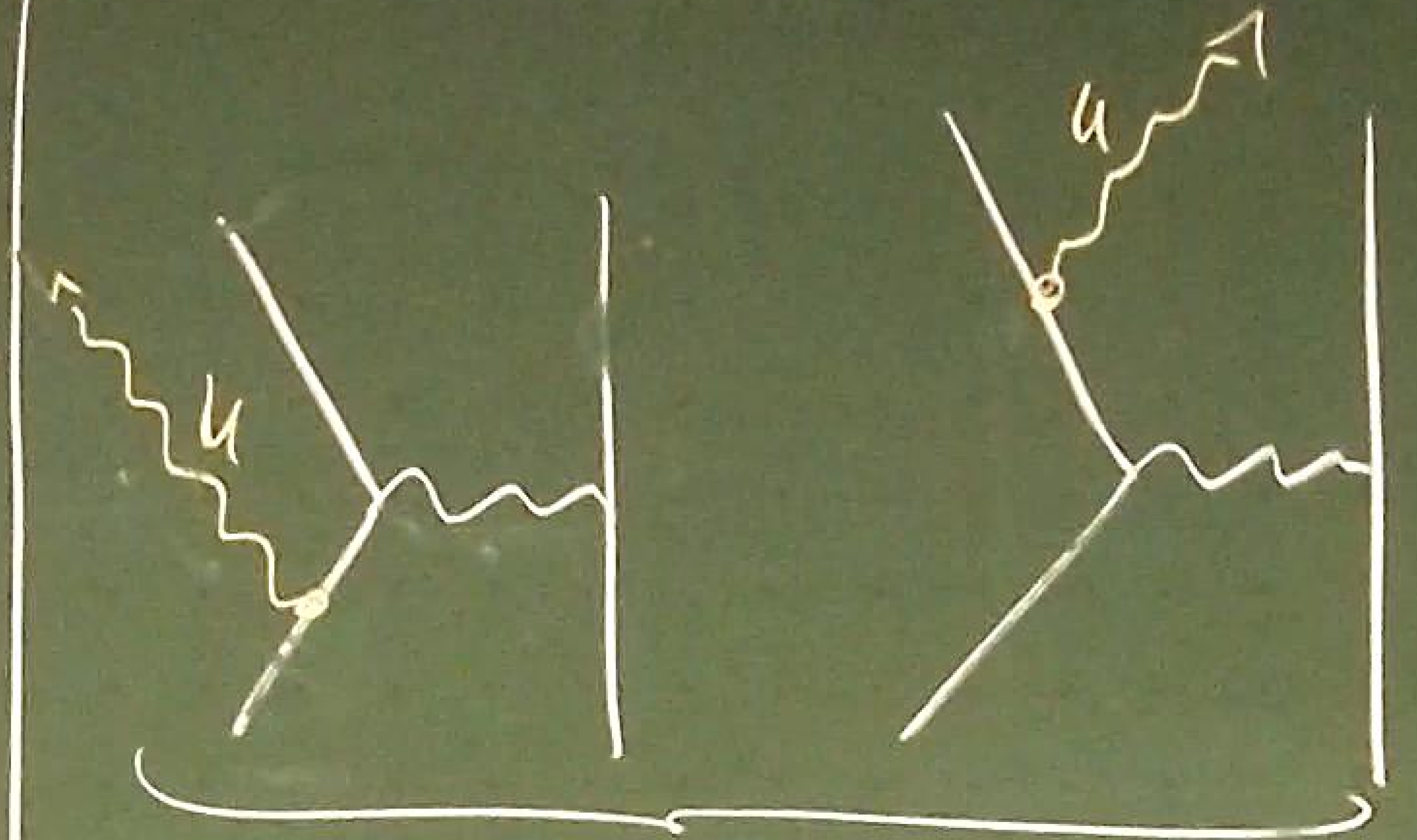
- UV div
- IR div



Vacuum polarization

- UV divergence

- external final state photons:



Bremsstrahlung

- IR - divergence

4| Spoiler!

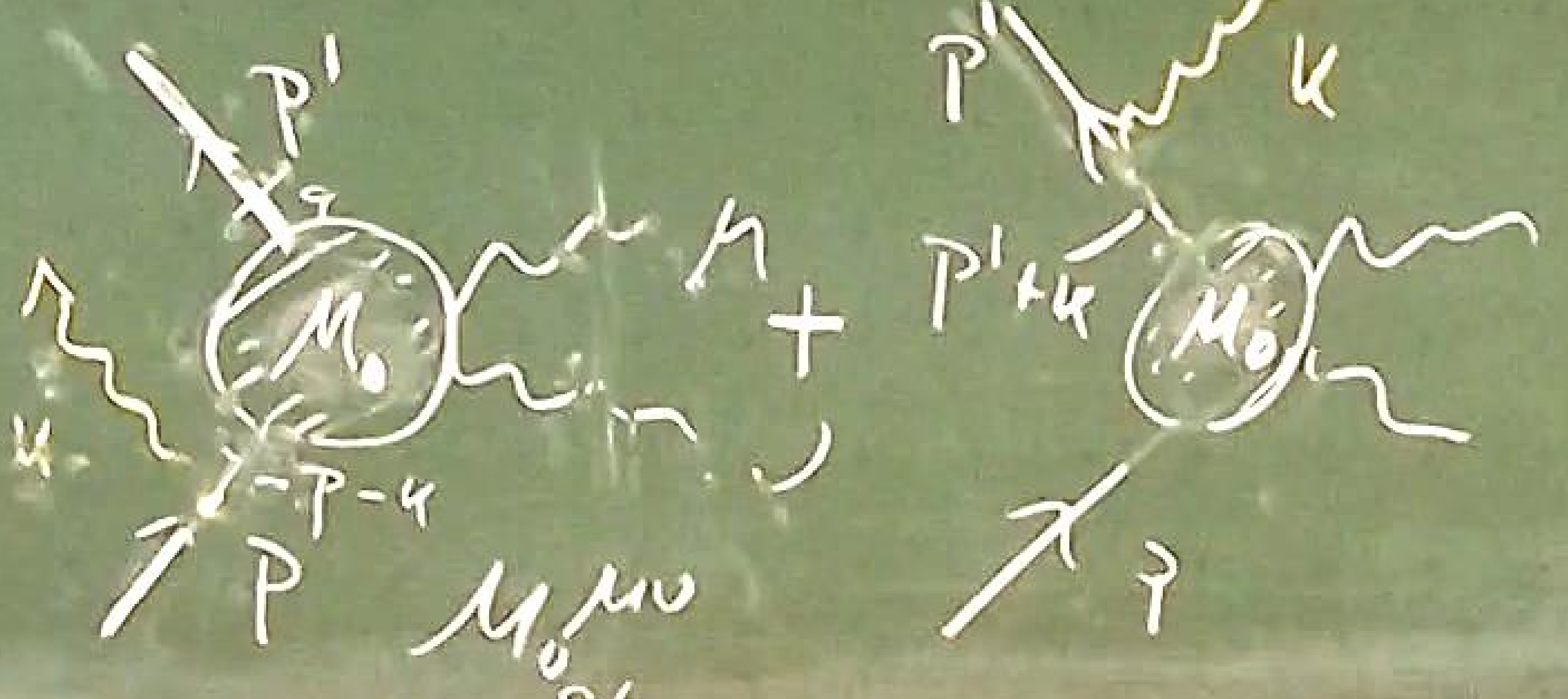
- UV div → cancel in observable quantities
- IR div → cancel with divergences of Bremsstrahlung diagrams

6.2 Soft Bremsstrahlung

1| Soft = Low-energy photons ($k \approx 0$)

2| Can be classically derived from Maxwell equations

3| & QFT



$$M = \bar{u}(p')M_0(p',p-k)\frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2+i\varepsilon}(-ie\gamma^\mu)u(p)\,\varepsilon^*_\mu(k)$$

$$+\,\bar{u}(p')(-ie\gamma^\mu)\frac{i(\not{p}'+\not{k}+m)}{(p'+k)^2-m^2+i\varepsilon}M(p'+k,p)u(p)\,\varepsilon^*_\mu(k)$$

4| Simplifications:

- $(p'\pm k)^2 - m^2 = \pm 2p'k \swarrow$ $p^2 = m^2$, $k^2 = 0$
- Soft photons: $|\vec{k}| \ll |\vec{p}-\vec{p}'|$

$\rightarrow M_0(p',p-k) \approx M(p',p) \approx M_0(p'+k,p)$

$\rightarrow \not{p}-\not{k} \approx \not{p}$

- Dirac algebra:

$$(\not{p}+m)\gamma^\mu\varepsilon^*_\mu u(p) \overset{\circ}{=} 2p^\mu\varepsilon^*_\mu u(p)$$

$$\bar{u}(p')\gamma^\mu\varepsilon^*_\mu(\not{p}'+m) \overset{\circ}{=} \bar{u}(p')2p'^\mu\varepsilon^*_\mu$$

$$\left[\underbrace{\overset{\text{use}}{(\not{p}-m)}}\,u(p) = 0\right]$$

$$\sum v\underbrace{\bar{v}u}_{=0}$$

5|

$$iM = \underbrace{\bar{u}(p')M_0(p',p)u(p)}_{\text{elastic scattering}}\cdot\underbrace{\left[e\left(\frac{p'\varepsilon^*}{p'k}-\frac{p\varepsilon^*}{pk}\right)\right]}_{\text{bremsstrahlung}}$$

6| Scattering cross section:

$$d\sigma(p\to p'+\gamma) = d\sigma(p\to p')\cdot\underbrace{\int\frac{d^3k}{(2\pi)^3}\sum_r\frac{e^2}{2|\vec{k}|}\left|\frac{p'\varepsilon^r}{p'k}-\frac{p\varepsilon^r}{pk}\right|^2}_{dP_k(p\to p')}$$

7| Evaluate:

$$\int dP_k = \frac{\alpha}{\pi}\int_0^\infty dk\,\frac{1}{k}\underbrace{\boxed{\int\frac{d\Omega_k}{4\pi}\sum_r\left|\frac{p'\varepsilon^r}{p'\tilde{k}}-\frac{p\varepsilon^r}{p\tilde{k}}\right|^2 \qquad \tilde{k}=\frac{k}{|\vec{k}|}=\begin{pmatrix}1\\ \hat{k}\end{pmatrix}}}_{\vartheta(p,p')}$$

$$=\frac{\alpha}{\pi}\vartheta(p,p')\Big[\underbrace{\log(\infty)}_{}-\underbrace{\log(0)}_{}\Big]$$

8| Approximations:

i| **Problem 1:** $|\vec{k}| \ll |\underbrace{\vec{p}-\vec{p}'}_{\vec{q}}|$

$|\vec{k}| \approx \|\vec{q}\|$

→ Introduce upper cutoff $|\vec{q}|$

ii| **Problem 2:** IR-divergence

"Solution": Regularisation with finite photon mass $\mu>0$.

$$\frac{1}{k}=\frac{1}{E_k}\longmapsto\frac{1}{\sqrt{\mu^2+\vec{k}^2}}$$

$$\to\int_0^{|\vec{q}|}dk\,\frac{1}{\sqrt{\mu^2+k^2}}=\log\left(\frac{\sqrt{\mu^2+|\vec{q}|^2}+|\vec{q}|}{\mu}\right)\overset{\mu\to0}{\sim}\log\left(2\frac{|\vec{q}|}{\mu}\right)\sim\log\left(\frac{|\vec{q}|}{\mu}\right)=\frac{1}{2}\log\left(\frac{|\vec{q}|^2}{\mu^2}\right)$$

2| General form:

$$\Gamma^\mu(p',p)=f(p^\mu,p'^\mu,\gamma^\mu,m,e,\mathbb{1})$$

3| Restrictions:

1| Lorentz covariance: Γ^μ transforms like γ^μ

$$\boxed{\Gamma^\mu = A\cdot\gamma^\mu+\tilde{B}\,p^\mu+\tilde{C}\,p'^\mu}$$

iii] Relativistic limit:

$$E_p, E_{p'} \gg m$$

$$\vartheta(p,p') \overset{*}{=} 2\log\left(\frac{-q^2}{m^2}\right)$$

$$-q^2 = -\underbrace{\underbrace{(p'-p)^2}}_{<0} \geq 0$$

$$\simeq |\vec{q}|^2$$

9] Result

$$\boxed{d\sigma(p\to p'+\gamma) \simeq d\sigma(p\to p')\,\frac{\alpha}{\pi}\,\boxed{\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{\mu^2}\right)}}$$

$$\boxed{\begin{matrix}\mu\to 0\\ E_{p,p'}\gg m\end{matrix}}$$

Sudakov double logarithm

10] Two problems.

- Depends on physical photon mass ↯
- Logarithmic divergence for $-q^2\to\infty$

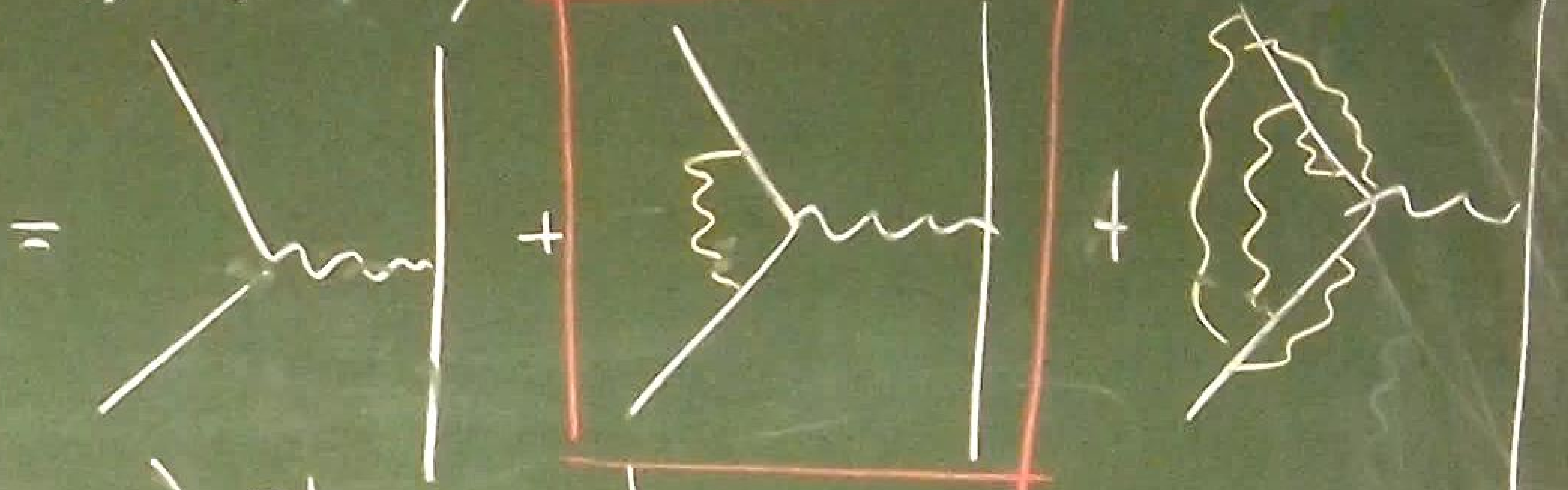
  ↯ Probability.

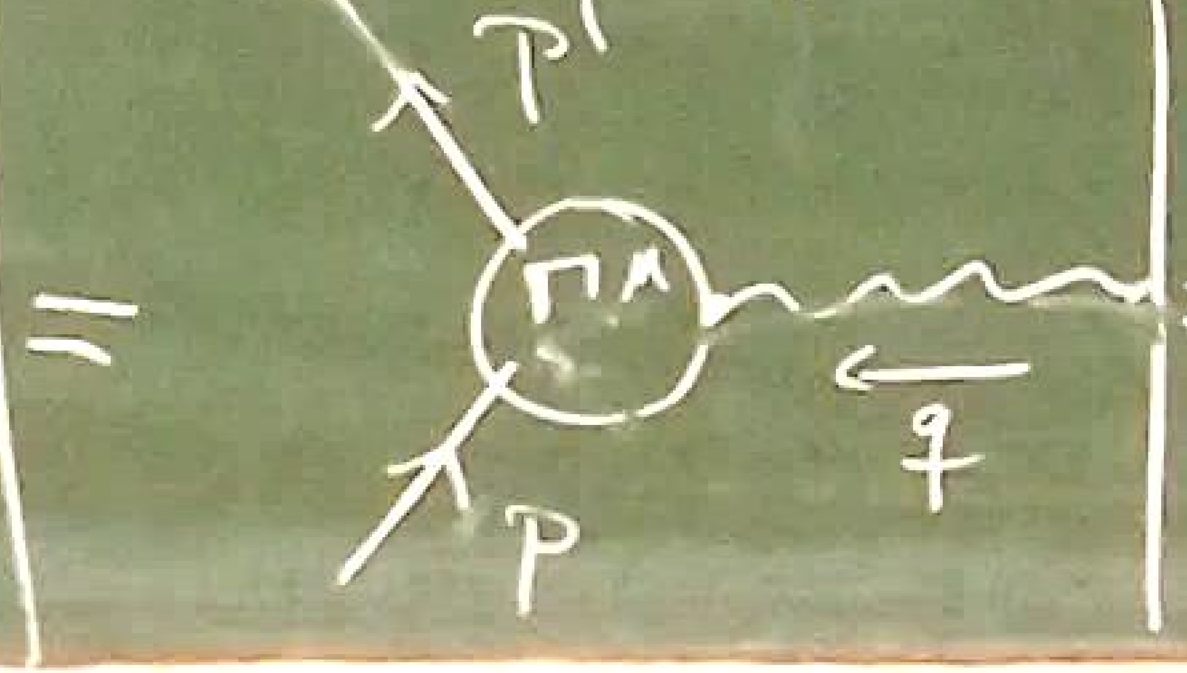
6.3 The Electron Vertex Function

6.3.1 Formal Structure

1] Scattering amplitude:

$$i\mathcal{M}(e^-(p)\mu^-(k) \to e^-(p')\mu^-(k'))$$





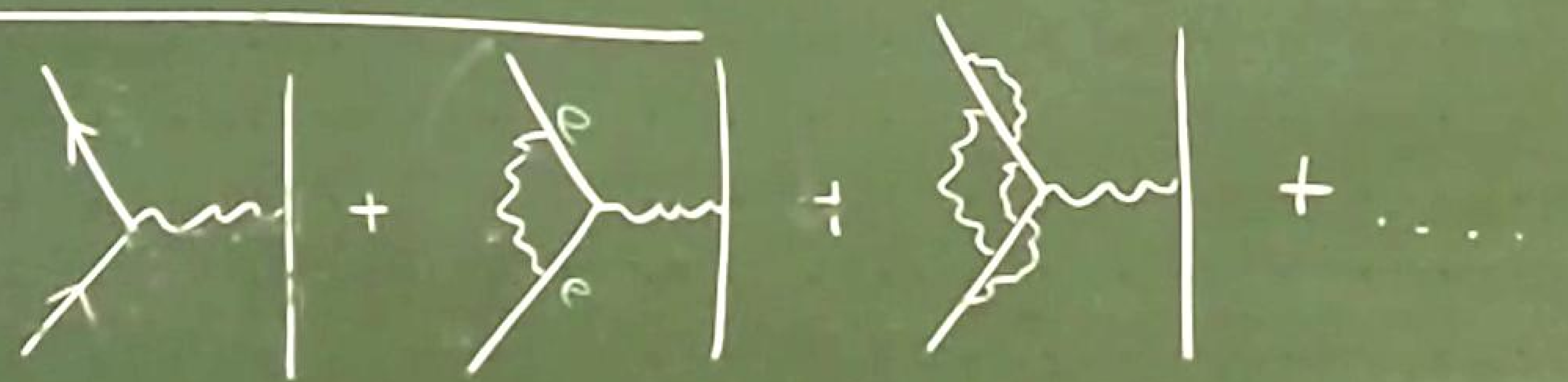
$$= ie^2\left(\bar{u}_e(p')\Gamma^\mu(p',p)u_e(p)\right)\frac{1}{q^2}\left(\bar{u}_m(k')\gamma_\mu u_m(k)\right)$$

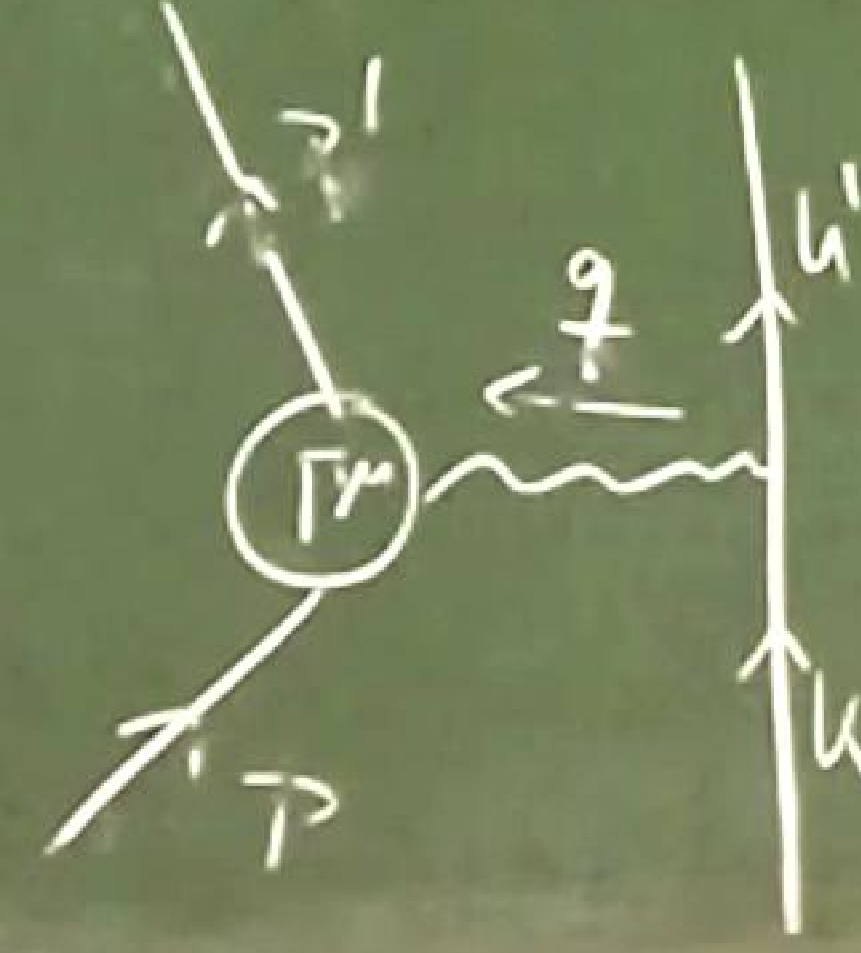
Recap:

6.3 Electron vertex function

6.3.1 Formal structure

1]

$i\mathcal{M} =$ 

$=$ 

$= ie^2 \underbrace{\bar{u}(p')\Gamma^\mu u(p)}_{e} \frac{1}{q^2} \underbrace{\bar{u}(k)\gamma_\mu u(k)}_{\text{muon (heavy)}}$

$m \to \infty$

2] General form of $\Gamma^\mu(p,p')$

$\Gamma^\mu(p,p') = f(p^\mu, p'^\mu, \gamma^\mu, m, e, c)$

3] Restrictions

$\Big\{$ All equations below are valid if evaluated as $\bar{u}\,\Gamma^\mu\, u$!

i] Lorentz covariance.

$$\Gamma^\mu = \underline{A\gamma^\mu + B(p'^\mu + p^\mu)} + C(p'^\mu - p^\mu)$$

ii] $(p^2, pp', p'^2, p_\mu\gamma^\mu = \not{p},)$

$\sum_s v_s \bar{v}_s$

$(\not{p} - m)u = 0 \Rightarrow \boxed{\begin{array}{l} \not{p}u(p) = m\,u(p) \\ \bar{u}\,\not{p}' = \bar{u}(p')\,m \end{array}}$

$\Rightarrow \underset{\in A, B, C}{X} = X(p^\mu, p'^\mu, m, e, c)\cdot \mathbb{1}$

III) $q^2 = (p'-p)^2 = 2(m^2 - pp')$

$$X = X(q^2, m, e, \epsilon)$$

IV) **Ward identity**: U(1) gauge-symmetry of QED

$$q_\mu \Gamma^\mu \overset{*}{=} 0$$

{ Noether's theorem for QFT

$$\left.\begin{array}{l} \to 0 = q_\mu \Gamma^\mu = A \underbrace{\overset{\not{q}}{q_\mu \gamma^\mu}}_{=0} + B \underbrace{q_\mu (p'^\mu + p^\mu)}_{=0} + C q^2 \\ \bar{u}(\not{p}' - \not{p}) u = (m-m)\bar{u} u = 0 \quad p'^2 = m^2 = p^2 \end{array}\right\} \Rightarrow C = 0$$

4) **Gordon identity**:

$$\bar{u}(p') \frac{p'^\mu + p^\mu}{2m} u(p) \hat{=} \bar{u}(p') \gamma^\mu u(p) - \bar{u}(p') \frac{i\sigma^{\mu\nu} q_\nu}{2m} u(p)$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

5) $$\boxed{\Gamma^\mu(p, p') = \underline{\gamma^\mu F_1(q^2)} + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2)} = \underline{\gamma^\mu} + O(\alpha)$$

$F_i(q^2)$: Form factors

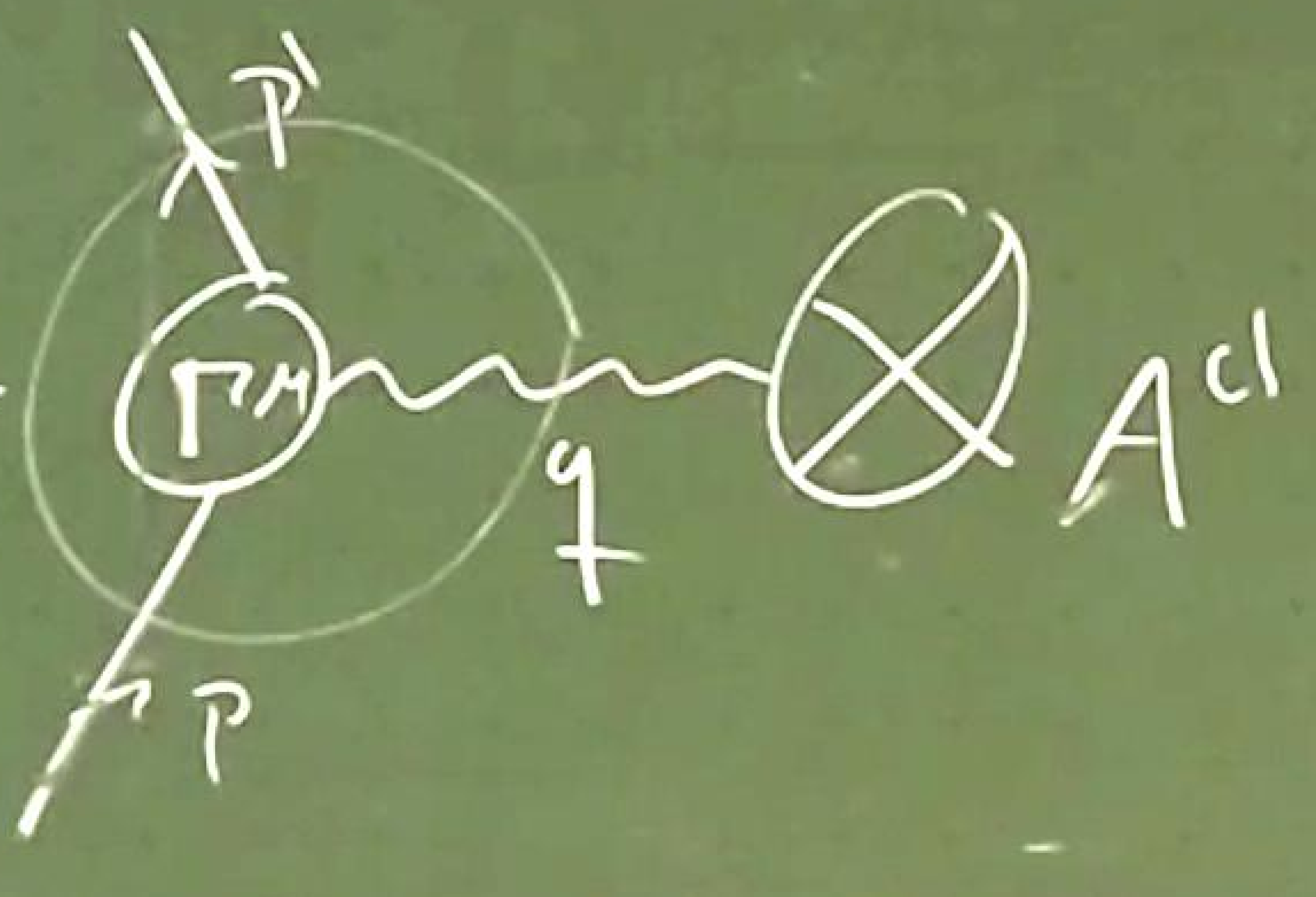
$$\underline{F_1(q^2) = 1 + O(\alpha)} \qquad \underline{F_2(q^2) = 0 + O(\alpha)}$$

6.3.2 Landé g-factor

1. **Setting:** ⊗ Classical external field $A^{cl}_\mu(x)$

$$H_{int} = e\int d^3x\, \bar{\psi}(x)\gamma^\mu\psi(x)A^{cl}_\mu(x)$$

$i\mathcal{M}(2\pi)\delta(p'^0-p^0) =$



$q = p'-p$

2) Electric charge:

i) $A^{cl}_\mu(x) = (\phi(\vec{x}), 0,0,0) \Rightarrow A_\mu(q) = (2\pi\delta(q^0)\phi(\vec{q}), 0,0,0)$

ii) $i\mathcal{M} = -ie\,\bar{u}(p')\Gamma^0(p',p)u(p)\cdot\phi(\vec{q})$

iii) $\phi(\vec{x})$ slowly varying $\rightarrow$ $\phi(\vec{q})$ concentrated $\vec{q}=0$

$\rightarrow$ take limit $\vec{q}\rightarrow 0$

$$i\mathcal{M} \simeq -ie\,\bar{u}\,F_1(0)\gamma^0 u(p)\cdot\phi(\vec{q})$$

$|\vec{p}| \ll m^2$

$$\simeq \underbrace{-ie\,F_1(0)\,\phi(\vec{q})}_{\hat{V}(\vec{q})}\cdot \underline{2m\,\xi'^\dagger\xi}$$

iv) Born approximation with potential

$i\mathcal{M} = -i\hat{V}(\vec{q})$

$$V(\vec{x}) = e\,F_1(0)\,\phi(\vec{x})$$

v) $\Rightarrow F_1(0) = 1$

$$F_1(0) = \underbrace{F_1^{(0)}(0)}_{1} + \alpha F_1^{(1)}(0) + \alpha^2 \dots$$

$$= 1 + O(\alpha)$$

$$\rightarrow \boxed{F_1^{(n)}(0) = 0 \quad \text{for } n \geq 1}$$

3] Magnetic moment

i] $A^{cl}_\mu(x) = (0, \vec{A}(x))$

$\to A^{cl}_\mu(q) = (0, 2\pi\delta(q^0)\vec{A}(\vec{q}))$

ii] $i\mathcal{M} = -ie\bar{u}\Gamma^\mu u\, A^{cl}_\mu(\vec{q})$

$= ie\bar{u}(p')\left[\underbrace{\gamma^i F_1(q^2) + \frac{i\sigma^{i\nu}}{2m}q_\nu F_2(q^2)}_{q=0,\ \overset{=0}{}\ |\vec{p}|^2 \ll 1}\right]u(p)A^i_{cl}(\vec{q})$

iii] $\bar{u}(p')\gamma^i u(p) \overset{0}{=} \boxed{\frac{p^i+p'^i}{2m}}\, 2m\xi'^\dagger\xi + 2m\xi'^\dagger\left(\frac{-i}{2m}\varepsilon^{ijk}q^j\sigma^k\right)\xi$

$(p-eA)^2 = pA + Ap$

iv] $\frac{iq_\nu}{2m}\bar{u}(p')\sigma^{i\nu}u(p) \overset{0}{\approx} 2m\xi'^\dagger\left(\frac{-i}{2m}\varepsilon^{ijk}q^j\sigma^k\right)\xi$

v] Summary

$i\mathcal{M} \overset{q\to 0}{\approx} -ie\,\xi'^\dagger\left\{\frac{-1}{2m}\sigma^k[F_1(0)+F_2(0)]\right\}\xi\cdot\underbrace{\left[-i\varepsilon^{ijk}q^jA_{cl}(\vec{q})\right]}_{B^k_{cl}(\vec{q})}(2m)$

vi] Born approximation. $\vec{B}_{cl} = \nabla\times\vec{A}_{cl}$

$V(\vec{x}) = -\langle\vec{\mu}\rangle\cdot\vec{B}$

$\langle\vec{\mu}\rangle = \frac{e}{2m}[F_1(0)+F_2(0)]\,\xi'^\dagger\frac{\sigma^k}{2}\xi$

$= g\cdot\overset{\frac{e}{2m}}{\boxed{\mu_B}}\langle\vec{S}\rangle$

⇒ Landé factor:

$g = 2[\overbrace{F_1(0)}^{1}+F_2(0)]$

$= 2+2F_2(0)$

$= \underset{\text{Dirac equation}}{2} + \underbrace{2\alpha F_2^{(1)}(0)+O(\alpha^2)}_{\text{Anomalous magnetic moment}}$

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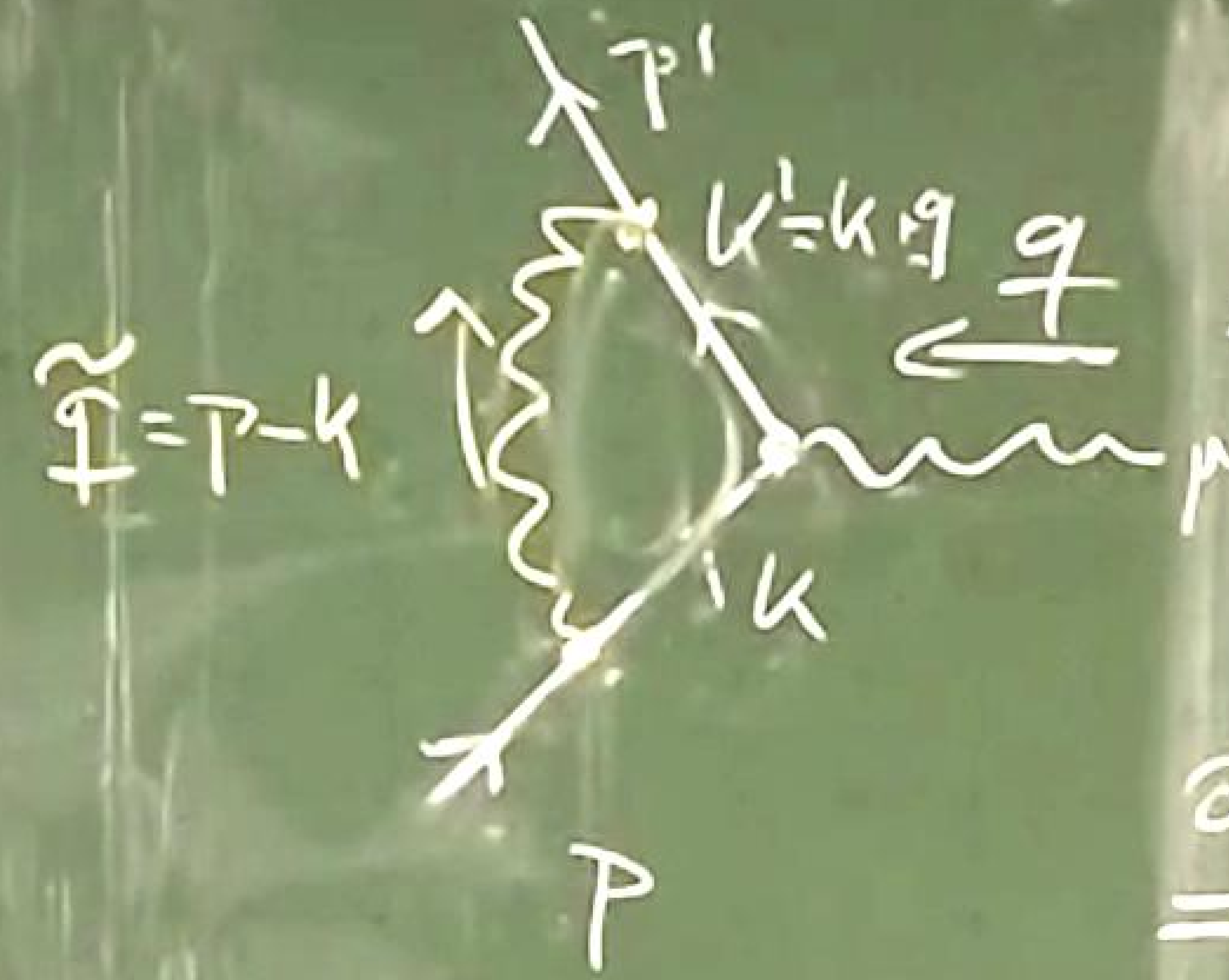
6.3.3 Evaluation

1. Scattering amplitude:

$$\bar{u}(p') \propto [\Gamma^{(1)}(p',p)]^{\mu} u(p)$$

$=$



$$= \int \frac{d^4k}{(2\pi)^4} \frac{-i g_{\nu\rho}}{\tilde{q}^2 + i\varepsilon} \bar{u}(p')(-ie\gamma^{\nu}) \frac{i(\not{k}' + m)}{k'^2 - m^2 + i\varepsilon} \gamma^{\mu} \frac{i(\not{k} + m)}{k^2 - m^2 + i\varepsilon} (-ie\gamma^{\rho}) u(p)$$

$$\overset{\circ}{=} 2ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p')[\not{k}\gamma^{\mu}\not{k}' + m^2\gamma^{\mu} - 2m(k+k')^{\mu}] u(p)}{\underbrace{(\tilde{q}^2 + i\varepsilon)}_{A_1} \underbrace{(k'^2 - m^2 + i\varepsilon)}_{A_2} \underbrace{(k^2 - m^2 + i\varepsilon)}_{A_3}}$$

2] Feynman Parameters:

$$\boxed{\frac{1}{A_1 \cdots A_n} = \left(\prod_{i=1}^{n} \int_0^1 dx_i\right) \delta\left(\sum_{i=1}^{n} x_i - 1\right) \frac{(n-1)!}{[\underbrace{x_1 A_1 + \ldots + x_n A_n}_{D}]^n}}$$

x_i : Feynman parameters

3] $$\frac{1}{A_1 A_2 A_3} = \int_0^1 dx\,dy\,dz\, \delta(x+y+z-1) \frac{2}{D^3}$$

$$D \overset{\circ}{=} k^2 + 2k(yq - zp) + yq^2 + zp^2 - (x+y)m^2 + i\varepsilon$$

$$= \underbrace{l}_{k+yq-zp}{}^2 - \underbrace{\Delta}_{-xyq^2 + (1-z)^2 m^2 > 0} + i\varepsilon$$

$\underline{4]}$ $\bar{u}(p')\,[\;—\,''\,—\;]\;u(p)$

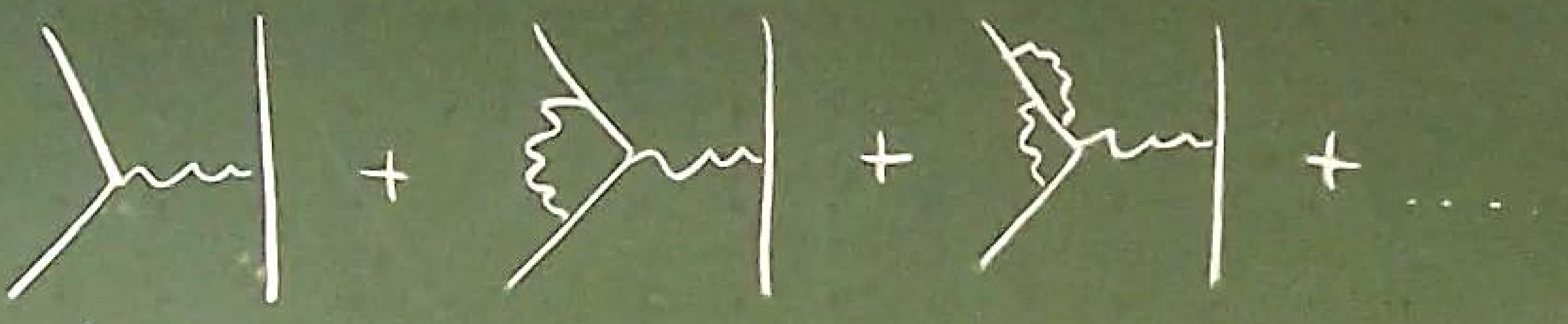
$$\overset{\circ}{\sim}\; u(p')\left\{-\frac{1}{2}\gamma^\mu l^2+\left[-y\cancel{q}+z\cancel{p}\right]\gamma^\mu\left[(1-z)\cancel{q}+z\cancel{p}\right]+m^2\gamma^\mu-2m\left[(1-2y)\,q^\mu+2z\,p^\mu\right]\right\}u(p)$$

Recap:

6.3. The Electron Vertex Function

6.3.1. Formal Structure

$i\mathcal{M} =$



$\equiv$

Form factors

$$\boxed{\Gamma^\mu(p,p') = \gamma^\mu \underset{\substack{\uparrow \\ 1+\mathcal{O}(\alpha)}}{F_1(q^2)} + \frac{i\sigma^{\mu\nu}q_\nu}{2m} \underset{\substack{\uparrow \\ 0+\mathcal{O}(\alpha)}}{F_2(q^2)}}$$

6.3.2. The Landé g-factor

- Response of electric charge $\Rightarrow$ $\boxed{F_1(0) = 1 = \underbrace{F_1^{(0)}(0)}_{1} + \alpha \underbrace{F_1^{(1)}(0)}_{0} + \underbrace{\alpha^2}_{0}}$

- Response of magnetic moment

$$\langle\vec{\mu}\rangle = g \cdot \mu_B \langle\vec{S}\rangle$$

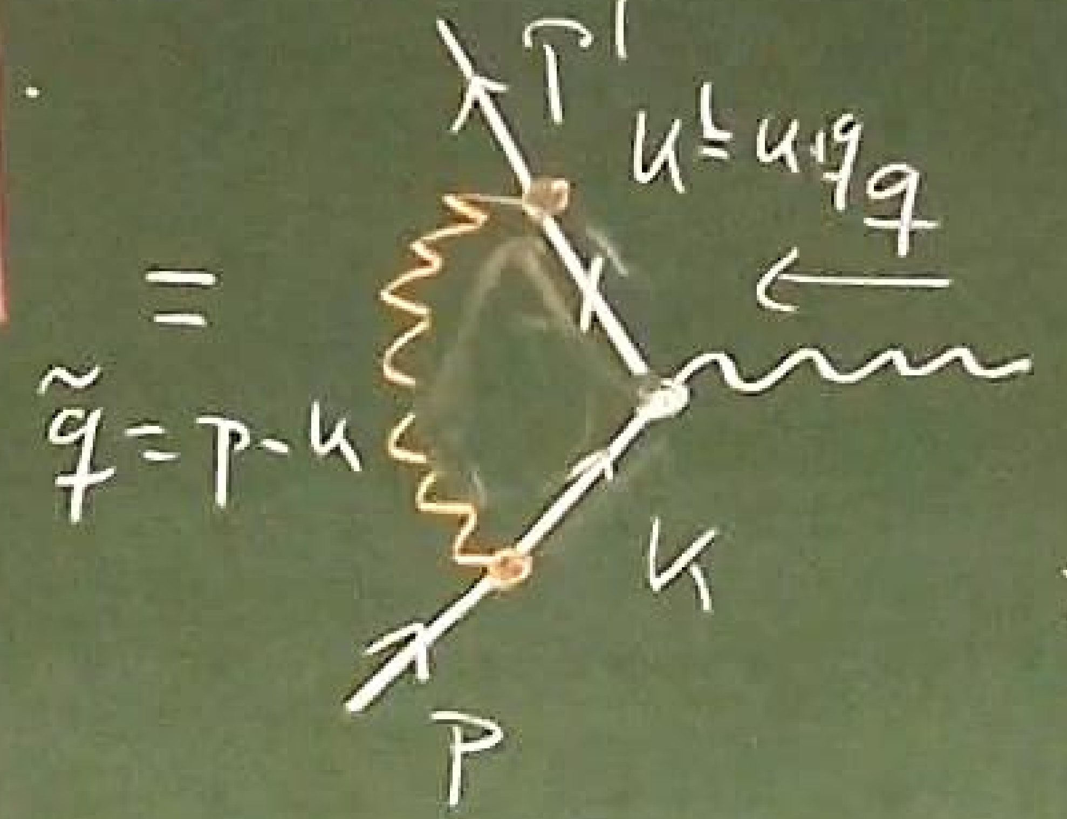
Landé factor, $\mu_B = \frac{e}{2m}$: Bohr magneton

$$\Rightarrow \boxed{g = 2 + 2F_2(0) = \underbrace{2}_{\text{Dirac g.}} + \underbrace{2\alpha F_2^{(1)}(0) + \mathcal{O}(\alpha^2)}_{\text{anomalous mag. moment}}}$$

6.3.3. Evaluation

1] $\bar{u}(p')\left[\alpha\Gamma^{(1)}(p',p)\right]^\mu u(p)$

$=$



$$= \cdots \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\nu \gamma^\mu \gamma_\nu}{A_1 A_2 A_3}$$

↑ Feynman rules

Dirac propagators

Simplify with contraction identities

2] Feynman parameters

3] $$\frac{1}{\underbrace{(\tilde{q}^2+i\varepsilon)}_{A_1}\underbrace{(k'^2-m^2+i\varepsilon)}_{A_2}\underbrace{(k^2-m^2+i\varepsilon)}_{A_3}} = \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2}{D^3}$$

$$\boxed{D = \ell^2 - \Delta + i\varepsilon}$$

$$\boxed{\begin{aligned} \ell &= k + yq - zp \\ \Delta &= -xyq^2 + (1-z)^2m^2 \end{aligned}}$$

4] $\boxed{\text{Numerator}} = \bar{u}(p')\left[\underbrace{\not{k}\gamma^\mu\not{k}'}_{} + m^2\gamma^\mu - 2m(k+k')^\mu\right]u(p) \overset{*}{=} \bar{u}(p')\left\{-\tfrac{1}{2}\gamma^\mu l^2 + [-y\not{q} + z\not{p}]\gamma^\mu[(1-y)\not{q} + z\not{p}] + m^2\gamma^\mu - 2m\left[(1-2y)q^\mu + 2zp^\mu\right]\right\}u(p)$

$* =$ only valid in $\int d^4l$ integral ! $\boxed{l^\rho l^\sigma}$:

$\begin{cases} k' = k + q \\ k = l - yq + zp \end{cases}$

$x+y+z=1$

$= \bar{u}(p')\left\{\gamma^\mu\underbrace{\left[-\tfrac{1}{2}l^2 + (1-x)(1-y)q^2 + (1-2z-z^2)m^2\right]}_{A} + (p'+p)^\mu\underbrace{[mz(z-1)]}_{B} + q^\mu\underbrace{[m(z-2)(x-y)]}_{C}\right\}u(p)$

$(p'-p)^\mu$

$\gamma^\mu + \dots\ \sigma^{\mu\nu}$

5] $C = 0$

$L'^{\mu\nu} = \int \frac{d^4l}{(2\pi)^4}\frac{l'^\mu l'^\nu}{D(l^2)} = \int\frac{d^4l'}{(2\pi)^4}\frac{l'^\mu l'^\nu}{D(l'^2)} = L^{\mu\nu}$

$\Rightarrow L^{\mu\nu} = g^{\mu\nu}C(l^2)$

$\Rightarrow g_{\mu\nu}L^{\mu\nu} = 4C(l^2) = \int\frac{d^4l}{(2\pi)^4}\frac{l^2}{D(l^2)}$

$\Rightarrow \boxed{L^{\mu\nu} = \int\frac{d^4l}{(2\pi)^4}\frac{g^{\mu\nu}l^2}{4D(l^2)}}$

6] Gordon identity. $\longrightarrow$

$\bar{u}(p')\,a\Gamma^{\mu}u = 2ie^2\int\frac{d^4l}{(2\pi)^4}\int_0^1 dx\int_0^1 dy\int_0^1 dz\,\delta(x+y+z-1)\frac{2}{D(l^2)^3}$

$\bar{u}(p')\left\{\gamma^\mu\left[-\tfrac{1}{2}l^2 + (1-x)(1-y)q^2 + (1-4z+z^2)m^2\right] + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\left[2m^2 z(1-z)\right]\right\}u(p)$

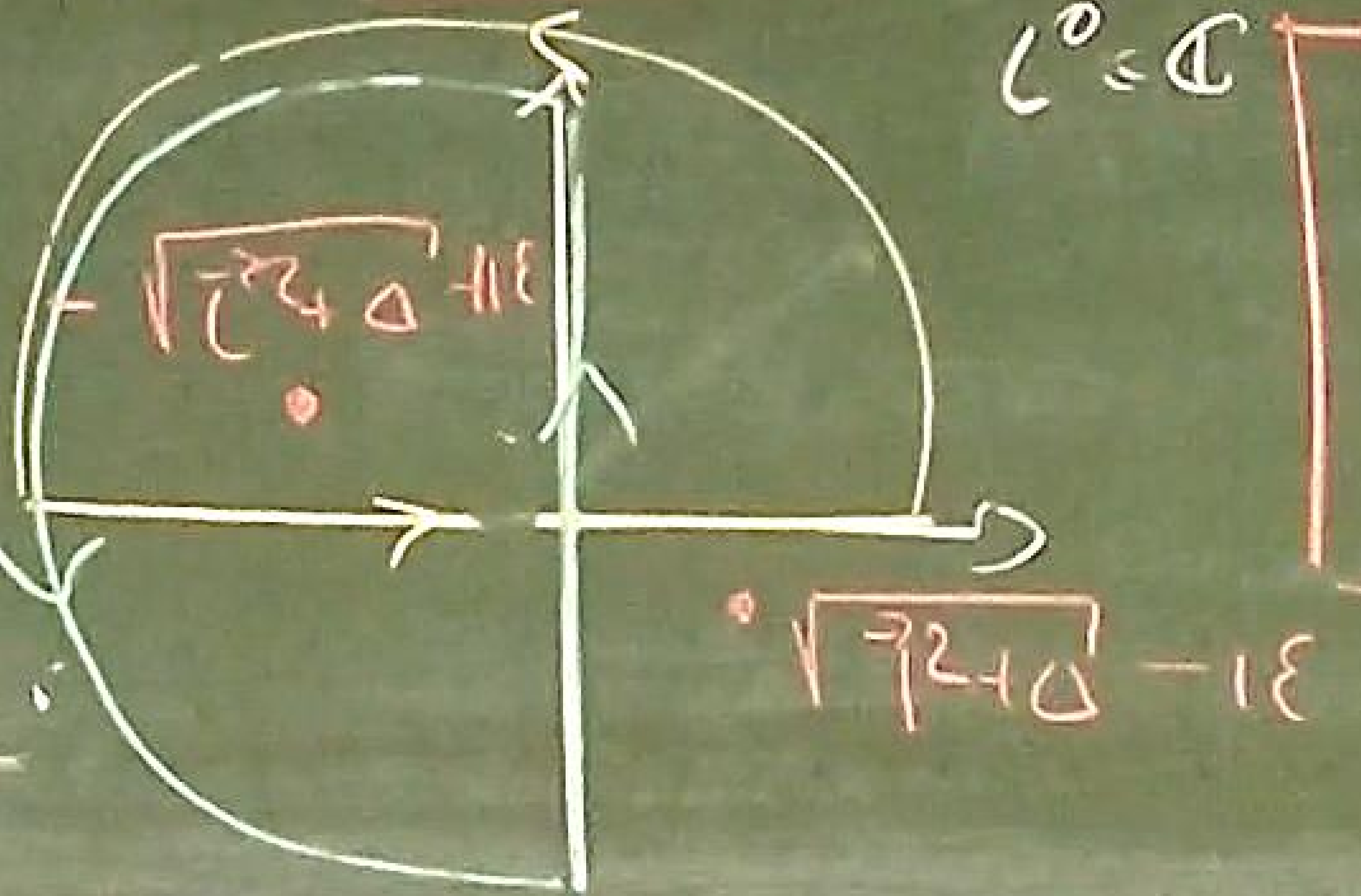
$l \in \mathbb{R}^{1,3}$

$l_E \in \mathbb{R}^4$

7] Momentum integral:

i] $l^2 = l_0^2 - \vec{l}^2$

Solution: Wick rotation:



$l^0 \in \mathbb{C}$

$$\boxed{\begin{aligned} &l^0 = il_E^0,\quad \vec{l} = \vec{l}_E \\ &\Rightarrow l^2 = -l_E^{0\,2} - \vec{l}^2 = -l_E^2 \end{aligned}}$$

ii] Then$\overset{m>3}{}$ m>2.

$$\lim_{\varepsilon\to 0}\int\frac{d^4l}{(2\pi)^4}\frac{l^2\,1}{(l^2-\Delta+i\varepsilon)^m}\overset{WR}{=}\frac{i}{(-1)^m}\frac{1}{(2\pi)^4}\int d^4l_E\frac{1}{(l_E^2+\Delta)^m}$$

$$=\frac{i(-1)^m}{(2\pi)^4}\underbrace{\int d\Omega_4}_{2\pi^2}\underbrace{\int_0^\infty dl_E\frac{l_E^3}{(l_E^2+\Delta)^m}}_{=\frac{1}{2(m-1)(m-2)\Delta^{m-2}}} \quad 5$$

$$=\frac{i(-1)^m}{(4\pi)^4}\frac{1}{(m-1)(m-2)\Delta^{m-2}} \qquad (*)$$

$$=\frac{i(-1)^{m-1}}{(4\pi)^4}\frac{2}{(m-1)(m-2)(m-3)\Delta^{m-3}} \qquad (**)$$

Problem: for m=3 diverges!

→ UV divergence

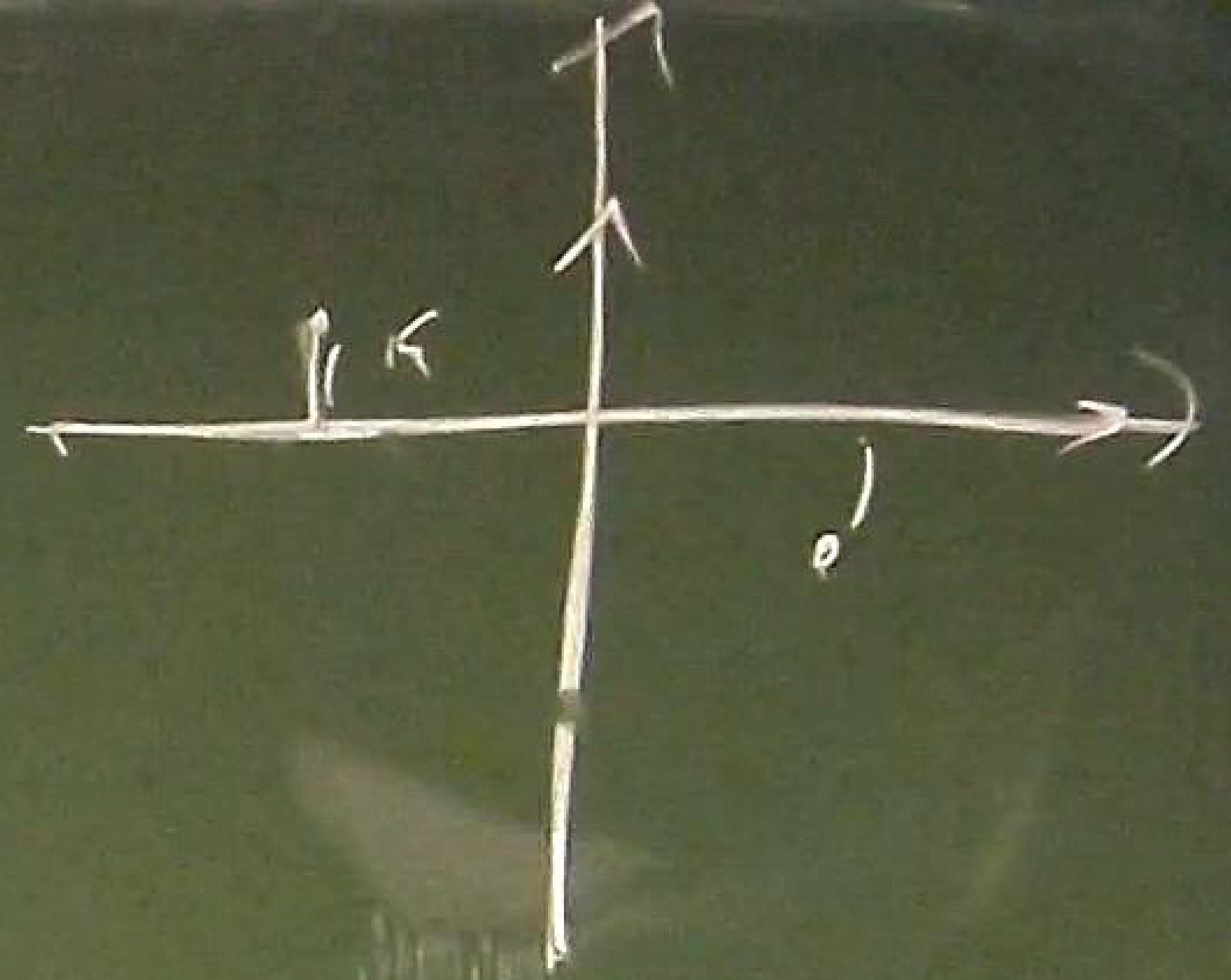
iii] Fix: Pauli-Villars regularization

$$\frac{-ig_{\mu\nu}}{q^2+i\varepsilon}\longmapsto\frac{-ig_{\mu\nu}}{q^2+i\varepsilon}-\boxed{\frac{ig_{\mu\nu}}{q^2-\Lambda^2+i\varepsilon}}$$

for $\Lambda\to\infty$.

Hope: Λ should drop out in physical quantities

$$\longrightarrow \Delta_\Lambda=-xyq^2+(1-z)^2m^2+z\Lambda^2$$



IV] $m=3$

- $Eq\,(*) \longmapsto Eq\,(*) - \overset{\to 0 \text{ for } \Lambda \to \infty}{\overbrace{\mathcal{O}(\Lambda^{-2})}}$

- $Eq\,(**) \longmapsto$

$$\lim_{\varepsilon\to 0} \int \frac{d^4l}{(2\pi)^4} \left[\frac{l^2}{(l^2-\Delta+i\varepsilon)^3} - \frac{l^2}{(l^2-\Delta_\Lambda+i\varepsilon)^3} \right]$$

$$\overset{G}{=} \frac{i}{(4\pi)^2} \log\left(\frac{\Delta_\Lambda}{\Delta}\right) \xrightarrow{\Lambda\to\infty} \frac{i}{(4\pi)^2} \log\left(\frac{z\Lambda^2}{\Delta}\right)$$

8] Result

$$\bar{u}(p')\,\propto\Gamma^{(1)\mu}\,u(p) = \frac{\alpha}{2\pi}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1)$$

$$\bar{u}(p') \Big\{ \gamma^\mu \Big[\underbrace{\log\left(\frac{z\Lambda^2}{\Delta}\right) + \frac{(1-x)(1-y)q^2}{\Delta} + \frac{(1-4z+z^2)m^2}{\Delta}}_{\text{"}F_1^{(1)}(q^2)\text{"}} \Big] + \frac{i\sigma^{\mu\nu}q_\nu}{2m} \underbrace{\Big[\frac{2m^2 z(1-z)}{\Delta} \Big]}_{\text{"}F_2^{(1)}(q^2)\text{"}} \Big\} u(p)$$

z

9] $\propto F_1$:

i] Problem 1: $F_1(0)=1,\ F_1^{(0)}(0)=1,\ \underline{F_1^{(1)}(0)=0}$

But. $F_1^{(1)} \neq 0$ ↯

Fix 1 $\quad F^{(1)}(q^2) \longmapsto F_1^{(1)}(q^2) - F_1^{(1)}(0)$

ii] Problem 2: IR divergence from $\tilde{q} \to 0$

$$\propto q^2=0 \quad \int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1)\, \frac{1-4z+z^2}{(1-z)^2} = \int_0^1 dz \left(\int_0^{1-z} dy \right) \frac{-2+(1-z)(3-z)}{(1-z)^2} = \overbrace{\int_0^1 dz \frac{-2}{1-z}}^{\infty} + \text{finit}$$

$1-z$

Fix: Add small photon mass $\mu \to 0$

$$\Delta \mapsto \Delta_\mu = -xyq^2 + (1-z)^2 m^2 + z\mu^2$$

iii| Fix 1 + Fix 2:

$$F_1(q^2) = 1 + \alpha F_1^{(1)}(q^2) + O(\alpha^2)$$

$$F_1^{(1)}(q^2) = \frac{1}{2\pi}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1)$$

$$\times \left[\log\left(\underbrace{\frac{m^2(1-z)^2}{m^2(1-z)^2 - q^2xy}}_{\text{Fix 1}} \right) + \frac{m^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy + \underset{\text{Fix 2}}{\boxed{z\mu^2}}} - \underbrace{\frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + \underset{\text{Fix 2}}{\boxed{z\mu^2}}}}_{\text{Fix 1}} \right]$$

10| $\sphericalangle F_2$:

$$F_2(q^2) = \alpha F_2^{(1)}(q^2) + O(\alpha^2) \quad \text{with} \quad F_2^{(1)}(q^2) = \frac{1}{2\pi}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1) \left[\frac{2m^2 z(1-z)}{m^2(1-z)^2 - q^2xy} \right]$$

$$F_2(0) = \frac{\alpha}{\pi}\int \ldots \delta \frac{2z}{1-z} + O(\alpha^2)$$

$$= \frac{\alpha}{2\pi}\int_0^1 dz \int_0^{1-z} dy \frac{2z}{1-z} + O(\alpha^2)$$

$$= \frac{\alpha}{2\pi} + O(\alpha^2)$$

$$g = 2 + 2F_2(0)$$

$$\alpha^2 = \frac{1}{137^2} \approx 0.5 \cdot 10^{-4}$$

Anomalous magnetic moment:

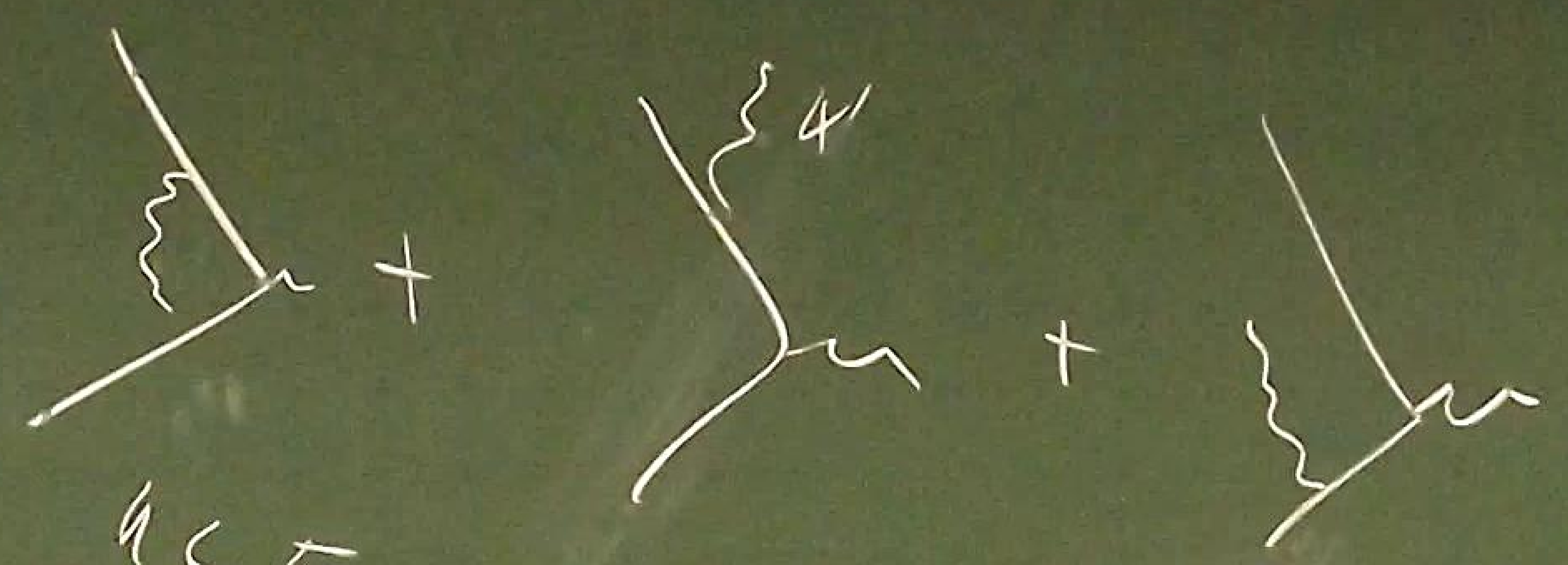
$$a_e^{QED} = \frac{g-2}{2} = F_2(0) = \frac{\alpha}{2\pi} + O(\alpha^2)$$

$$\approx 0.0011|614$$

$$a_e^{exp} \approx 0.0011|597$$

$$\boxed{\begin{aligned} a_e^{SM} &= 0.00115965218\,|\,2031 \\ a_e^{exp} &= 0.00115965218\,|\,073 \end{aligned}}$$

6.3.4. The Infrared Divergencies



$k < E_{min}$

$e p + \gamma$ | $e p'$

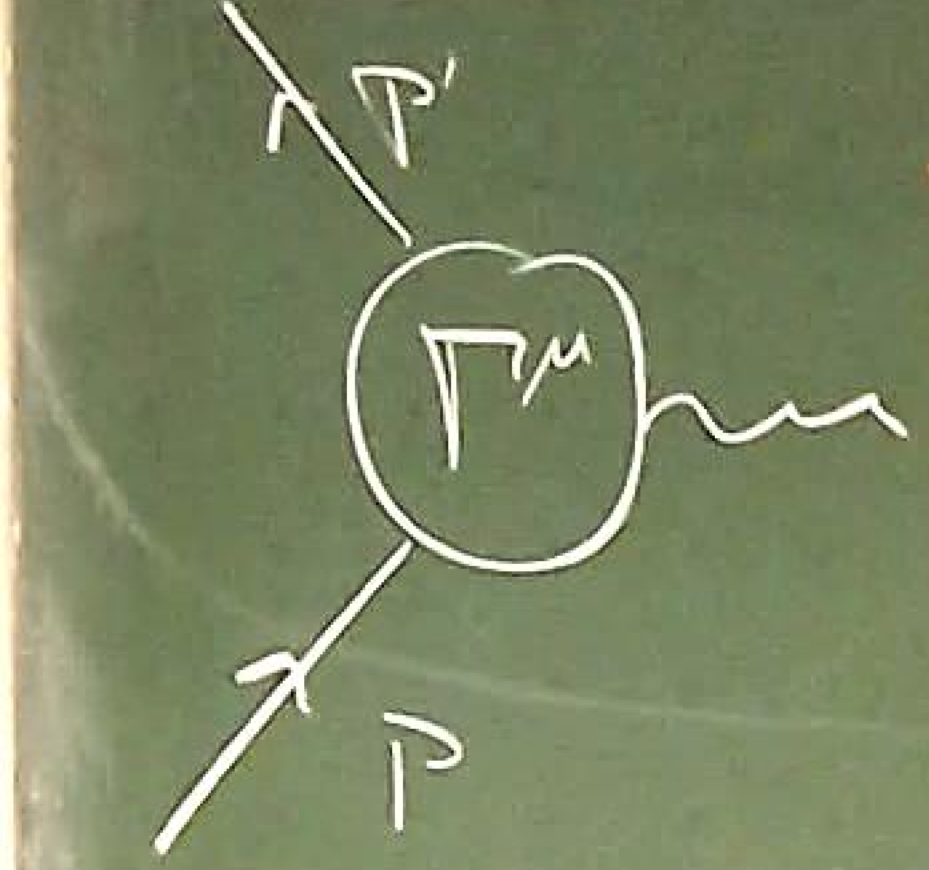
$\uparrow$

$e p$ | $e p$ $\quad \alpha \log \frac{q^2}{m^2} \log \frac{q^2}{\mu} \quad - \ - \ - \ - \ -$

$$\frac{d\sigma}{d\Omega} \sim \left.\frac{d\sigma}{d\Omega}\right|_0 \frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{E_{min}}\right)$$

Recap

6.3. The Electron Vertex Function

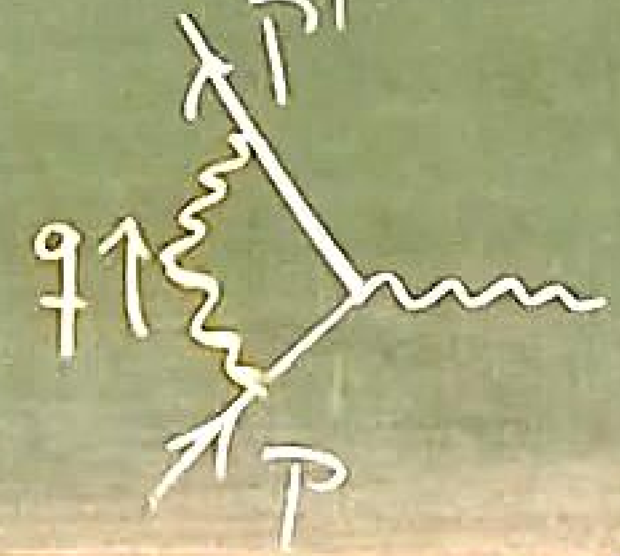


$$\boxed{\Gamma^\mu(p',p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2)}$$

$$\boxed{g = 2 + 2F_2(0)}$$

Anomalous magnetic moment: $a_e = \frac{g-2}{2} = F_2(0) = \underset{\uparrow}{\frac{\alpha}{2\pi}} + O(\alpha^2)$

Schwinger term



- $F_1(q^2) = 1 + \alpha F_1^{(1)}(q^2) + O(\alpha^2)$

$$F_1^{(1)}(q^2) = \frac{1}{2\pi}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1)\left[\log\left(\frac{m^2(1-z)^2}{m^2(1-z)^2 - q^2xy}\right) + \frac{m^2(1-4z+z^2)+q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy + z\mu^2} - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + \underset{\uparrow}{z\mu^2}}\right]$$

Small photon mass $\mu \to 0$ to regulate IR divergence

Problem: $F_1^{(1)} \xrightarrow{\mu \to 0} \infty$

How to solve this?

- $F_2(q^2) = \alpha F_2^{(1)}(q^2) + O(\alpha^2)$

$$F_2^{(1)}(q^2) = \underbrace{\frac{1}{2\pi}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1)\left[\frac{2m^2 z(1-z)}{m^2(1-z)^2 - q^2xy}\right]}_{\text{No divergence !}}$$

6.3.4 The Infrared Divergence

1| Goal: $|F_1(q^2)| \xrightarrow{\mu\to 0} \infty$?

2| Show:

$$F_1(q^2) = 1 - \frac{\alpha}{\textcircled{2}\pi} f_{IR}(q^2) \log\left(\frac{\overset{\{q^2, m^2\}}{\Lambda}}{\mu^2}\right) + O(\alpha^2)$$

$$f_{IR}(q^2) = \int_0^1 d\xi \, \frac{m^2 - q^2/2}{m^2 - q^2 \xi (1-\xi)} - 1 \geq 0$$

3| Cross section:

$$\frac{d\sigma(p\to p')}{d\Omega} \underset{\uparrow}{\sim} \left(\frac{d\sigma}{d\Omega}\right)_0 \cdot \left[1 - \frac{\alpha}{\pi} f_{IR}(q^2) \log\left(\frac{\Lambda}{\mu^2}\right) + O(\alpha^2)\right]$$

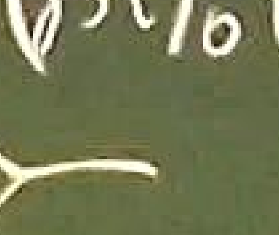
$$|M|^2 \sim |\Gamma^\mu|^2 \sim |F_1|^2$$

4| Limit $-q^2 \to \infty$

$$f_{IR}(q^2) \sim \int_0^1 d\xi \, \frac{-q^2/2}{-q^2 \xi(1-\xi) + m^2} \overset{\circ}{\sim} \log\left(\frac{-q^2}{m^2}\right)$$

$$\to F_1(-q^2 \to \infty) \overset{\mu\to 0}{\sim} 1 - \frac{\alpha}{2\pi} \underbrace{\log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right)}_{} + 0$$

5| $\cdot \ \frac{d\sigma(p\to p')}{d\Omega} \overset{\mu\to 0}{\sim} \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 \ominus \frac{\alpha}{\pi} \underline{\log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right)} + O(\alpha^2)\right]$



$\cdot \ \frac{d\sigma(p\to p'+\gamma)}{d\Omega} \overset{\mu\to 0}{\sim} \left(\frac{d\sigma}{d\Omega}\right)_0 \left[+ \frac{\alpha}{\pi} \underline{\log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right)} + O(\alpha^2)\right]$

detection threshold of detector

$\gamma(k < E_{min})$

6|

$$\left(\frac{d\sigma(p\to p')}{d\Omega}\right)_{\text{measured}} = \frac{d\sigma(p\to p)}{d\Omega} + \frac{d\sigma(p\to p'+\gamma)}{d\Omega}$$

7] General q:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} \overset{\mu\to 0}{=} \left(\frac{d\sigma}{d\Omega}\right)_0 \Bigg[\underbrace{\boxed{1-\frac{\alpha}{\pi} f_{IR}(q^2)\log\left(\frac{A}{\mu^2}\right)}}_{\text{Elastic scattering}} + \underbrace{\boxed{\begin{array}{l}\frac{\alpha}{2\pi}\,\mathcal{J}(p,p')\,\log\left(\frac{E_{\min}^2}{\mu^2}\right)\\ \quad , \\ \int\frac{d\Omega_k}{4\pi}\sum_r\left|\frac{p'\varepsilon^r}{p'q}-\frac{p\varepsilon^r}{pq}\right|^2\end{array}}}_{\text{Bremsstrahlung}} + O(\alpha^2)\Bigg]$$

8] $\mathcal{J}(p,p') \overset{*}{=} 2 f_{IR}(q^2)$

9]

$$\boxed{\begin{array}{l}\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} \overset{\mu\to 0}{\sim} \left[1-\frac{\alpha}{\pi} f_{IR}(q^2)\log\left(\frac{A}{E_{\min}^2}\right)+O(\alpha^2)\right]\left(\frac{d\sigma}{d\Omega}\right)_0 \\ \overset{q\to\infty}{\sim} \left[1\ominus\frac{\alpha}{\pi}\boxed{\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{E_{\min}^2}\right)}+O(\alpha^2)\right]\left(\frac{d\sigma}{d\Omega}\right)_0\end{array}}$$

Independend of μ

Sudakov double logarithm

Correct expression:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\substack{\text{measured}\\ p\to p'}} \sim \left(\frac{d\sigma}{d\Omega}\right)_0 \underbrace{\exp\left[-\frac{\alpha}{\pi} f_{IR}(q^2)\log\left(\frac{-q^2}{E_{\min}^2}\right)\right]}_{\text{Sudakov form factor}}$$

6.4. Field Strength Renormalization

6.4.1. Structure of Two-Point Correlators in Interacting Theories

$\sphericalangle$ ϕ^4-theory.

1| Goal: $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$

2| Interpretation for free theory:

$$\langle 0|T\underset{a^\dagger+a}{\phi(x)}\underset{a^\dagger+a}{\phi(y)}|0\rangle$$

$x^0>y^0$

= Amplitude of particle to propagate from y to x

3| Mathematical preliminaries:

i| $\sphericalangle$ Hilbert space $\mathcal{H}_{int}$ of interacting theory H

ii| Basis of $\mathcal{H}_{int}$.

$[H,\vec{P}]=0 \to |\lambda_{\vec p}\rangle$ eigenstates of H with $E_{\vec p}(\lambda)$ and momentum $\vec{P}$

iii| $\sphericalangle$ $P^\mu=\begin{pmatrix}H\\ \vec{P}\end{pmatrix}$ $\quad P^\mu|\lambda_{\vec p}\rangle=\begin{pmatrix}E_{\vec p}(\lambda)\\ \vec{p}\end{pmatrix}|\lambda_{\vec p}\rangle$

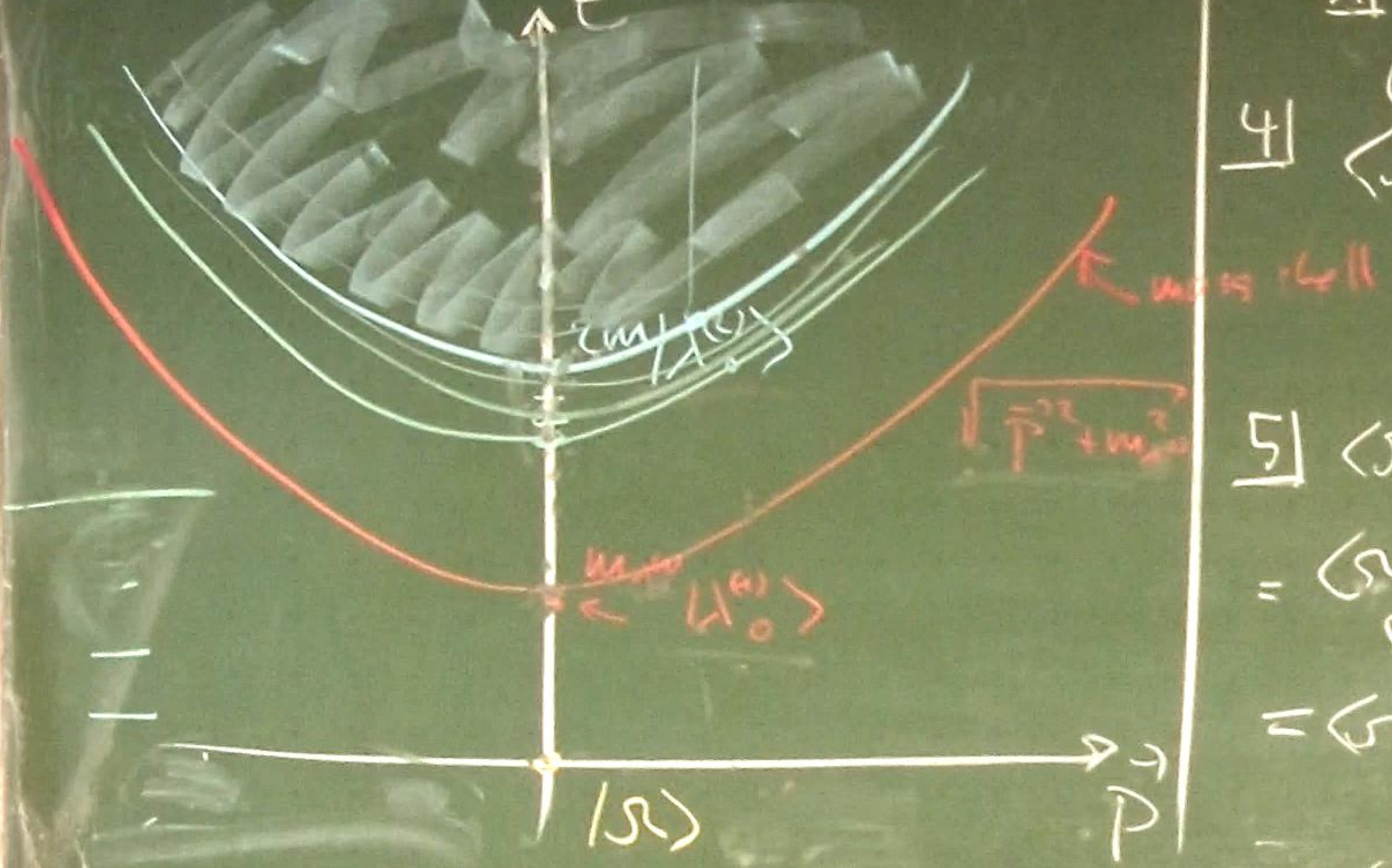
→ Boost $\Lambda_{\vec p}\in SO^+(1,3)$

$$\Lambda_{\vec p}\begin{pmatrix}m_\lambda\\ \vec{0}\end{pmatrix}=\begin{pmatrix}E_p(\lambda)\\ \vec{p}\end{pmatrix} \Rightarrow E_p(\lambda)=\sqrt{|\vec{p}|^2+m_\lambda^2}$$

→ $\forall\, |\lambda_{\vec p}\rangle\ \exists\, \Lambda_{\vec p}\ \exists\, |\lambda_0\rangle$:

$$\boxed{\begin{array}{l} \bullet\ |\lambda_{\vec p}\rangle = U(\Lambda_{\vec p})|\lambda_0\rangle \\ \bullet\ \begin{cases} H|\lambda_0\rangle = m_\lambda|\lambda_0\rangle \\ P|\lambda_0\rangle = 0 \end{cases} \\ \bullet\ \begin{cases} H|\lambda_{\vec p}\rangle = E_{\vec p}(\lambda)|\lambda_{\vec p}\rangle \\ P|\lambda_{\vec p}\rangle = \vec{p}\,|\lambda_{\vec p}\rangle \end{cases} \end{array}}$$

IV Typical spectrum of $P^\mu = (H, \vec P)$



3]

$$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda{}' \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}(\lambda)} |\lambda_{\vec p}\rangle\langle\lambda_{\vec p}|$$

$(x^0 > y^0)$

4] $\langle\Omega| \phi(x) \phi(y) |\Omega\rangle$

$$= \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} \underline{\langle\Omega| \phi(x) |\lambda_{\vec p}\rangle} \langle\lambda_{\vec p}| \phi(y) |\Omega\rangle + \underline{\text{const}}$$

5] $\langle\Omega| \phi(x) |\lambda_{\vec p}\rangle$

$$= \langle\Omega| e^{iPx} \phi(0) e^{-iPx} |\lambda_{\vec p}\rangle$$

$$= \langle\Omega| \phi(0) |\lambda_{\vec p}\rangle \, e^{-ipx}\Big|_{p^0 = E_{\vec p}(\lambda)}$$

ϕ scalar

$$= \langle\Omega| \phi(0) |\lambda_0\rangle \, e^{-ipx}\Big|_{p^0 = E_{\vec p}(\lambda)}$$

$$U^\dagger(\Lambda) \phi(0) U(\Lambda) = \phi(\Lambda^\mu{}_\nu 0^\nu) = \phi(0)$$

6] $\langle\Omega| \phi(x) \phi(y) |\Omega\rangle$

$$= \sum_\lambda |\langle\Omega| \phi(0) |\lambda_0\rangle|^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} e^{-ip(x-y)}\Big|_{p^0 = E_p(\lambda)}$$

$$\overset{x^0 > y^0}{=} \sum_\lambda \text{— | —} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_\lambda^2 + i\varepsilon} e^{-ip(x-y)}$$

$$\overset{x^0 < y^0}{=} \langle\Omega| \phi(y) \phi(x) |\Omega\rangle$$

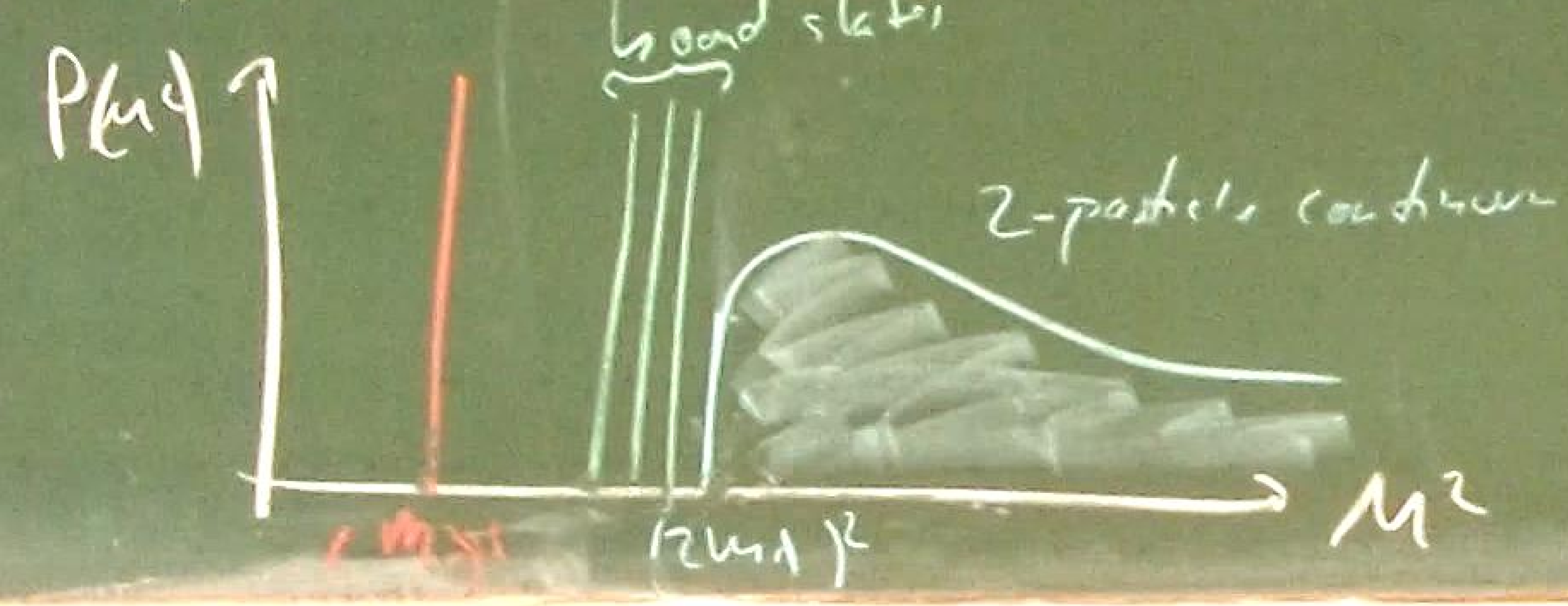
7) → Källén-Lehmann spectral representation

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int\frac{dM^2}{2\pi}\,\rho(M^2)\,D_F(x-y;M^2)$$

Spectral density:

$$\rho(M^2) = 2\pi\sum_\lambda \delta(M^2-m_\lambda^2)\,|\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$

8) Typical spectral density:



Recap:

6.4. Field Strength Renormalization

6.4.1 Structure of Two-Point Correlators in interacting theories

7] Källén-Lehmann spectral representation:

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^{\infty}\frac{dM^2}{2\pi}\,\underbrace{\rho(M^2)}\,\underbrace{D_F(x-y,M^2)}$$

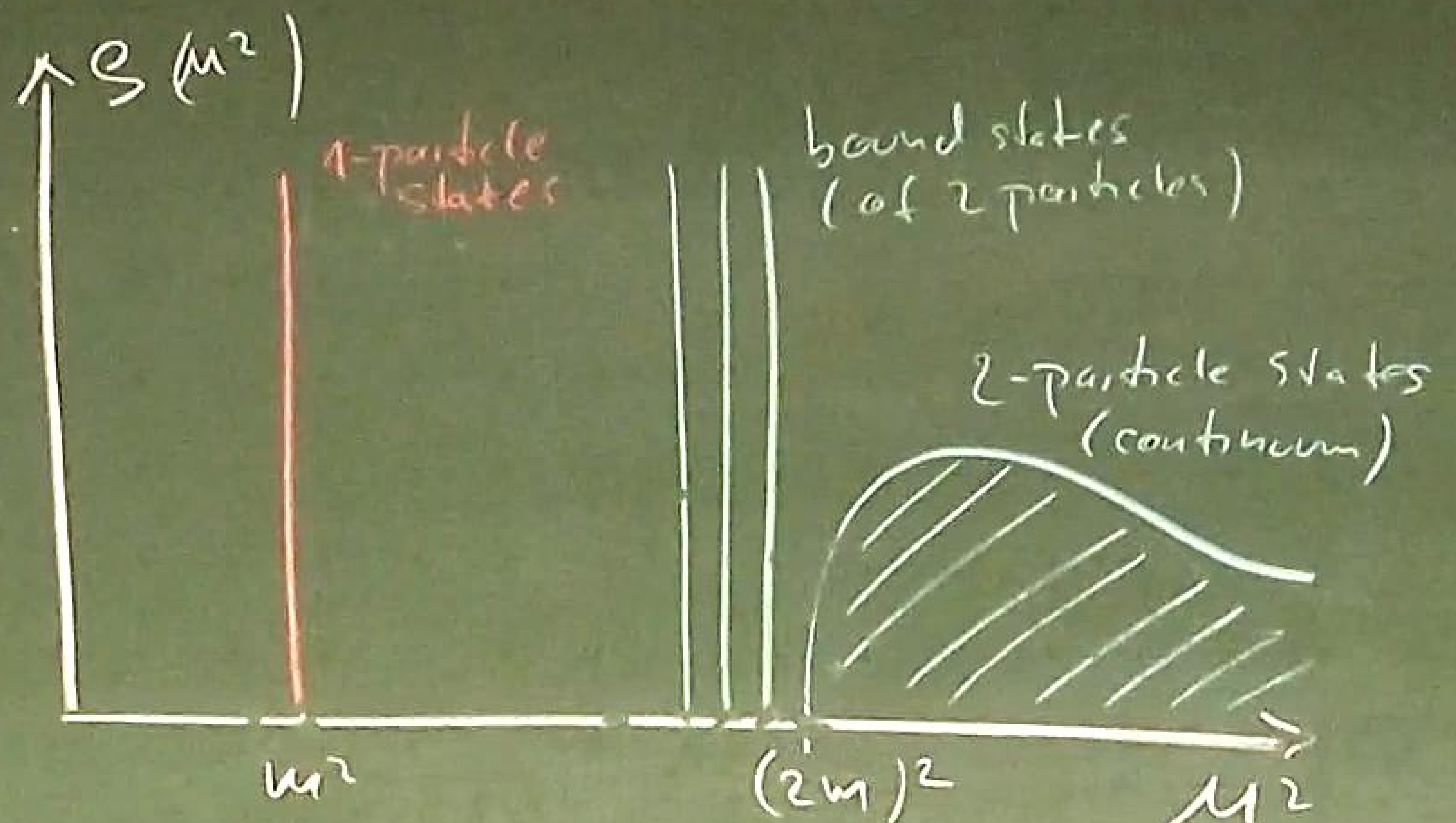
Spectral density: ←

Feynman propagator for mass M

$$\rho(M^2) = 2\pi\sum_\lambda \delta(M^2-m_\lambda^2)\,|\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$

eigenenergy of $|\lambda_0\rangle$ $\quad \vec{p}=0$

8] Typical spectral density:



$$\rho(M^2) = 2\pi\,\delta(M^2-m^2)\cdot Z + \left\{\begin{array}{l}\text{multi particle}\\ \text{states for}\\ M^2 \gtrsim (2m)^2\end{array}\right\}$$

with: $\lambda = 1$

- **Field strength renormalization**

  $Z = |\langle\Omega|\phi(0)|1_0\rangle|^2$

- **Physical mass:**

  $m = m_1$ (given by $H|1_0\rangle = m_1|1_0\rangle$)

- **Bare mass:**

  m_0 (given by $H = \ldots \frac{1}{2}m_0^2\phi^2 \ldots$)

- **Free theory:**

  $Z = |\langle 0|\phi(0)|\underbrace{\vec{p}=0\rangle}_{\sqrt{2E_{\vec{p}}}\,a_{\vec{p}}^\dagger|0\rangle}|^2 = 1$

  $m = m_0$

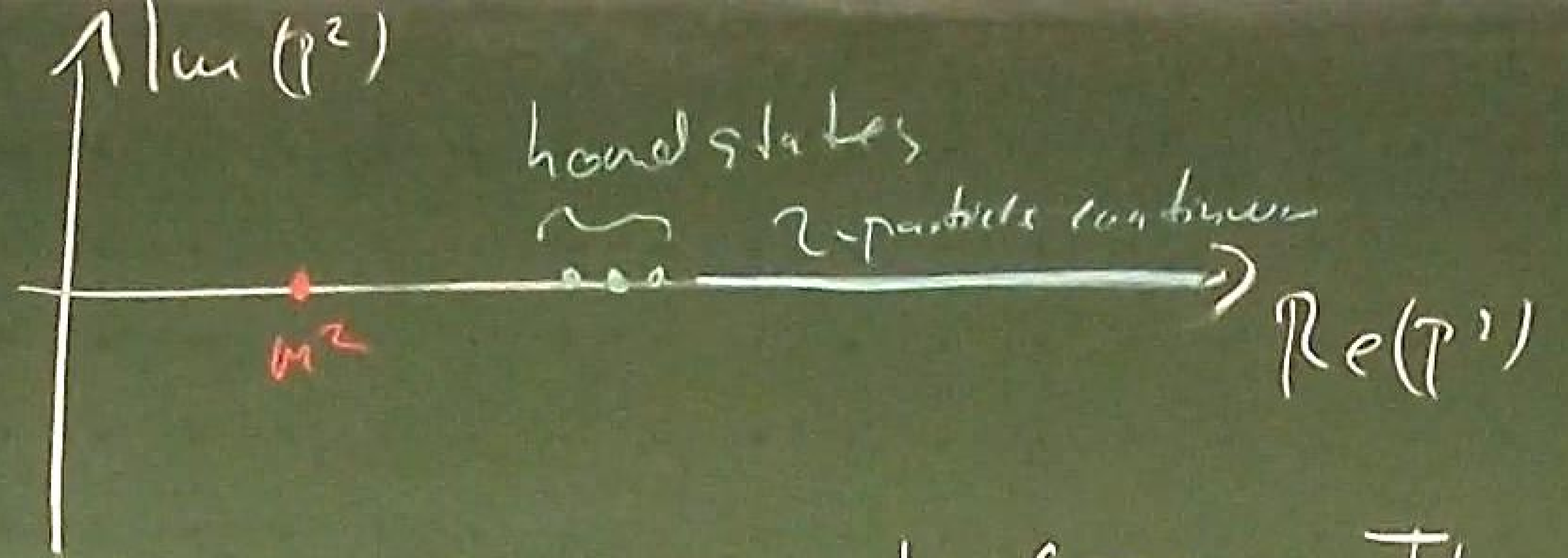
9) Fourier transform:

$$\int d^4x\, e^{ipx}\langle\Omega|T\phi(x)\phi(0)|\Omega\rangle$$

$$= \int \frac{dM^2}{2\pi}\frac{i\rho(M^2)}{p^2-M^2+i\varepsilon}$$

$$= \frac{iZ}{p^2-m^2+i\varepsilon} + \int_{\sim(2m)^2}^{\infty}\frac{dM^2}{2\pi}\frac{i\rho(M^2)}{p^2-M^2+i\varepsilon}$$

$$\overset{\text{free}}{=} \frac{i\cdot 1}{p^2-m_0^2+i\varepsilon}$$



6.4.2 Application to QED: The Electron Self-Energy

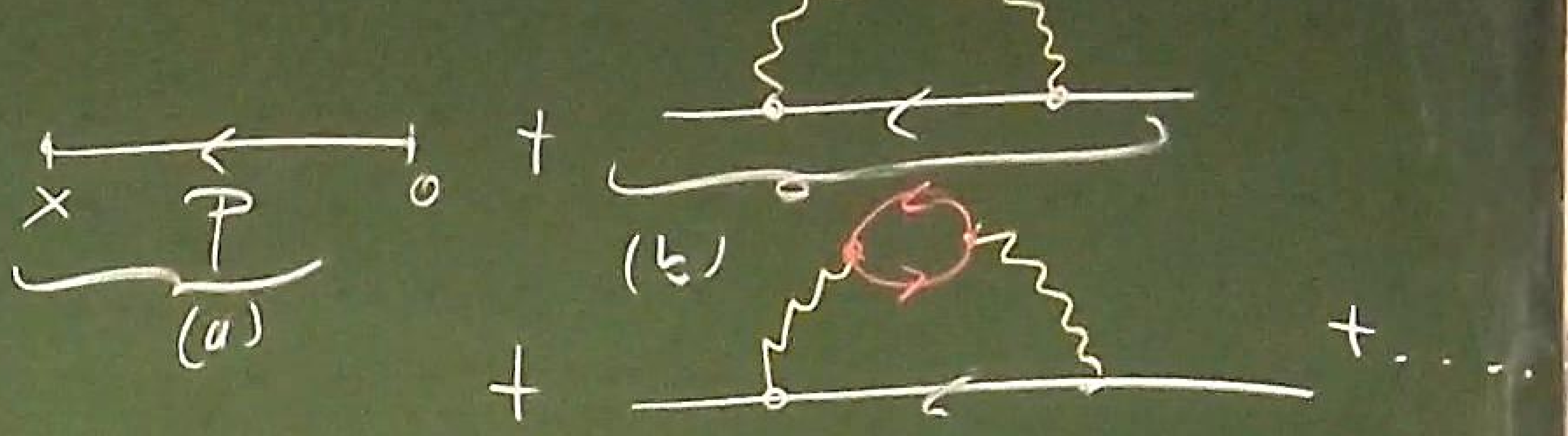
1) ϕ^4 theory → QED

$$\int d^4x\, e^{ipx}\langle\Omega|T\psi(x)\bar\psi(0)|\Omega\rangle \overset{*}{=} \frac{iZ_2(\not p+m)}{p^2-m^2+i\varepsilon} + \ldots$$

2) On the other side:

$$\int d^4x\, e^{ipx}\langle\Omega|T\psi(x)\bar\psi(0)|\Omega\rangle$$

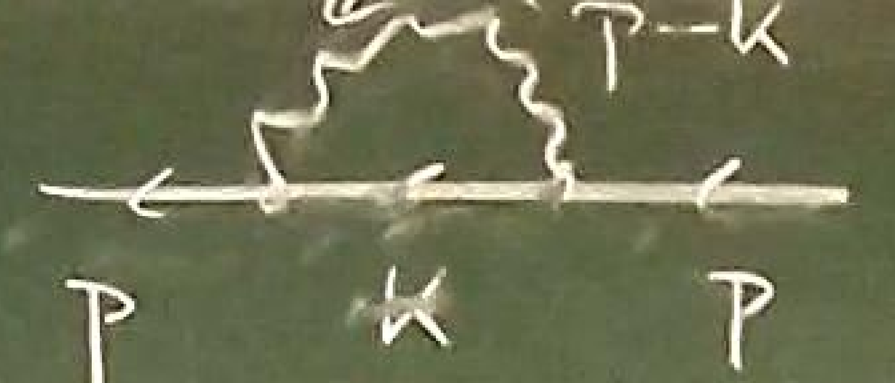
=



3) α^0-order:

$$(a) = \frac{i(\not p+m_0)}{p^2-m_0^2+i\varepsilon}$$

4] α^1-order.



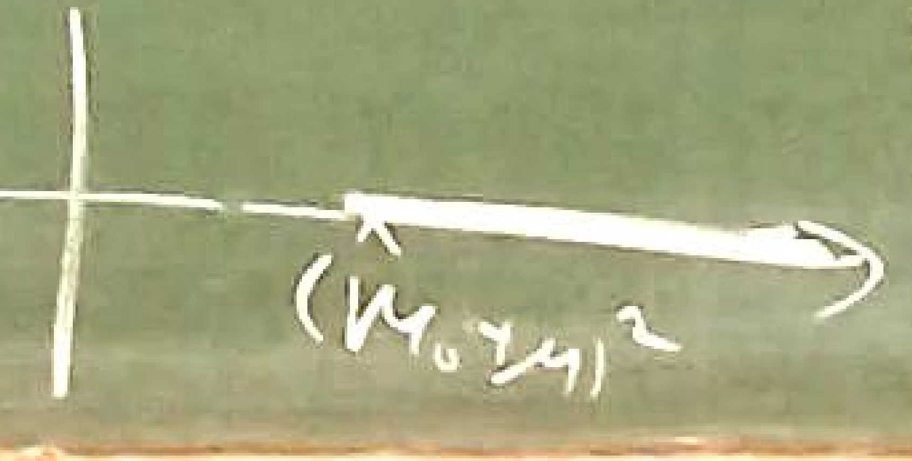
$$(b) = \frac{i(\not p + m_0)}{p^2 - m_0^2 + i\varepsilon} \underbrace{\left[(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu \frac{i(\not k + m_0)}{k^2 - m_0^2 + i\varepsilon} \gamma_\mu \frac{-i}{(p-k)^2 + i\varepsilon} \right]}_{-i\Sigma_2(p)} \frac{i(\not p + m_0)}{p^2 - m_0^2 + i\varepsilon}$$

→ IR and UV divergence

$$\Sigma_2(p) \overset{\Lambda \to \infty}{\sim} \frac{\alpha}{2\pi} \int_0^1 dx \, (2m_0 - x\not p) \log\left[\frac{x\Lambda^2}{(1-x)m_0^2 + x\mu^2 - x(1-x)p^2} \right]$$

↑

Problemset 10



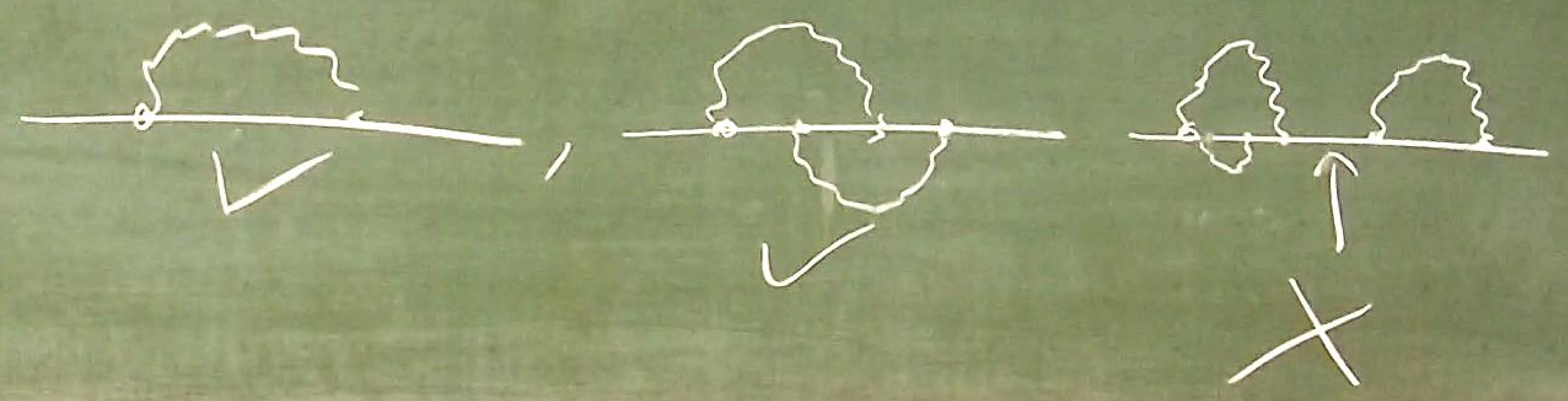
$p^2 \geq (m_0+\mu)^2$

5] Summation to all orders in α:

i] Definition:

One-particle irreducible (1PI) diagrams
= bridgeless graphs

Example:



Define

$$-i\Sigma(p) := \{\text{Sum of all 1PI diagrams}\}$$

$$=$$

$$= -i\Sigma_2(p) + \mathcal{O}(\alpha^2)$$

$$\Sigma(p) = f(\underbrace{\gamma^\mu p_\mu}_{\not p}) + g(\underbrace{p^\mu p_\mu}_{p^2 = \not p^2}) + c(\underbrace{\gamma^\mu \gamma_\mu}_{4})$$

$$= \Sigma(\not p)$$

ii) $\int d^4x\, e^{ipx} \langle \Omega | T \psi(x) \psi(0) | \Omega \rangle$

$$= \{\text{Sum of all one-particle diagrams}\}$$

$$=$$

$$= \frac{i\cancel{(\not p + m_0)}}{p^2 - m_0^2} + \frac{i\not p + m_0}{p^2 - m_0^2}\left(-i\Sigma(p)\right)\frac{i(\not p + m_0)}{p^2 - m^2} + \dots\dots$$

$$'' \quad (\not p - m_0)\cancel{(\not p + m_0)} \qquad \not p^2 = p^2 \cdot \mathbb{1}$$

$$= \frac{i}{\not p - m_0} \sum_{n=0}^{\infty} \left(\frac{\Sigma(\not p)}{\underbrace{\not p - m_0}_{q}} \right)^n = \frac{1}{\not p - m_0} \frac{1}{1 - \frac{\Sigma(\not p)}{\not p - m_0}}$$

$$= \frac{1}{\not p - m_0 - \Sigma(\not p)}$$

6) Laurent series:

$$\frac{1}{\not p - m_0 - \Sigma(\not p)} \overset{!}{=} \frac{iZ_2}{\not p - m} + \dots\dots$$

→ Pole for $\not p = \mathbb{1} m = m$.

$$\boxed{m - m_0 = \Sigma(\not p = m)}$$

(implicit eq. for m)

→ Expand denominator around the root:

$$\not p - m_0 - \Sigma(\not p) = (\not p - m)\boxed{\left(\mathbb{1} - \frac{d\Sigma}{d\not p}\right)_{\not p = m}} + \mathcal{O}((\not p - m)^2)$$

6]

$$\Rightarrow \boxed{Z_2 = \left(1 - \frac{d\Sigma(\not p)}{d\not p}\Big|_{\not p = m}\right)^{-1}}$$

7] Results in leading order $O(\alpha)$:

d-10

i] Physical mass:

$$\delta m = m - m_0 = \Sigma(\not p = m)$$

$$= \underline{\Sigma_2(\not p = m) + O(\alpha^2)}$$

$$= \underline{\Sigma_2(\not p = m_0)} + O(\alpha^2)$$

$E^2 + \vec p^{\,2}$

$$\overset{\Lambda\to\infty}{\sim} \frac{3\alpha}{4\pi} m_0 \log\left(\frac{\Lambda^2}{m_0^2}\right) \xrightarrow{\Lambda\to\infty} \infty$$

↑ PSet 10

⇒ Mass shift δm is UV divergent

ii] Field strength renormalization:

$$\delta Z_2 = Z_2 - 1 \overset{\uparrow}{=} \frac{d\Sigma_2}{d\not p}\Big|_{\not p = m_0} + O(\alpha^2)$$

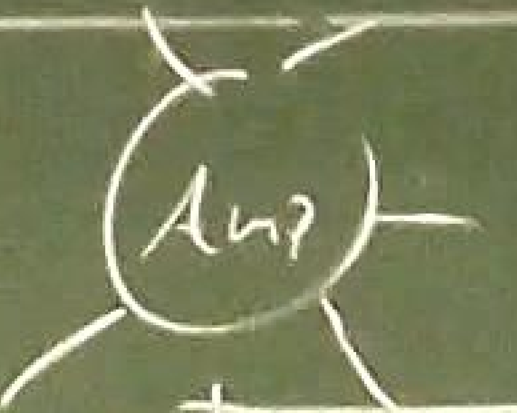
$$\frac{1}{1-x} = 1 + x + O(x^2)$$

$$= \frac{\alpha}{2\pi}\int_0^1 dx \left\{ -x\log\left[\frac{x\Lambda^2}{(1-x)^2 m^2 + x\mu^2}\right] + 2(2-x)\frac{x(1-x)m^2}{(1-x)^2 m^2 + x\mu^2}\right\}$$

→ Field strength renormalization is UV divergent.

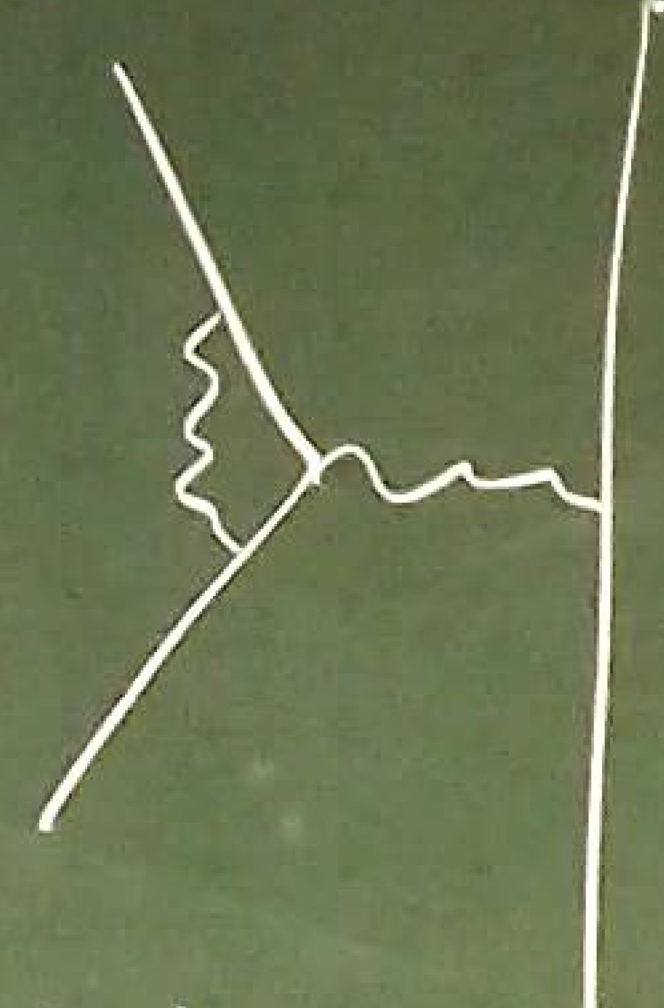
- $\delta Z_2 = -F_1^{(1)}(0)$

$\langle \mu \rangle = \sqrt{Z_2}^{\#}$

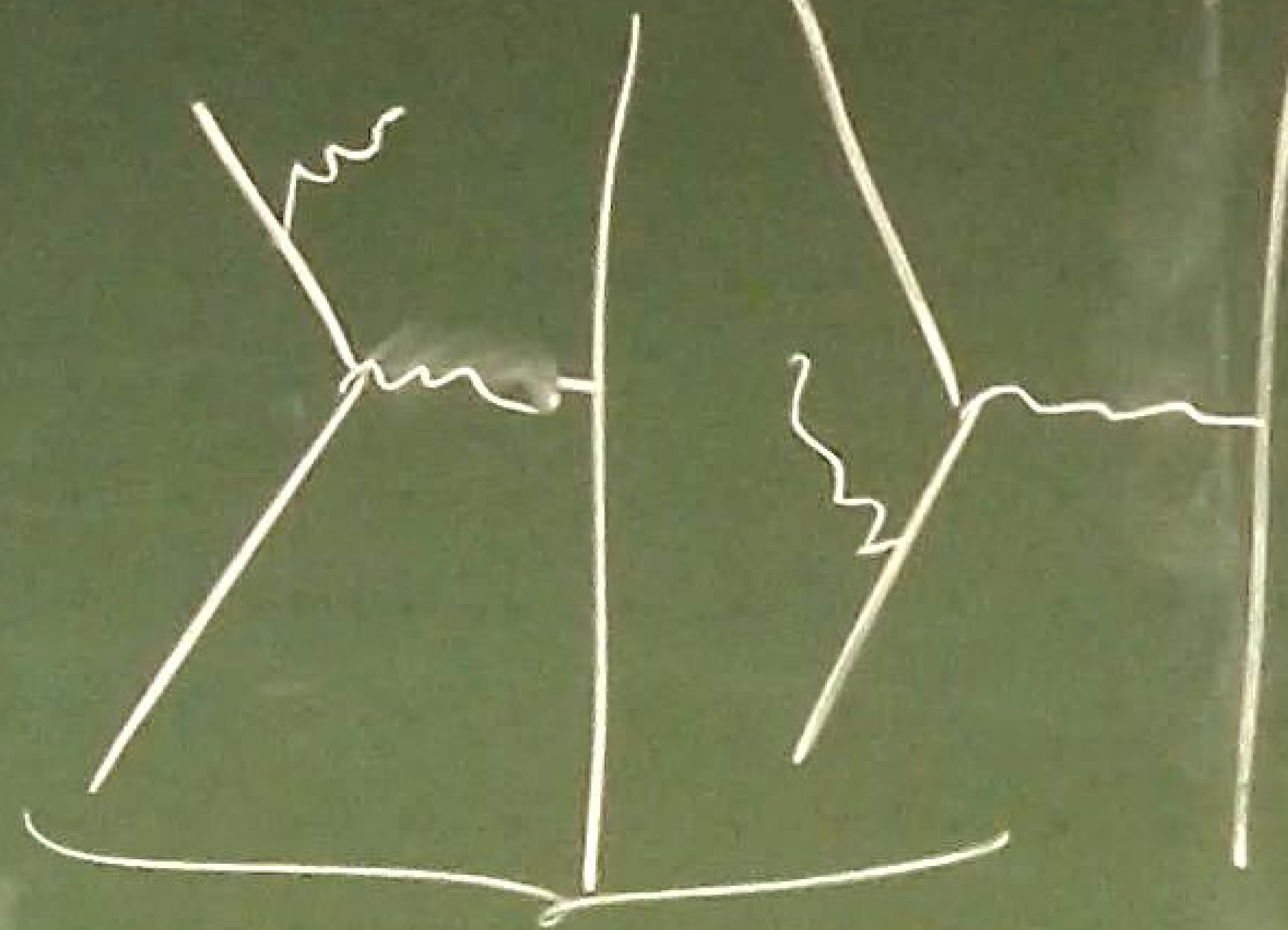


$$\Rightarrow F_1(q^2) = 1 + \boxed{F_1^{(1)}(q^2) \underline{+ \delta Z_2} - F_1^{(1)}(0)}$$

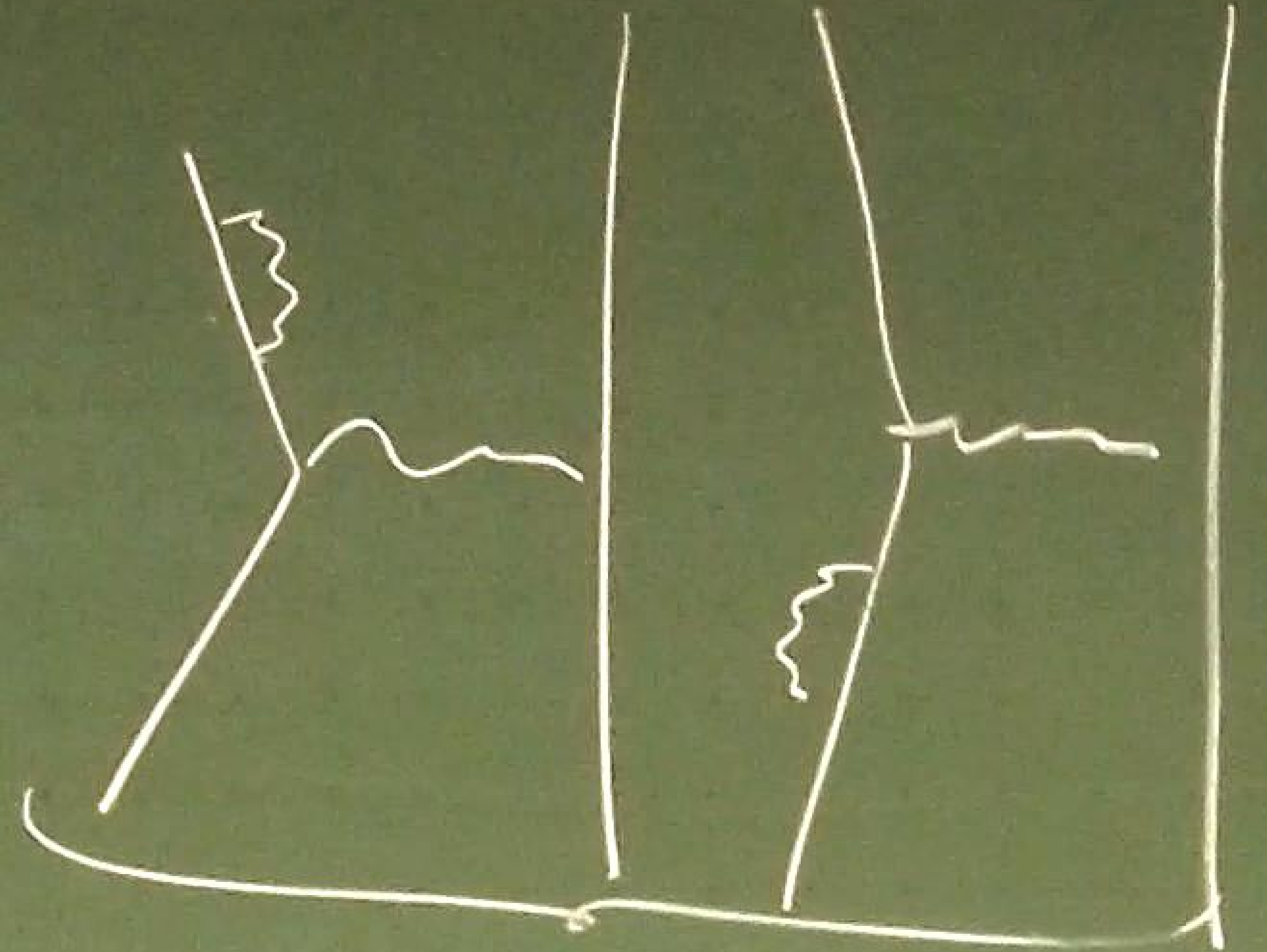
6.5. Electric Charge Renormalization



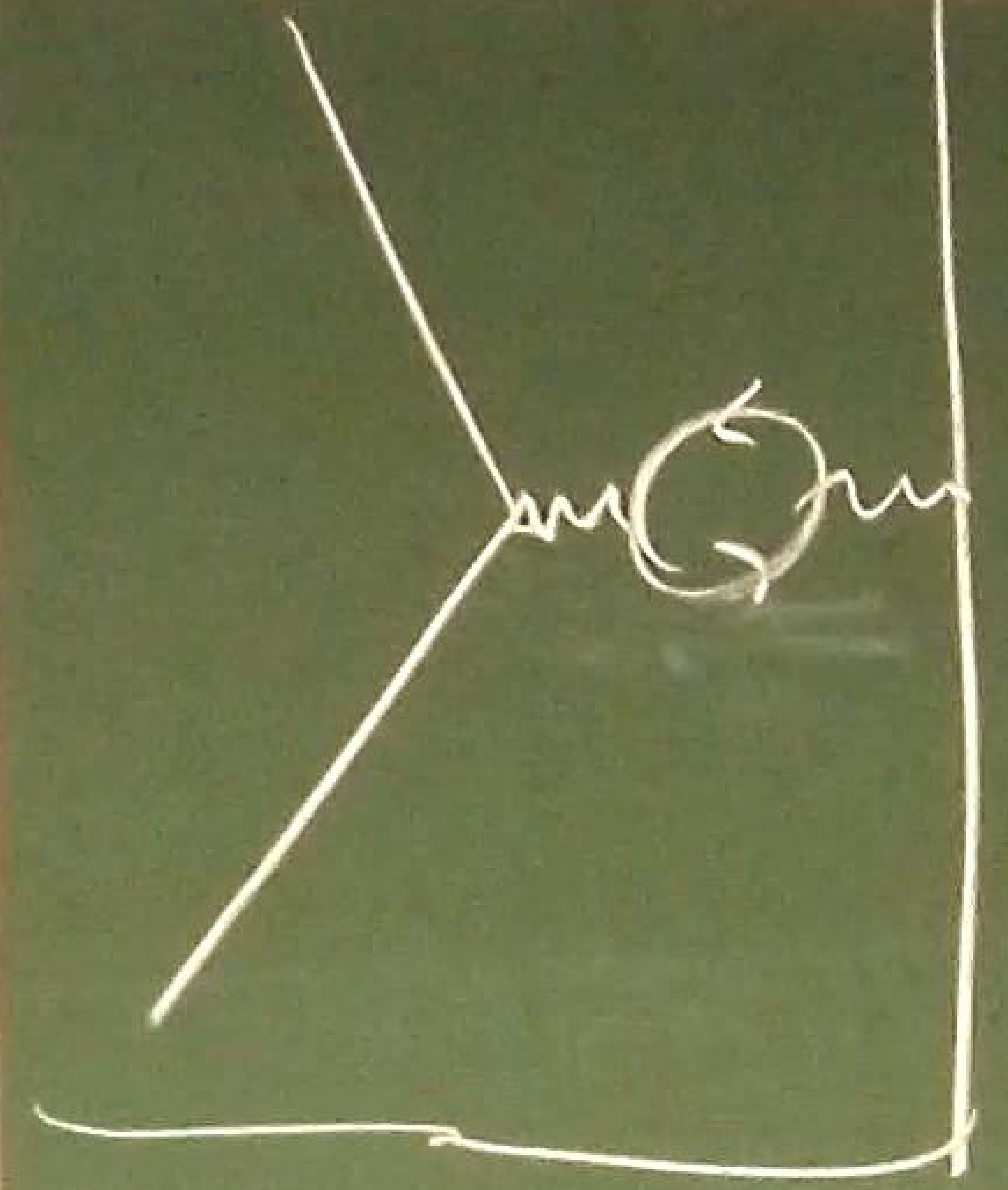
Vertex correction



Soft bremsstrahlung



Electron self-energy



- Photon self-energy
- Vacuum polarization diagram

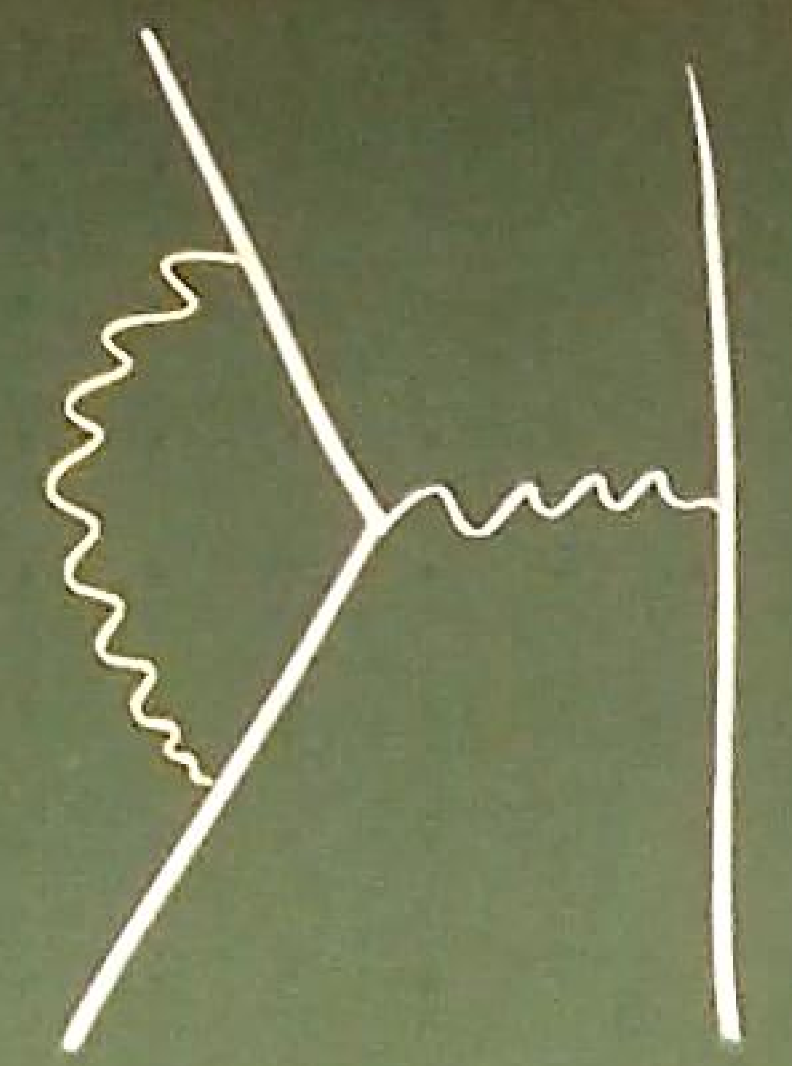
$$-i\gamma_\nu \frac{g^{\mu\nu}}{q^2} \frac{e_0^2}{(\quad q^2)} -i\gamma_\mu$$

$\alpha_{eff}(q^2)$

$S_{(\frac{1}{2})}$ $P_{(\frac{1}{2})}$

Recap:

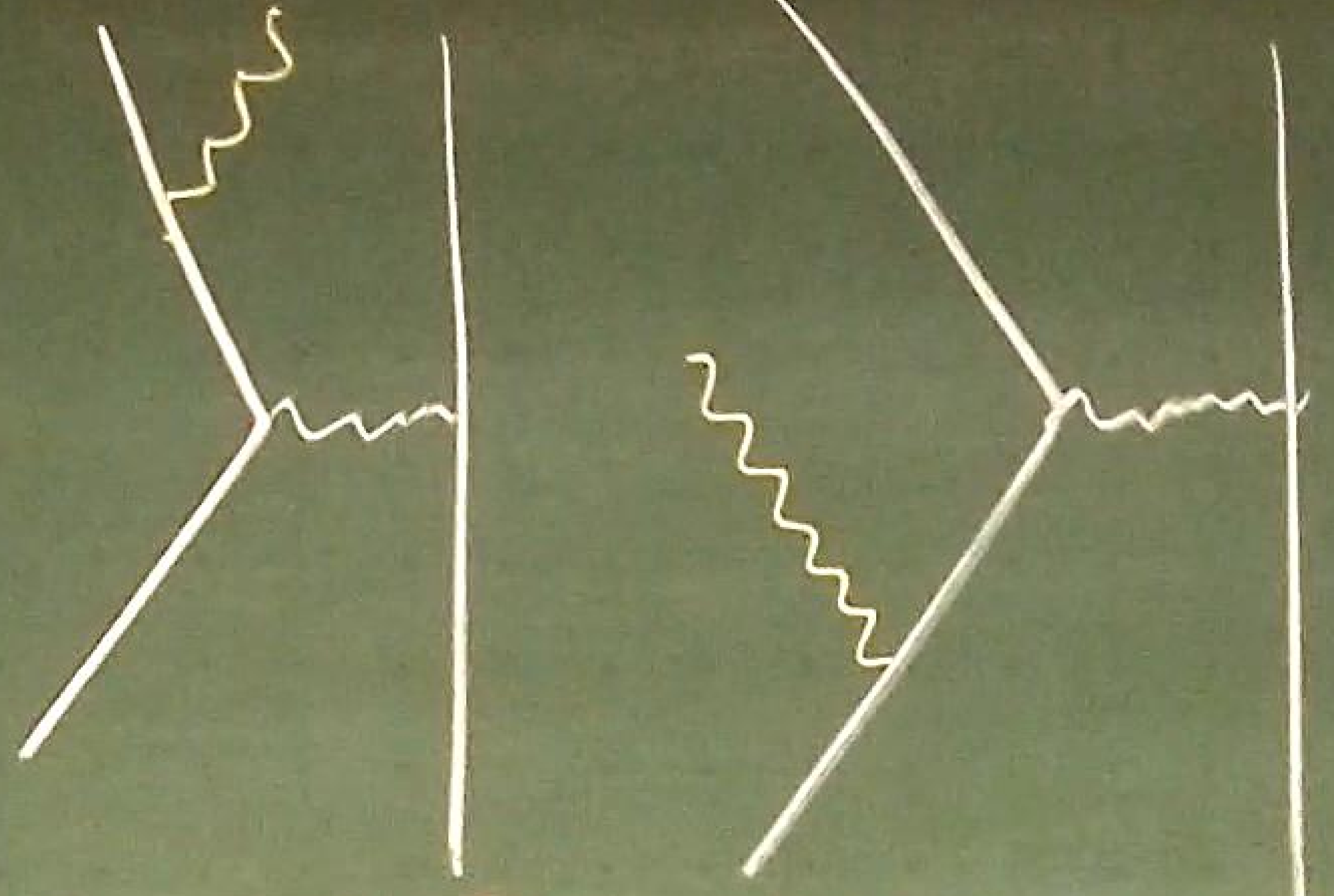
Radiative corrections:



Vertex correction

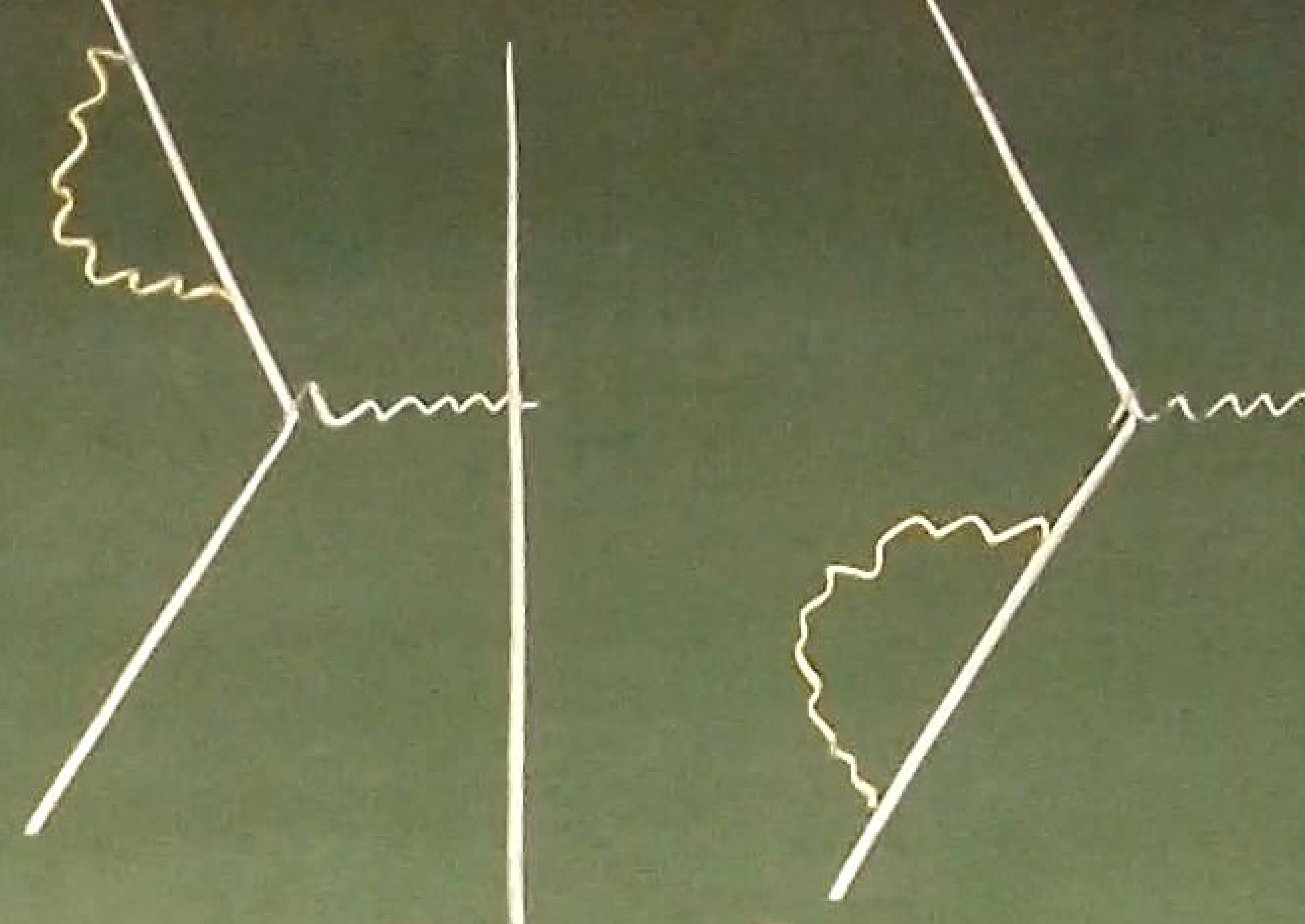
- Form factors F_1, F_2
- Anomalous magnetic moment:

$$\boxed{a = \frac{g-2}{2} = F_2(0) = \frac{\alpha}{2\pi} + \dots}$$



Soft Bremsstrahlung.

- Sudakov Form factor
- Cancelled IR div. of vertex correction



Electron Self-Energy.

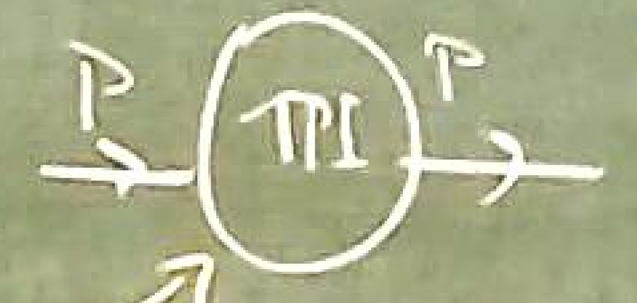
- Field strength renormalization.

$$\boxed{Z_2 = \left(1 - \frac{d\Sigma}{d\not{p}}\Big|_{\not{p}=m}\right)^{-1}}$$

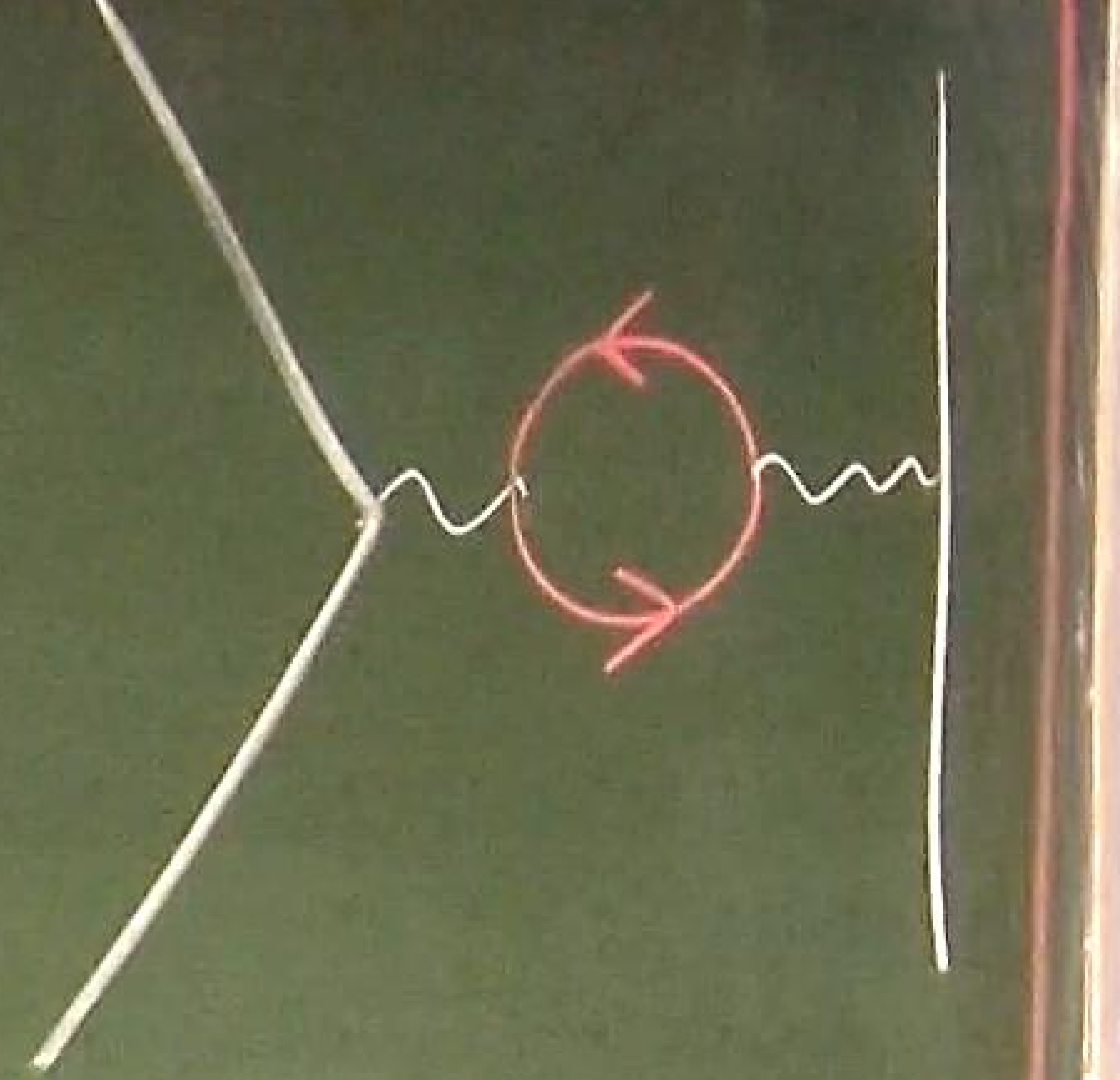
- Physical mass:

$$\boxed{m - m_0 = \Sigma(\not{p}=m)}$$

$-i\Sigma_1 =$



Sum of 1-particle irreducible diagrams



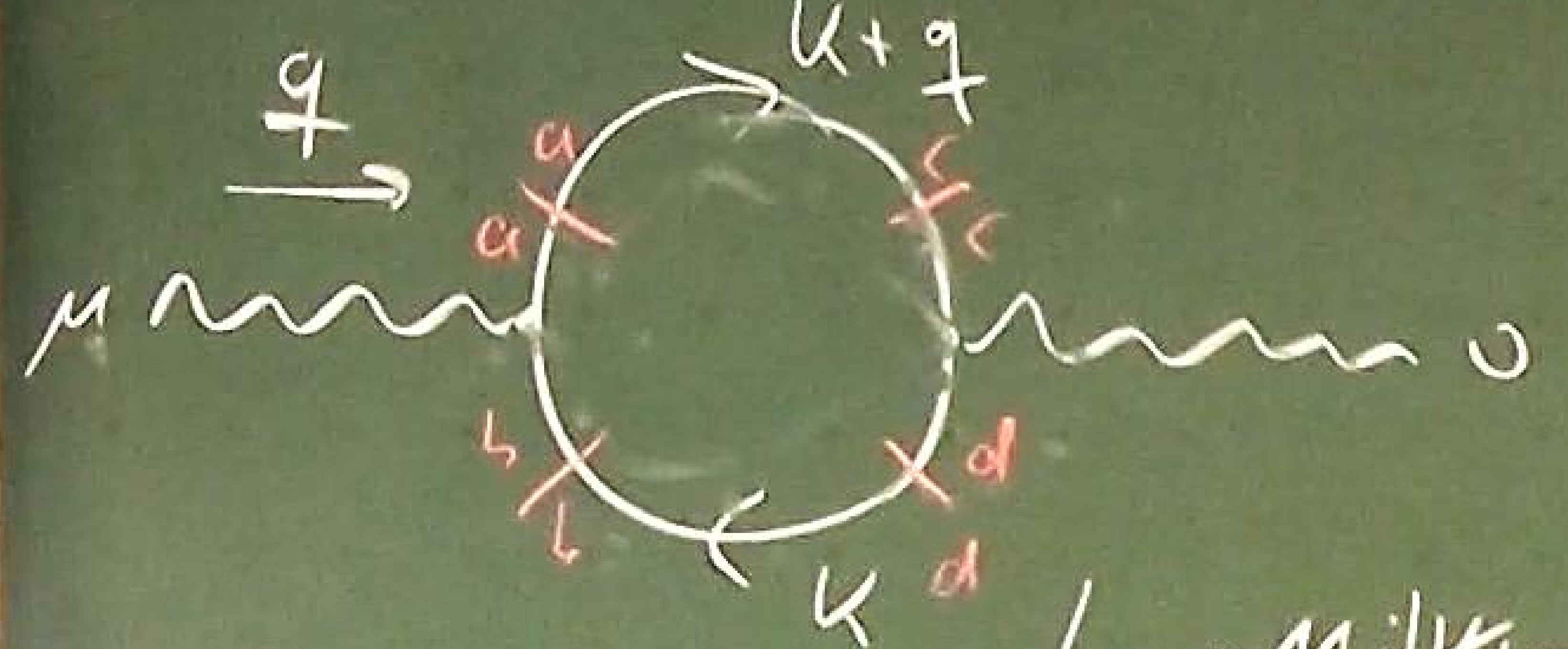
Vacuum-Polarization / Photon Self-Energy

⇓

Today

6.5. Electric Charge Renormalization

1| One-loop correction:

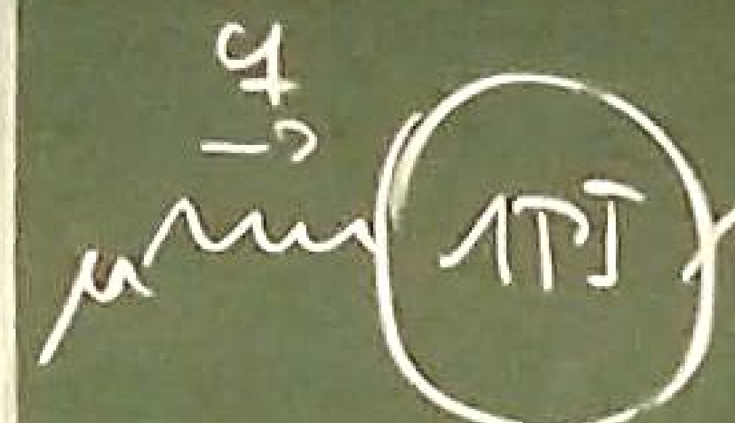


$$=(-1)(-ie)^2\int\frac{d^4k}{(2\pi)^4}\left(\gamma^\mu_{ab}\frac{i(\cancel{k}+m)_{bd}}{k^2-m^2}\gamma^\nu_{dc}\frac{i(\cancel{k}+\cancel{q}+m)_{ca}}{(k+q)^2-m^2}\right)$$

$$=(-1)(-ie)^2\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left[\gamma^\mu\frac{i(\cancel{k}+m)}{k^2-m^2}\gamma^\nu\frac{i(\cancel{k}+\cancel{q}+m)}{(k+q)^2-m^2}\right]\boxed{\equiv i\Pi_2^{\mu\nu}(q)}$$

- $\bar\psi\psi\bar\psi\psi = \bar\psi\psi\psi\bar\psi = -\psi\bar\psi\psi\bar\psi$

2| ☆ Sum of all 1PI diagrams:



$$\equiv i\Pi^{\mu\nu}(q) = i\left[\Pi_2^{\mu\nu}(q) + O(\alpha^2)\right]$$

i| Only tensors available: $g^{\mu\nu}; q^\mu q^\nu$

$$\hookrightarrow \Pi^{\mu\nu}(q) = A(q^2)\, g^{\mu\nu} + B(q^2)\, q^\mu q^\nu$$

ii| Ward identity: $q_\mu M^\mu \overset{*}{=} 0$

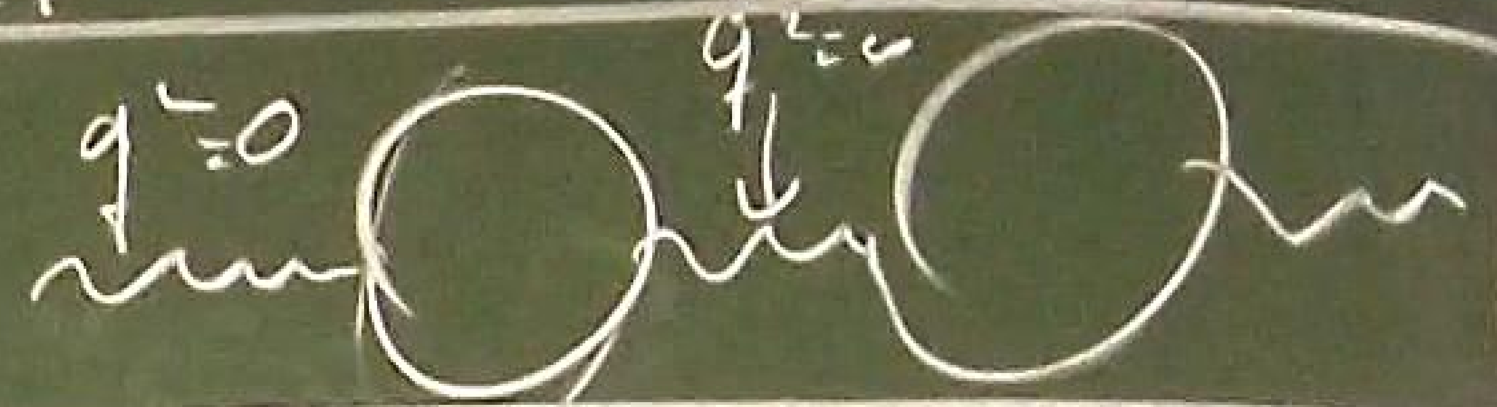
$$q_\mu \Pi^{\mu\nu}(q) \overset{*}{=} 0$$

$$\rightarrow B = -\frac{A}{q^2}$$

$$\rightarrow \Pi^{\mu\nu}(q) = \underline{(q^2 g^{\mu\nu} - q^\mu q^\nu)\frac{A}{q^2}}$$

iii| $\xrightarrow{*} \Pi^{\mu\nu}(q)$ has no pole at $q^2=0$

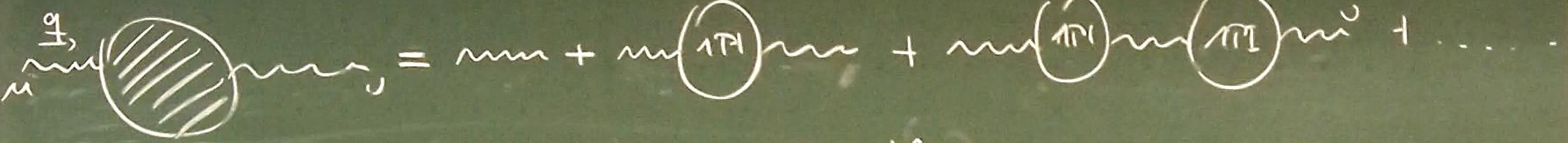
$$\frac{g^{\mu\nu}}{q^2+i\varepsilon}$$



$$\rightarrow \Pi(q^2) \equiv \frac{A(q^2)}{q^2} \text{ regular at } q^2=0$$

$$\hookrightarrow \boxed{\Pi^{\mu\nu}(q^2) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\,\Pi(q^2)}$$

3] X : Sum of all diagrams :



$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\left[i(q^2 g^{\rho\sigma} - q^\rho q^\sigma)\Pi(q^2)\right]\frac{-ig_{\sigma\nu}}{q^2} + \dots\dots$$

$$\Delta^\rho_\nu = \delta^\rho_\nu - \frac{q^\rho q_\nu}{q^2} \quad , \quad g^{\rho\sigma} g_{\sigma\nu} = \delta^\rho_\nu$$

$$i\Pi^{\rho\sigma}$$

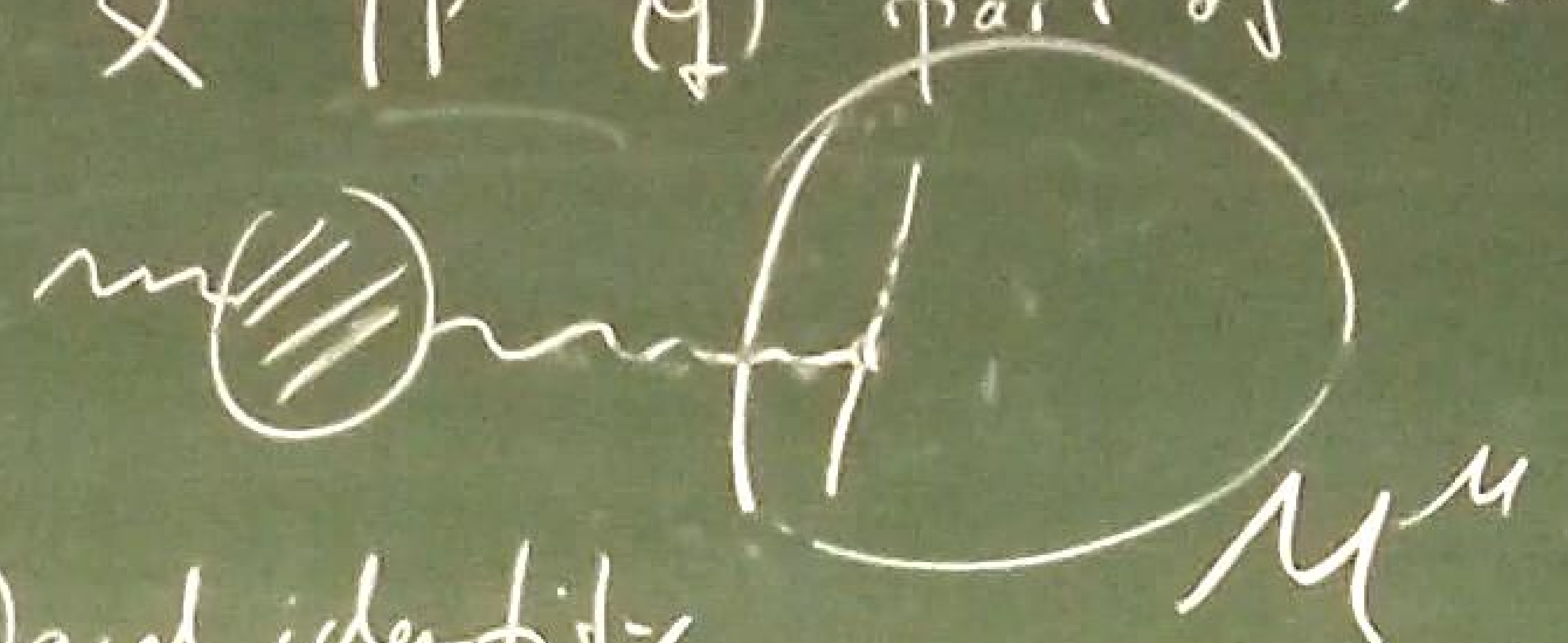
$$= -\frac{ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\Delta^\rho_\nu\,\Pi(q^2) + \frac{-ig_{\mu\rho}}{q^2}\Delta^\rho_\sigma\Delta^\sigma_\nu\,\Pi^2(q^2) + \dots$$

$$\Delta^\rho_\sigma\Delta^\sigma_\nu = \Delta^\rho_\nu$$

$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\Delta^\rho_\nu \boxed{\sum_{n=1}^{\infty}\Pi^n(q^2)} \longrightarrow \left(\frac{1}{1-\Pi(q^2)} - 1\right)$$

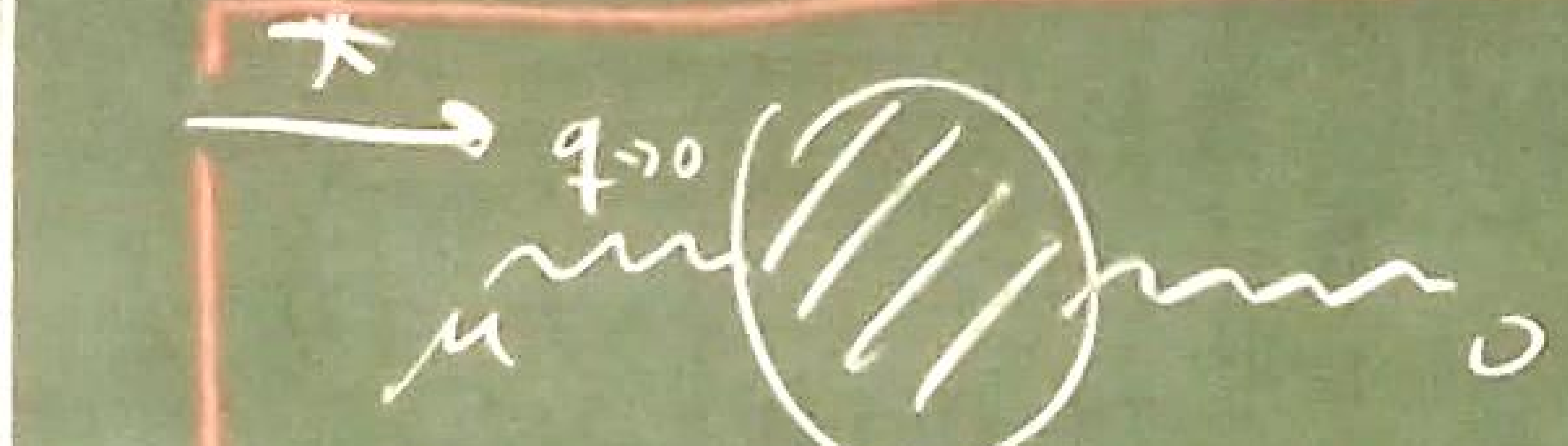
$$= \frac{-i}{q^2(1-\Pi(q^2))}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i}{q^2}\frac{q_\mu q_\nu}{q^2}$$

4] X $\Pi^{\mu\nu}(q)$ part of S-matrix computation.



$$q_\mu M^\mu \overset{*}{=} 0$$

Ward identity

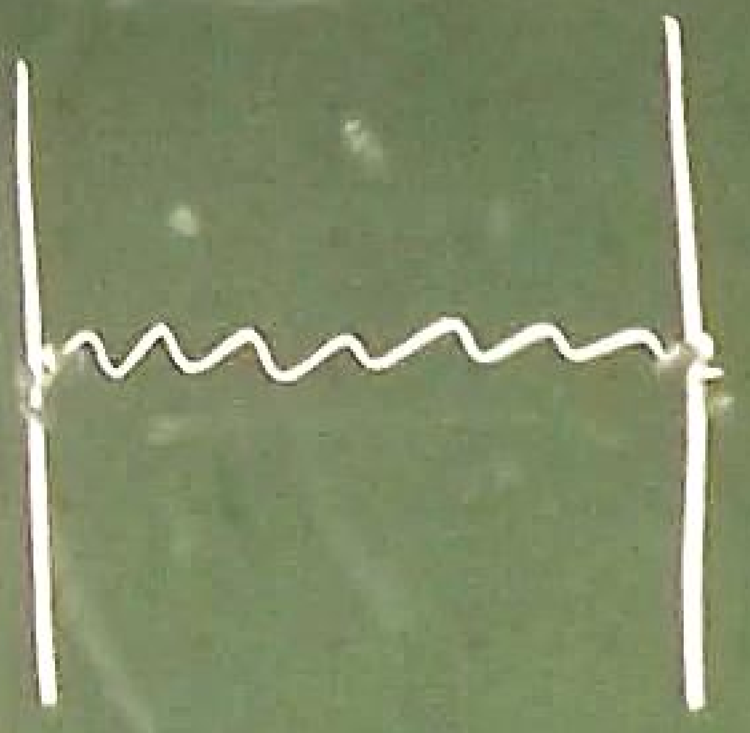


$$\boxed{\quad \hat{=} \quad \frac{-ig_{\mu\nu}}{q^2\left[1-\Pi(q^2)\right]}}$$

5) Charge renormalization.

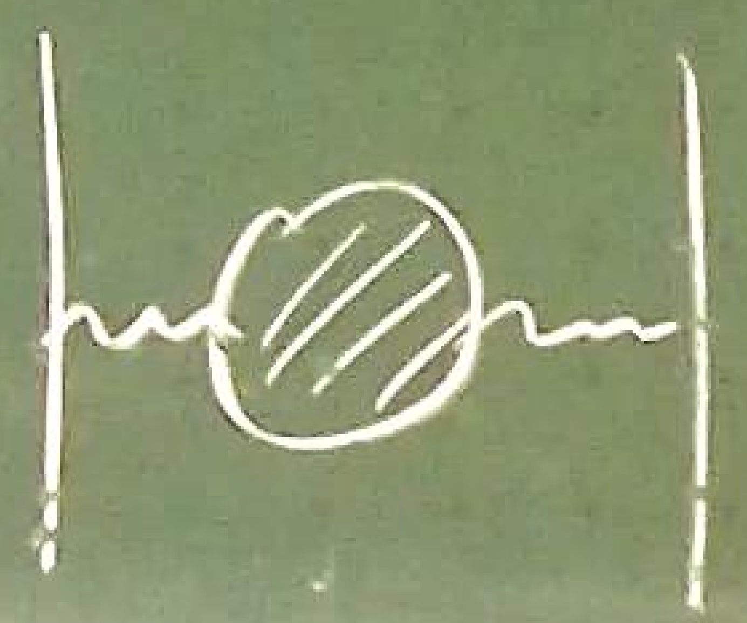
i) Define $$\boxed{Z_3 \equiv \frac{1}{1-\Pi(0)}}$$

& $q^2 \to 0$



$$= \frac{e^2 g_{\mu\nu}}{q^2}$$

$\rightarrow$



$$= \frac{\boxed{Z_3 e_0^2}\, g_{\mu\nu}}{q^2}$$

ii) $\rightarrow$ Charge renormalization:

Bare charge e_0 (given by $H = e_0 \bar{\psi} \gamma^\mu \psi A_\mu$)

Physical charge $e \equiv \sqrt{Z_3}\, e_0$

Fine structure constant: $\frac{e^2}{4\pi} = \alpha = Z_3 \alpha_0 = Z_3 \frac{e_0^2}{4\pi}$

Note: $\alpha = \alpha_0 + O(\alpha_0^2 \rightarrow O(\alpha^2) = O(\alpha_0^2)$

iii) $$\frac{-ig_{\mu\nu}}{q^2} \frac{e_0^2}{1-\Pi(q^2)} = \frac{-ig_{\mu\nu}}{q^2} \frac{e^2[1-\Pi(0)]}{1-\Pi(q^2)}$$

$$= \frac{-ig_{\mu\nu}}{q^2} \frac{e^2[1-\Pi_2(0)]}{1-\Pi_2(q^2)} + O(\alpha^2)$$

$(1-x) = (1+x)^{-1} + O(x^2)$

$$= \frac{-ig_{\mu\nu}}{q^2} \frac{e^2}{[1-\Pi_2(q^2)][1+\Pi_2(0)]} + O(\alpha^2)$$

$$= \frac{-ig_{\mu\nu}}{q^2} \frac{e^2}{1-[\Pi_2(q^2)-\Pi_2(0)]} + O(\alpha^2)$$

q^2-dependent fine structure constant.

$$\rightarrow \boxed{\alpha_{eff}(q^2) = \frac{e_0^2/4\pi}{1-\Pi(q)} = \frac{\alpha}{1-[\Pi_2(q^2)-\Pi_2(0)]} + O(\alpha^2)}$$

6| Computation of Π_2:

i| $$i\Pi_2^{\mu\nu}(q) = -(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\dots \right]$$

Trace identities

$$= -4e^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu(k+q)^\nu + k^\nu(k+q)^\mu - g^{\mu\nu}(k\cdot(k+q) - m^2)}{(k^2-m^2)\,((k+q)^2-m^2)}$$

Feynman parameters, $l = k + xq$, Wick rotation $l^0 = il_E^0$

$$\overset{\circ}{=} -4e^2 \int_0^1 dx \int \frac{d^4l_E}{(2\pi)^4} \frac{-\frac{2}{d} g^{\mu\nu} l_E^2 + g^{\mu\nu} l_E^2 - 2x(1-x)\, q^\mu q^\nu + g^{\mu\nu}(m^2 + x(1-x) q^2)}{(l_E^2 + \Delta)^2}$$

$$\Delta = m^2 - x(1-x) q^2$$

$$\boxed{\begin{aligned} l^\mu l^\nu &\triangleq \frac{1}{d} l^2 g^{\mu\nu} \\ g^{\mu\nu} g_{\mu\nu} &= d \end{aligned}}$$

ii| Strong UV-divergence: & $|l_E| < \Lambda$

$$i\Pi_2^{\mu\nu}(q) \sim e^2 g^{\mu\nu} \Lambda^2$$

$\rightarrow$ Regularization needed!

iii| Dimensional regularization:

1. Lower spacetime dimension $d \in \mathbb{N}$ until UV div. vanishes.
2. Generalize expressions to $d \in \mathbb{R}$
3. Take limit $d \nearrow 4$

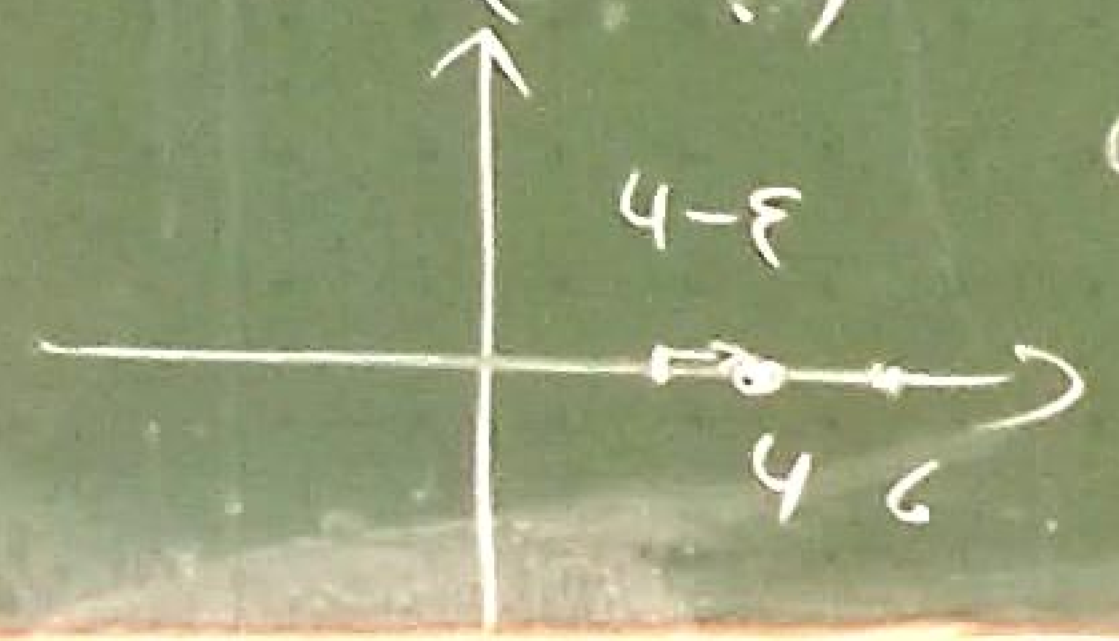
$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}$$

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^2}{(l_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1}$$

∡ $\underline{n=2}$:

$\Gamma(z)$ has poles $z=0,-1,-2,\dots$

$\rightarrow \Gamma(2-\frac{d}{2})$ has isolated poles at $d=4,6,8,\dots$



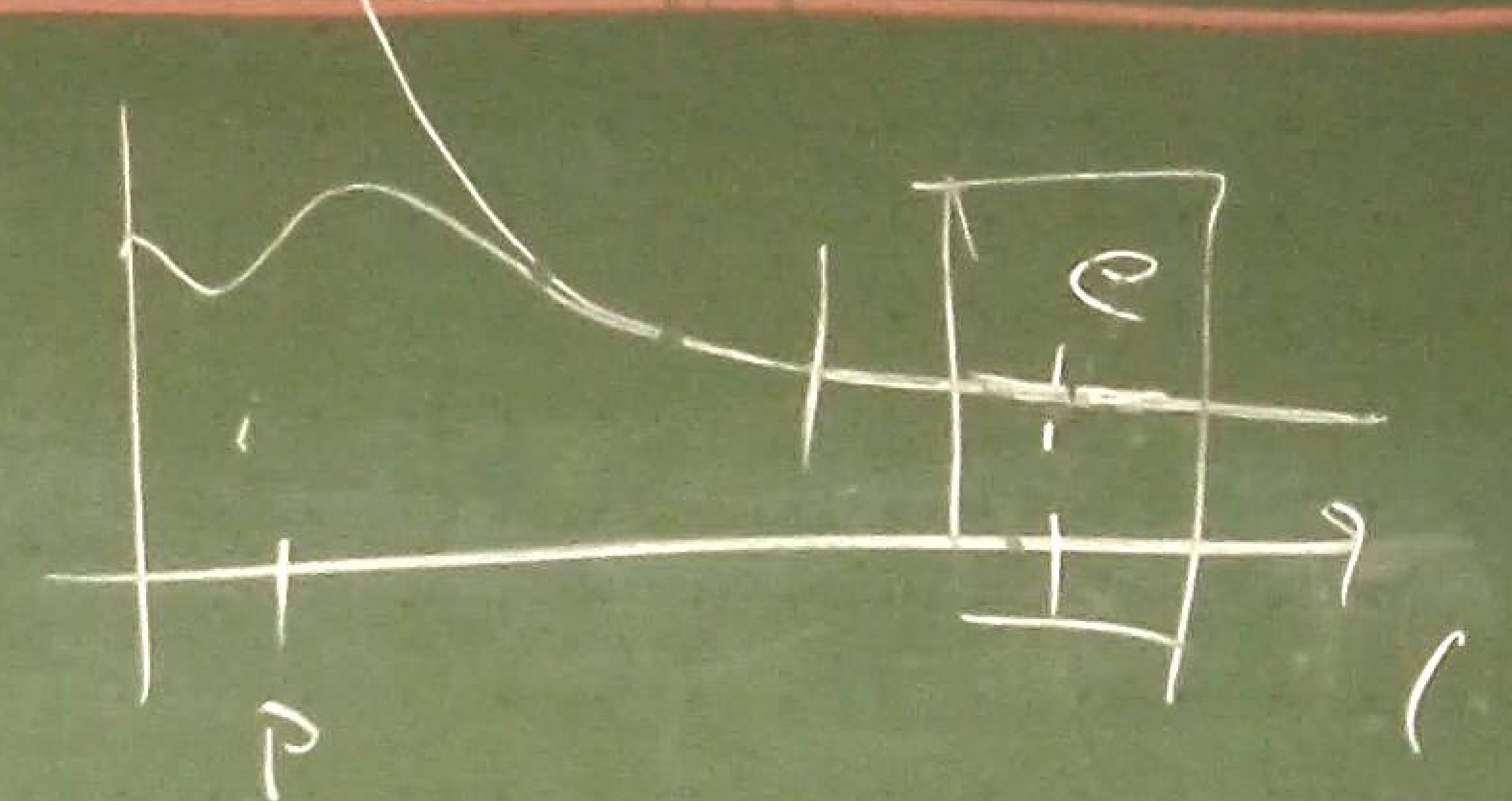
∡ $d=4-\varepsilon$

$$\Gamma(2-\frac{d}{2}) = \Gamma(\frac{\varepsilon}{2}) = \frac{2}{\varepsilon} - \underset{\substack{\uparrow \\ \text{Euler-Mascheroni} \\ \text{constant}}}{\gamma} + O(\varepsilon)$$

IV) $i\Pi_2^{\mu\nu}(q) \doteq (q^2 g^{\mu\nu} - q^\mu q^\nu)\, i\Pi_2(q^2)$

with

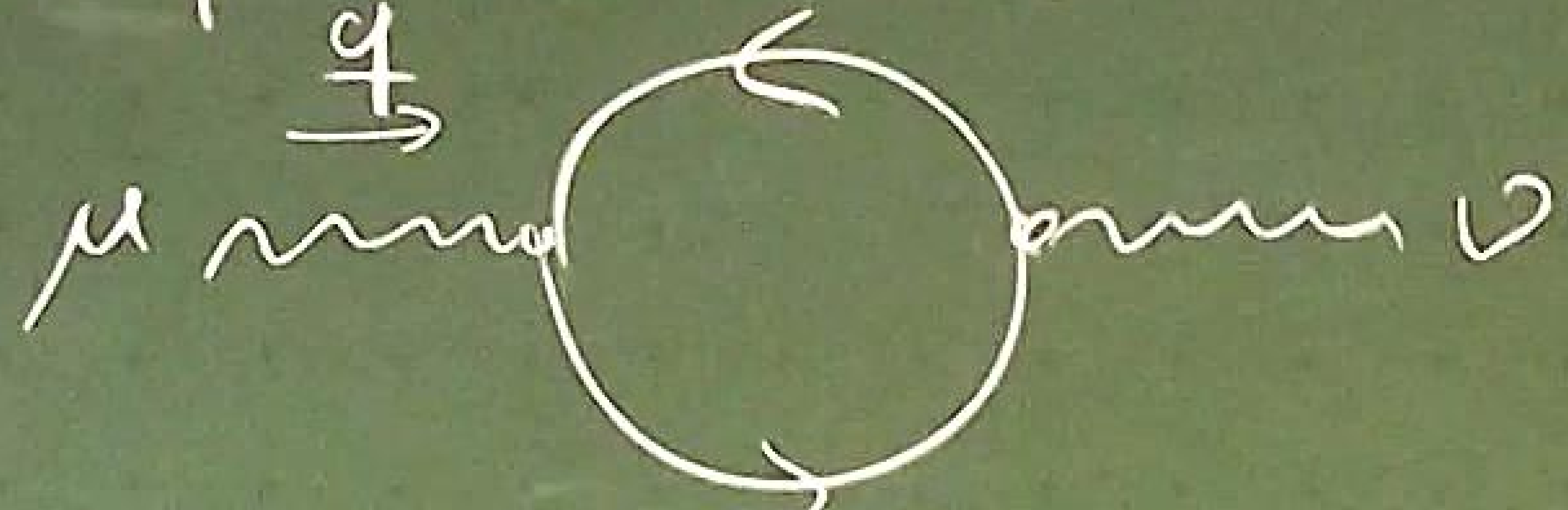
$$\Pi_2(q^2) = \frac{-8e^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{x(1-x)\,\Gamma(2-\frac{d}{2})}{(m^2 - x(1-x)q^2)^{2-\frac{d}{2}}}$$



Recap.

6.5. Electric Charge Renormalization

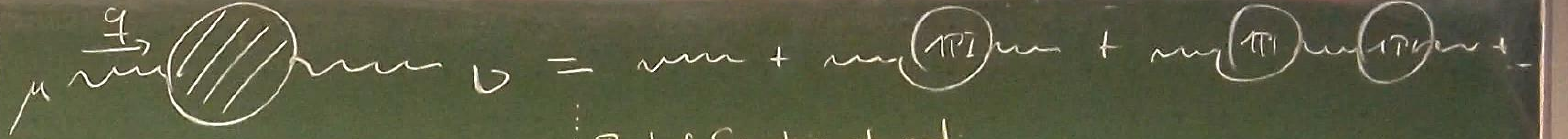
Vacuum polarization diagram:



$= i\Pi_2^{\mu\nu}(q)$

$$= i\boxed{\Pi^{\mu\nu}(q)} = i\left[\Pi_2^{\mu\nu}(q) + O(\alpha^2)\right]$$

$$\doteq (q^2 g^{\mu\nu} - q^\mu q^\nu)\cdot \Pi(q^2)$$

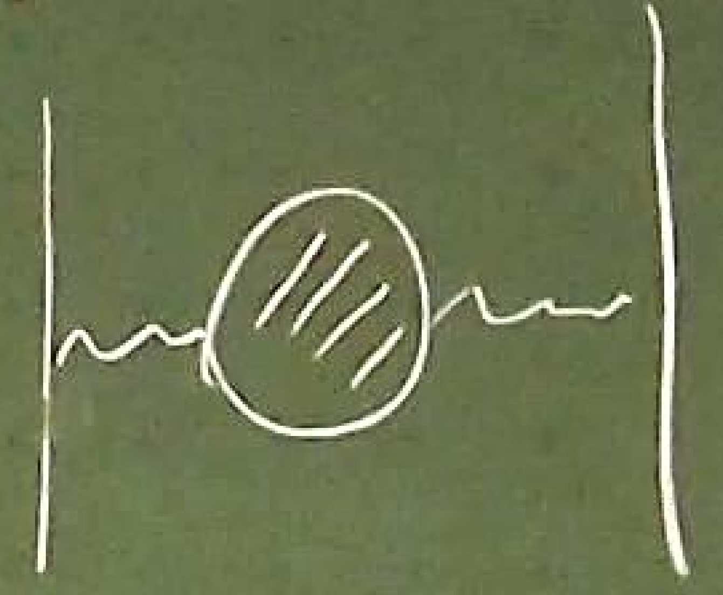


Part of S-matrix element:

$$= \frac{-ig_{\mu\nu}}{q^2\left[1-\Pi(q^2)\right]}$$

- $Z_3 = \frac{1}{1-\Pi(0)}$

(1) Physical charge: $\boxed{e = \sqrt{Z_3}\,\underset{\substack{\uparrow \\ \text{Bare charge}}}{e_0}}$



$$= -\frac{ig_{\mu\nu}}{q^2}\overset{Z_3 e_0^2}{\overset{\|}{e^2}}$$

(2) $\underset{\uparrow}{q}$-dependent fine-structure constant.

$$\boxed{\alpha_{\text{eff}}(q^2) = \frac{e_0^2/4\pi}{1-\Pi(q^2)} = \frac{\alpha}{1-\underbrace{\left[\Pi_2(q^2)-\Pi_2(0)\right]}_{\hat{\Pi}_2(q^2)}} + O(\alpha^2)}$$

6] Computation of $\Pi_2^{\mu\nu}$:

i] Feynman parameters, Substitution $l = k + xq$,

ii] Wick rotation $l^0 = i l_E^0$

iii] Dimensional regularization

$$d = 4 - \varepsilon$$

↑ spacetime dimension

- $\int \frac{d^d l_E}{(2\pi)^d} \frac{1\, l_E^2}{(l_E^2 + \Delta)^2} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-\frac{d}{2}-1)}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}-1} \boxed{\frac{d}{2}}$

- $\Gamma(2-\frac{d}{2}) = \Gamma(\frac{\varepsilon}{2}) = \frac{2}{\varepsilon} - \gamma + O(\varepsilon)$

$$\ln \Lambda \sim \frac{1}{\varepsilon}$$

iv] $i\Pi_2^{\mu\nu}(q) \doteq (q^2 g^{\mu\nu} - q^\mu q^\nu)\, i\Pi_2(q^2$

$$\Pi_2(q^2) = \frac{-8e^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{x(1-x)\,\Gamma(2-\frac{d}{2})}{[\underbrace{m^2 - x(1-x)q^2}_{=\Delta}]^{2-\frac{d}{2}}}$$

$$-\left(-\frac{2}{d} g^{\mu\nu} l_E^2 + g^{\mu\nu} l_E^2\right) = -\left(1 - \frac{2}{d}\right) g^{\mu\nu} l_E^2$$

$\downarrow \int d^d l_E$

$$= \underbrace{\underbrace{\left(-\frac{d}{2}+1\right)}_{z} \Gamma\Big(\underbrace{2-\frac{d}{2}-1}_{z}\Big)}_{\Gamma(z+1) = \Gamma(2-\frac{d}{2})}$$

$$z\,\Gamma(z) = \underbrace{z(z-1)!}_{z!}$$

VI] $\Pi_2(q^2) \doteq -\frac{2\alpha}{\pi} \int_0^1 dx\, x(1-x) \left[\frac{2}{\varepsilon} - \log(\Delta) - \gamma + \log(4\pi)\right] + O(\varepsilon)$

7] $O(\alpha)$ charge renormalization:

$$\frac{e^2 - e_0^2}{e_0^2} = Z_3 - 1 = \delta Z_3 = \frac{\Pi(0)}{1 - \Pi(0)}$$

$$= \Pi_2(0) + O(\alpha^4) \qquad \frac{1}{1-x} = 1 + x + \dots$$

$$\sim -\frac{2\alpha}{3\pi\varepsilon} \xrightarrow{\varepsilon \to 0} -\infty$$

8] $O(\alpha)$ q^2-dependence of $\alpha_{eff}(q^2)$:

$$\hat{\Pi}_2(q^2) = \Pi_2(q^2) - \Pi_2(0)$$

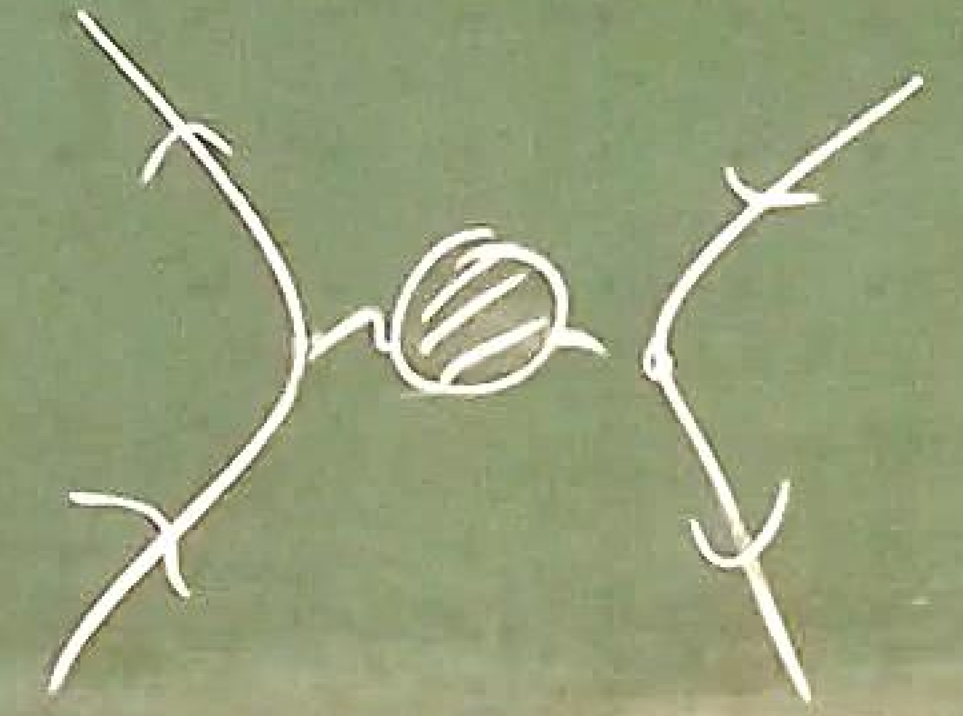
$$= -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\log\left[\frac{m^2}{m^2-x(1-x)q^2}\right]$$

9] Analysis + Interpretation of $\hat{\Pi}_2(q)$:

i] ✗ Effective potential in non-rel limit.

$(|q^2| \ll m^2,\quad q^2 = (p-p')^2 \approx -|\vec{p}-\vec{p}'|^2$

$= -\vec{q}^{\,2}$



$$\frac{e^2}{q^2[1-\hat{\Pi}(q^2)]}$$

ii] $$V(\vec{x}) = \underbrace{\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\vec{x}}}_{}\,\frac{-e^2}{|\vec{q}|^2\underbrace{(1-\hat{\Pi}_2(-\vec{q}^{\,2}))}_{}}$$

$$\frac{1}{1-x} = 1+x+\dots$$

$$1+\Pi_2(-\vec{q}^{\,2})+O(\alpha^2)$$

$-\vec{q}^{\,2} \ll m^2$

$$\overset{\circ}{=} \frac{-e^2}{\vec{q}^{\,2}}\left[1+\frac{\alpha}{15\pi m^2}\vec{q}^{\,2}\right]+O(\alpha^3)$$

$$\overset{\circ}{\approx} -\frac{\alpha}{|\vec{x}|}\boxed{-\frac{4\boxed{\alpha^2}}{15m^2}\delta^{(3)}(\vec{x})}+O(\alpha^3)$$

→ EM force becomes stronger for $|\vec{x}| \to 0$

iii] $$\Delta E = \int d^3x\, |\psi(x)|^2\left(-\frac{4\alpha^2}{15m^2}\delta^{(3)}(x)\right)$$

$$= -\frac{4\alpha^{\textcircled{2}}}{15m^2}|\psi(0)|^2 \overset{l=0}{<} 0$$

· Darwin term.

$$H_{\text{Darwin}} = \frac{\pi\boxed{\alpha}}{2m^2}\delta^{(3)}(\vec{x})$$

$$^2S_{\textcircled{\frac{1}{2}}} = {}^2P_{\textcircled{\frac{1}{2}}}$$

⇒ Part of Lamb shift.

- $\Delta E_{VT} \simeq -27\ \text{MHz}$
- $\Delta E_{ES} \simeq 1011\ \text{MHz}$
- $\Delta E_{AM} \simeq 68\ \text{MHz}$.

$$\boxed{\text{Lamb shift} \sim 1058\ \text{MHz}}$$

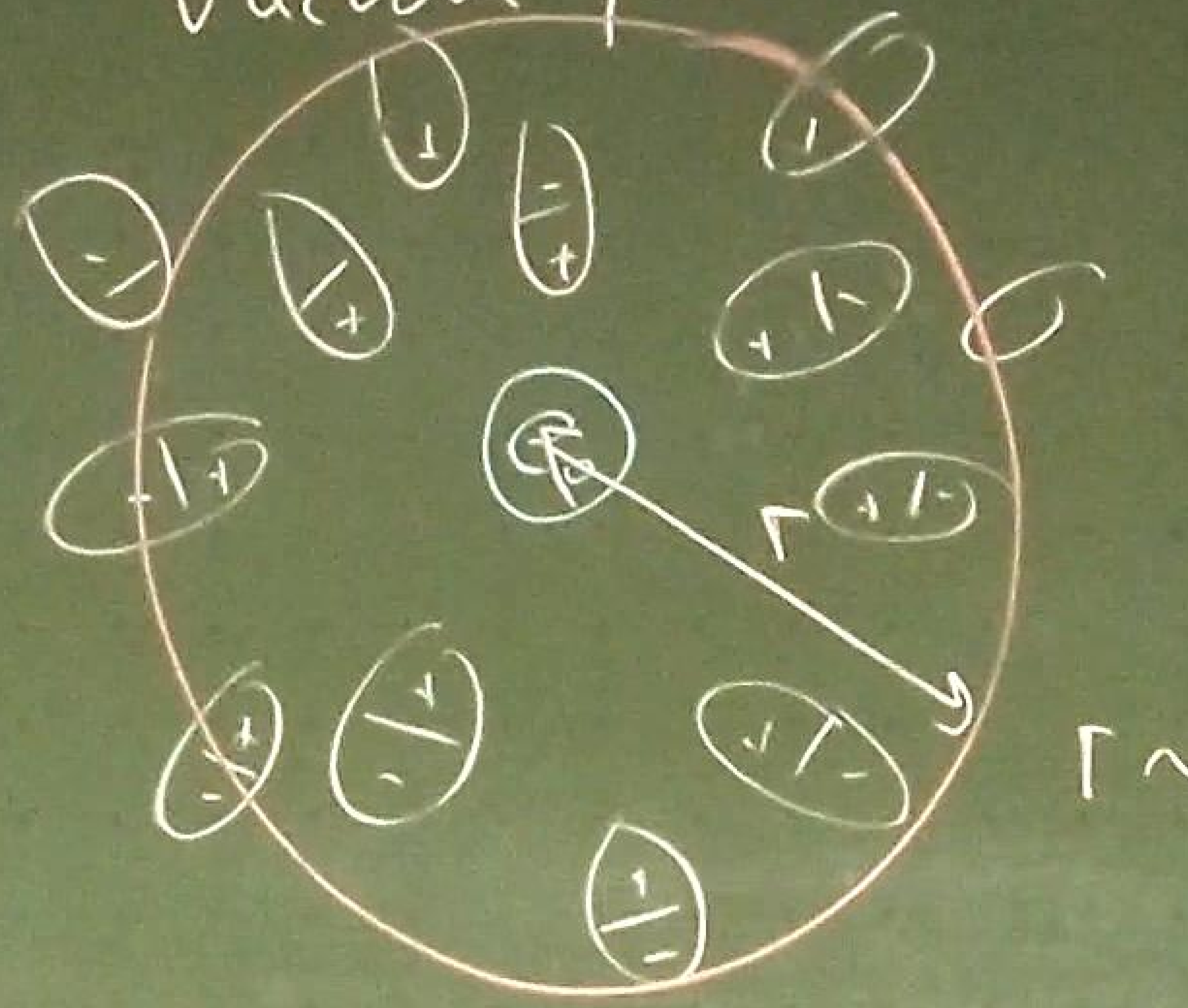
iv] More generally: Uehling potential:

$$V(r) = -\frac{\alpha}{r}\left(1 + \frac{\alpha}{4\sqrt{\pi}} \frac{e^{-2mr}}{(mr)^{3/2}} + \dots\right)$$

- $\lambda_C = \frac{h}{mc} = \frac{2\pi}{m}$
- $a_0 = \frac{\lambda_c}{2\pi\alpha} \approx 22 \cdot \lambda_c$

V] Interpretation:

Vacuum polarization



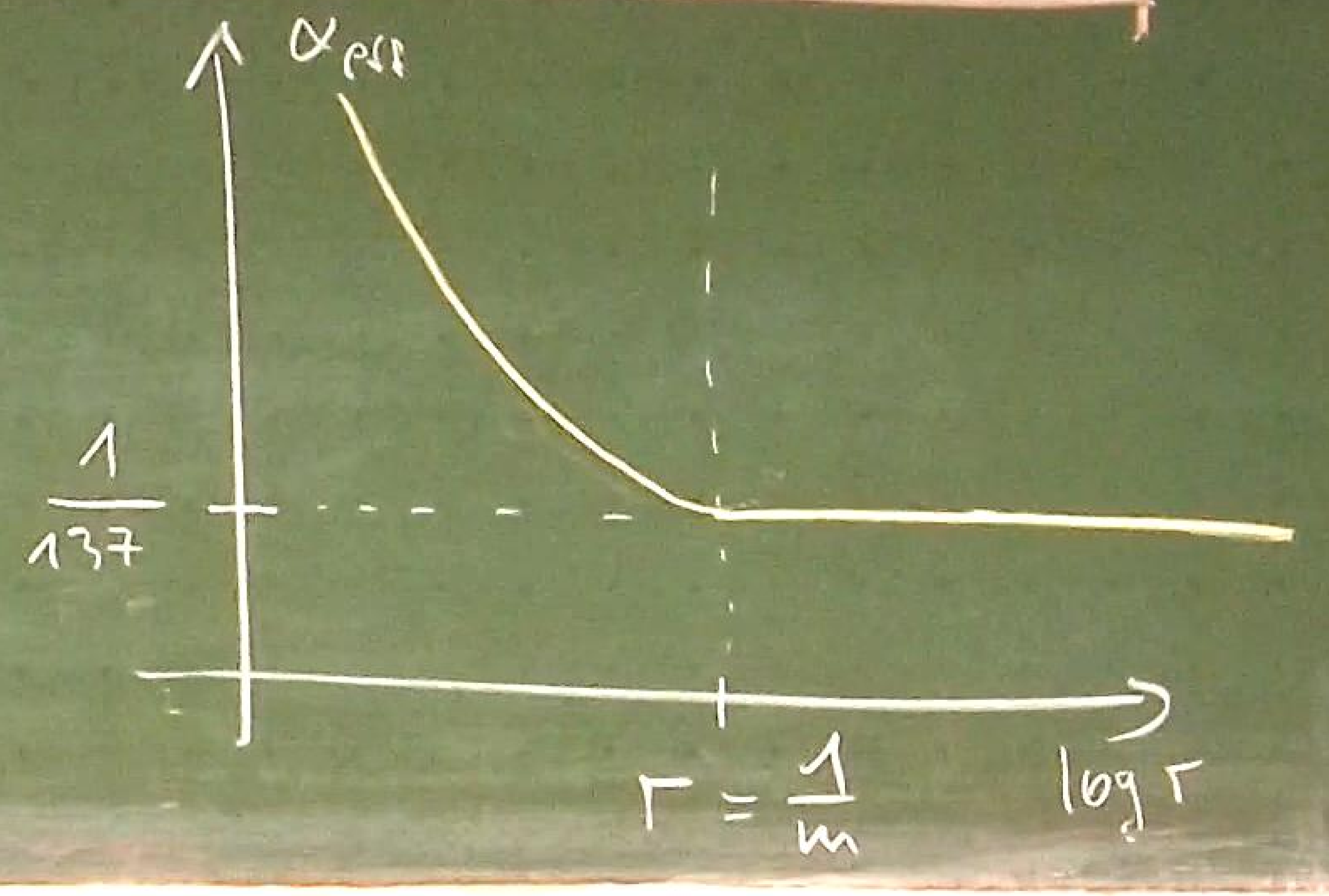
$$r \sim \frac{1}{q}$$

VI] ∢ Relativistic limit $-q^2 \gg m^2$

$$\hat{\Pi}_2(q^2) \doteq \frac{\alpha}{3\pi}\left[\log\left(\frac{-q^2}{m^2}\right) - \frac{5}{3} + O\left(\frac{m^2}{q^2}\right)\right]$$

$$\boxed{\xrightarrow{-q^2 \gg m^2} \alpha_{eff}(q) \approx \frac{\alpha^2}{1 - \frac{\alpha}{3\pi}\log\left(\frac{-q^2}{Am^2}\right)}}$$

$$A = e^{\frac{5}{3}}$$



Note:

$$1-\frac{\alpha}{3\pi}\log\left(\frac{\Lambda_L^2}{A m^2}\right) \overset{!}{=} 0$$

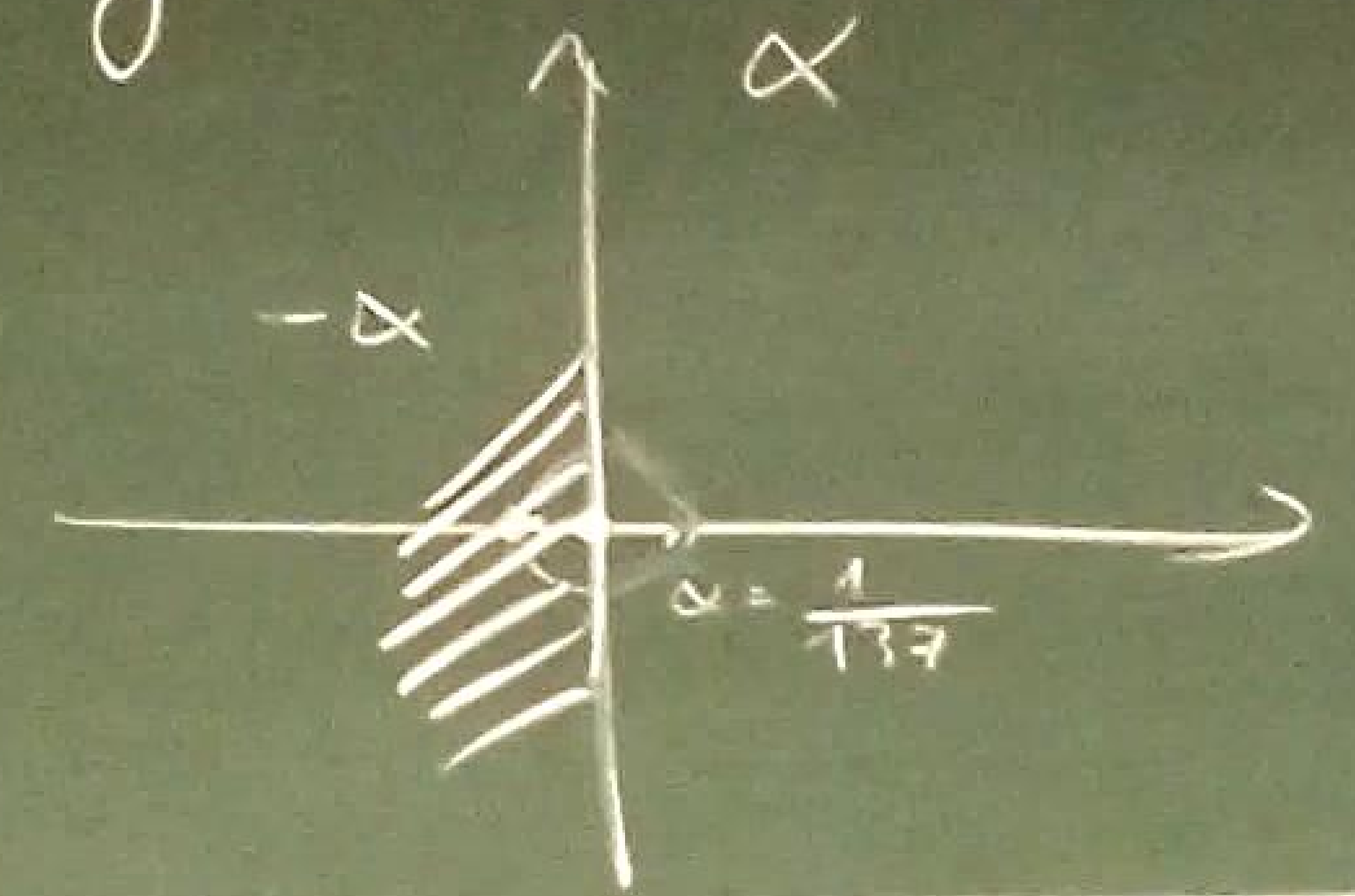
$A \sim O(1)$

$$\Rightarrow \boxed{\Lambda_L \sim m e^{\frac{3\pi}{2\alpha}} \sim 10^{286}\ eV}$$

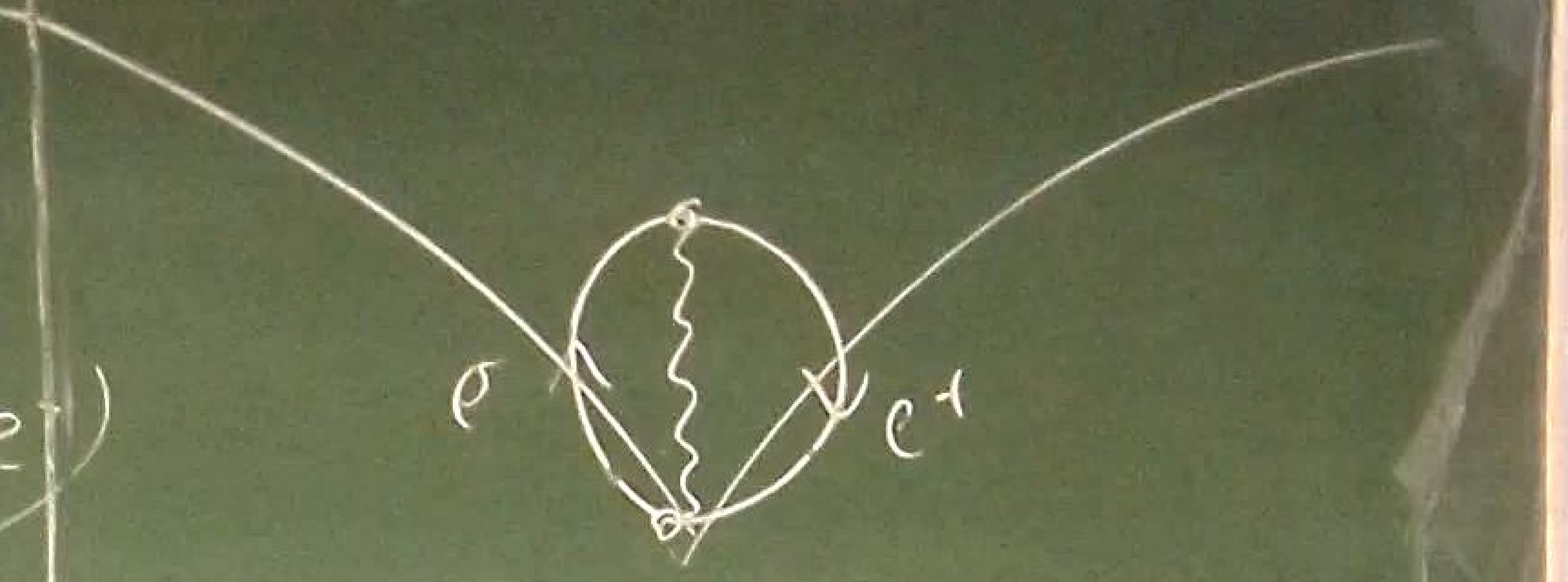
$$E_{LHC} \approx 10^{13}\ eV \ll E_{Pl} \approx 10^{28}\ eV \leq \Lambda_L \approx 10^{286}\ eV$$

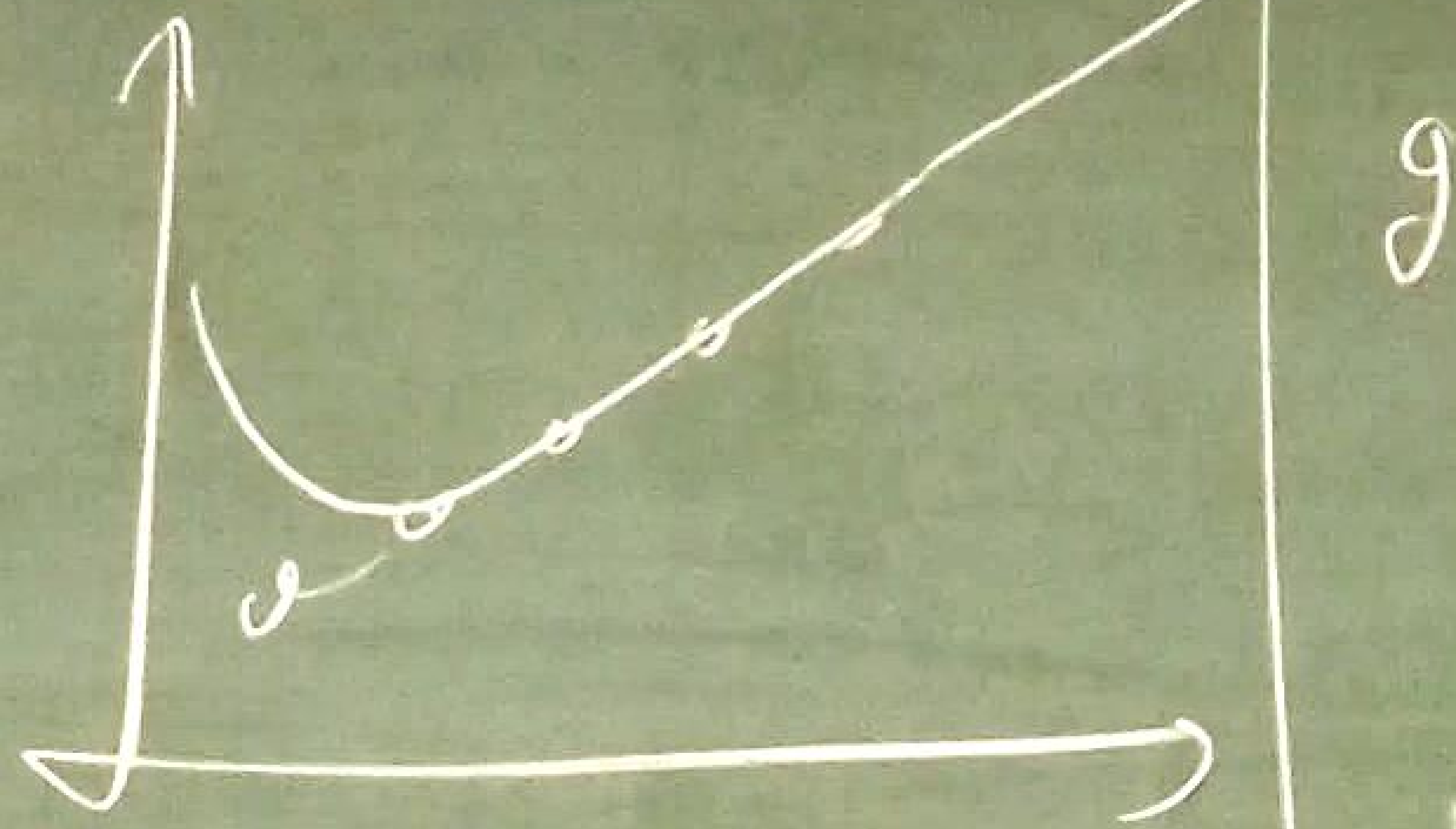
$\Rightarrow$ Landau pole

$$g(\alpha) = a_0 + a_1\alpha + a_2\alpha^2 + \dots$$



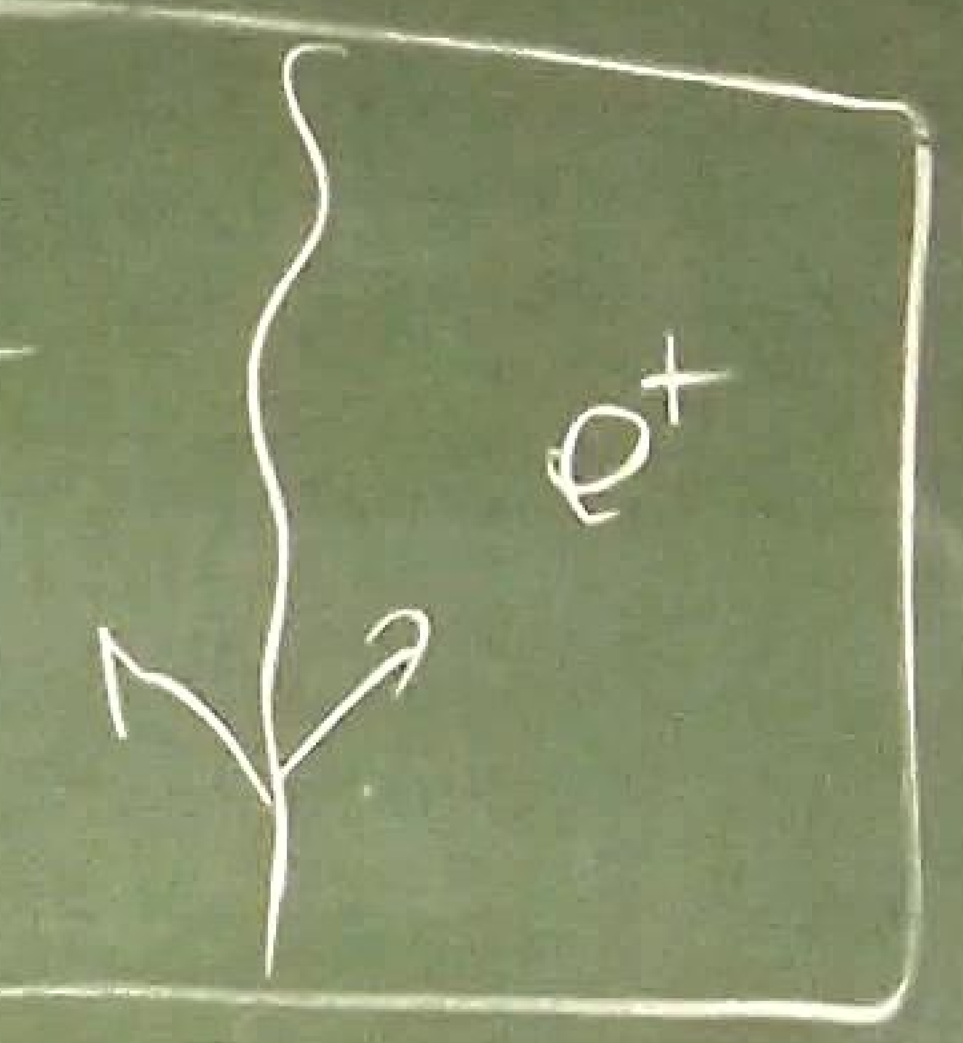
$$g \cdot f(\alpha) + \frac{e^2}{r} = V(e^-, e^+)$$





$$g = e^{-\frac{c}{\alpha}}$$

$$= 0 + 0 + 0 + 0$$



$$\alpha^N \sim e^{-\frac{c}{\alpha}}$$

$$\boxed{N \sim \frac{1}{\alpha} = 137}$$

7. Systematics of Renormalization

* IR-divergences

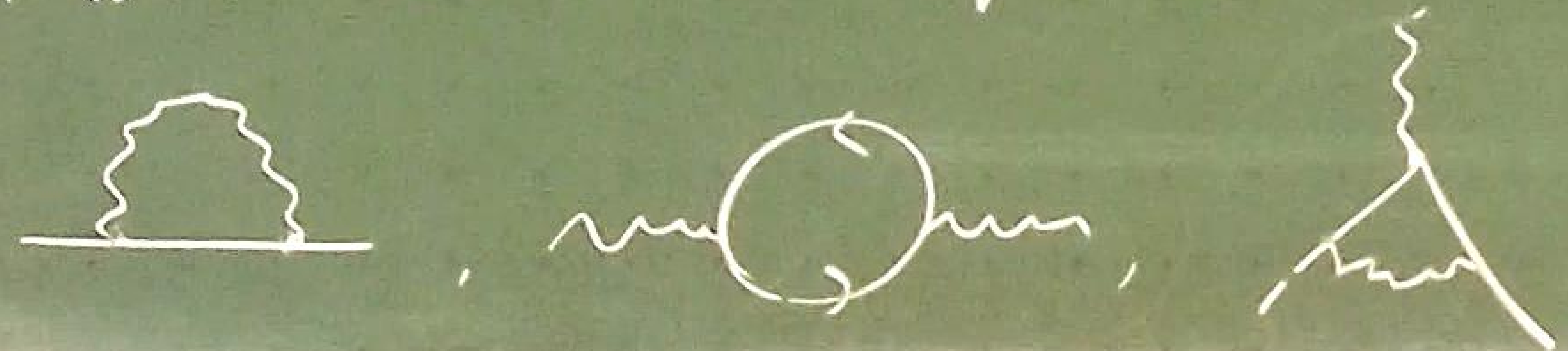
· Due to massless particle

· Cancel with soft bremsstrahlung

→ Not a fundamental problem

* UV-divergences:

· Due to unbounded high momenta:



· Diverging differences between physical an bare quantities

→ Fundamental problem

→ Study UV-divergences systematically

7. Systematics of Renormalization

1| Goal: Classify UV-divergencies in QED

2| Def:

N_e = # ext. e-lines

N_γ = # ext phot.-lines

P_e = # e-propagators: $\prod^{P_e} \frac{1}{\cancel{k}_i - m}$

P_γ = # phot.-prop.: $\prod_i^{P_\gamma} \frac{1}{k_i^2}$

V = # vertices

L = # independent Loops: $\prod_i^L \int \frac{d^4 k_i}{(2\pi)^4}$

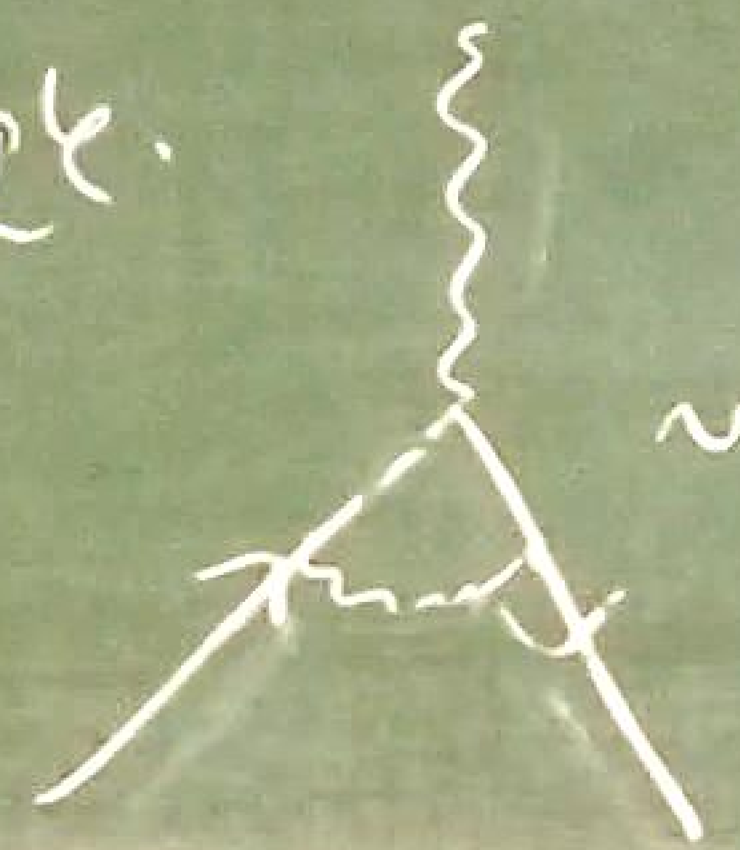
3| Superficial degree of divergence:

$$\boxed{D_{QED} = (3L + L) - (P_e + 2P_\gamma)}$$

Intuition:

$$D_{QED} \begin{cases} > 0 & : \text{Divergence with } \Lambda^D \\ = 0 & : \text{Divergence with } \log\Lambda \\ < 0 & . \text{ No divergence} \end{cases}$$

Example:



$\sim \log\Lambda$, $D_{QED} = 4-(2+2\cdot 1) = \underline{\underline{0}}$

However: Not always correct.

*  $\sim \log\Lambda$ $D_{QED} = 4-(2+2\cdot 0) = 2$

(Cancellations can reduce divergence)

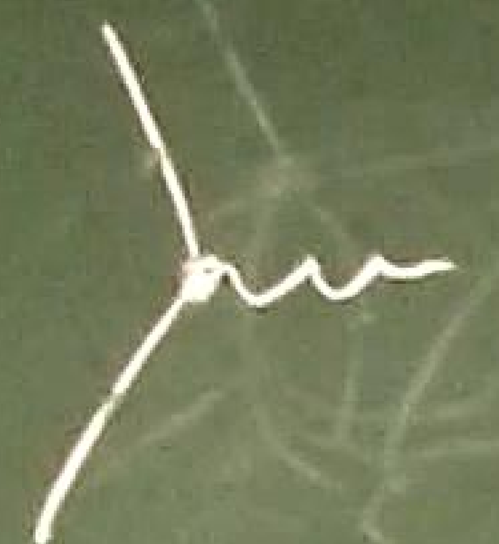
*  $\sim \log\Lambda$ $D_{QED} = 4-(2+2\cdot 2) = -2$

*  ~ 1 $D_{QED} = 40-(0+2\cdot 0) = 0$

4) $\cdot\ L = \underbrace{P_e + P_\gamma}_{E} - V + 1$

$\cdot\ V = 2P_\gamma + N_\gamma$

$= \frac{1}{2}(2P_e + N_e)$



$\Downarrow$

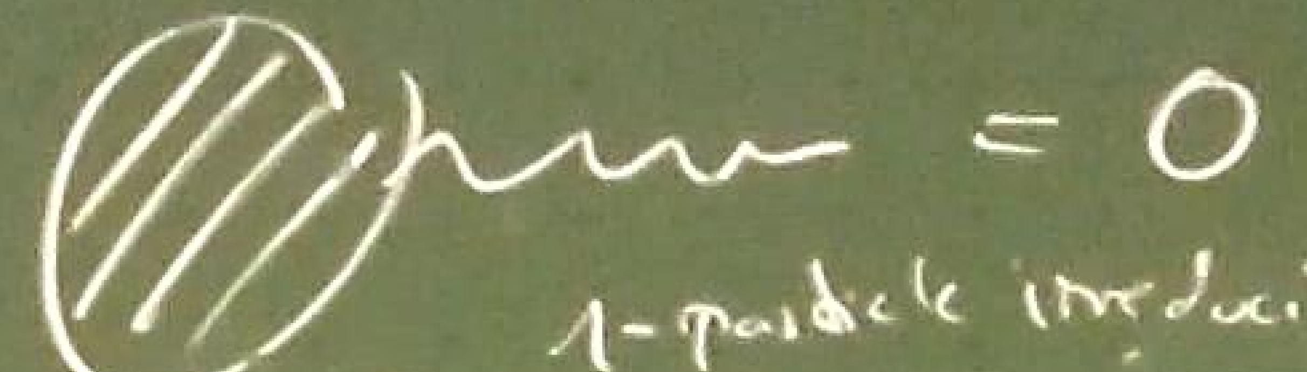
$$\boxed{D_{QED} = 4 - N_\gamma - \frac{3}{2}N_e}$$

→ Independent of number of vertices

5) Furry's theorem:

Feynman diagrams with an odd # of external photons as only external lines vanish.

Example:



$= 0$

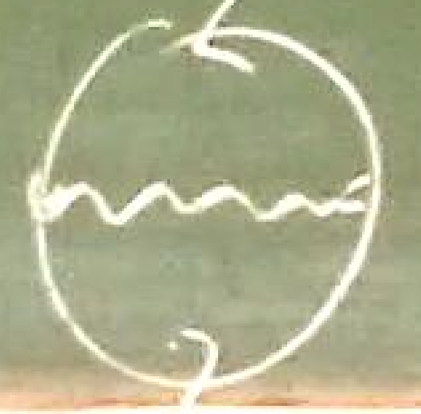
1-particle irreducible

6) Enumerate all amplitudes with $D_{QED} \geq 0$:

i) $N_e = 0$:

a) $N_\gamma = 0$:

~ badly divergent

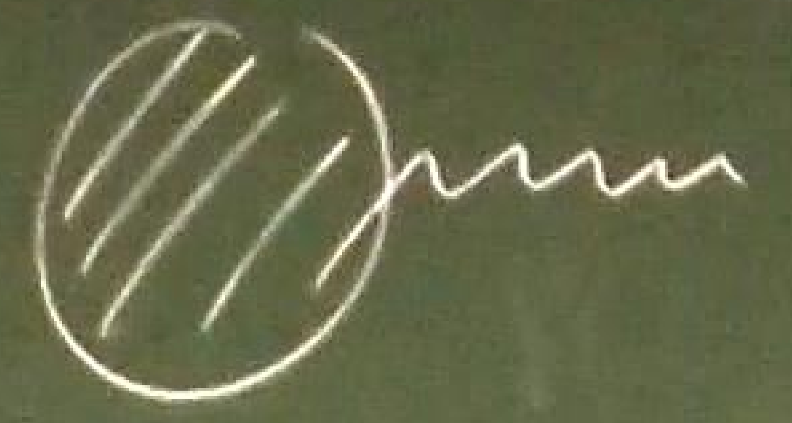


↳ Unobservable

Vacuum energy shift → ignore

$\to D = 4$

b) $N_\gamma = 1$ $\quad D = 3$



$\overset{\text{Furry}}{=} 0$

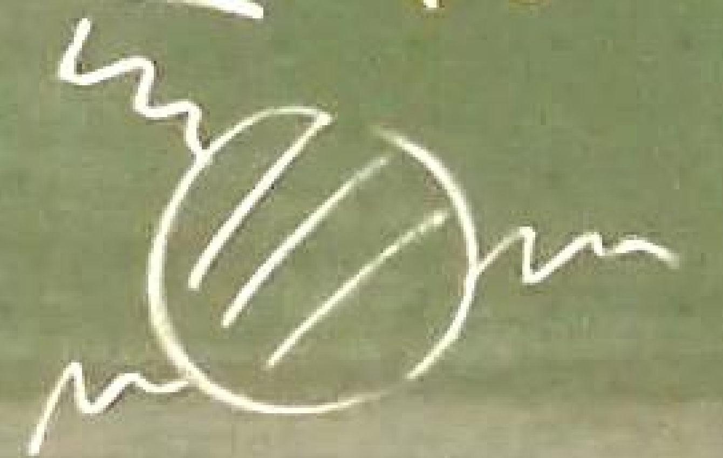
c) $N_\gamma = 2$ $\quad D = 2$

$= (g^{\mu\nu} q^2 - q^\mu q^\nu)\,\Pi(q^2)$

$\sim \quad$ —"— $\quad \frac{\text{const}}{\varepsilon}$

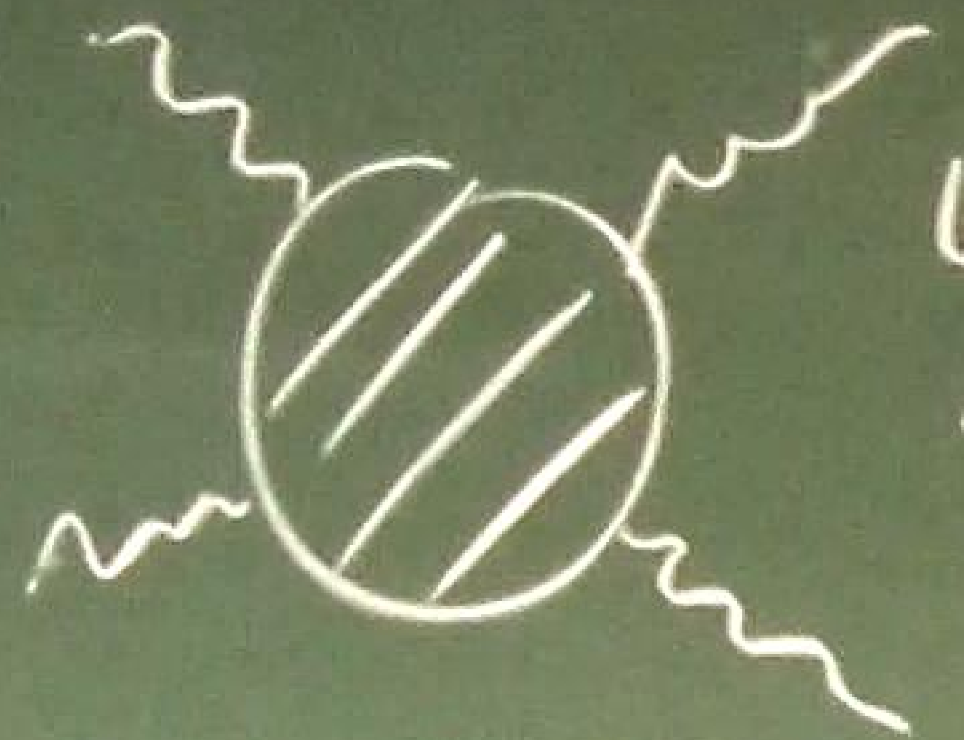
$\sim \quad$ —"— $\quad \underbrace{\log\Lambda \cdot \text{const.}}_{a_0(\Lambda)}$

d) $N_\gamma = 3$ $\quad D = 1$



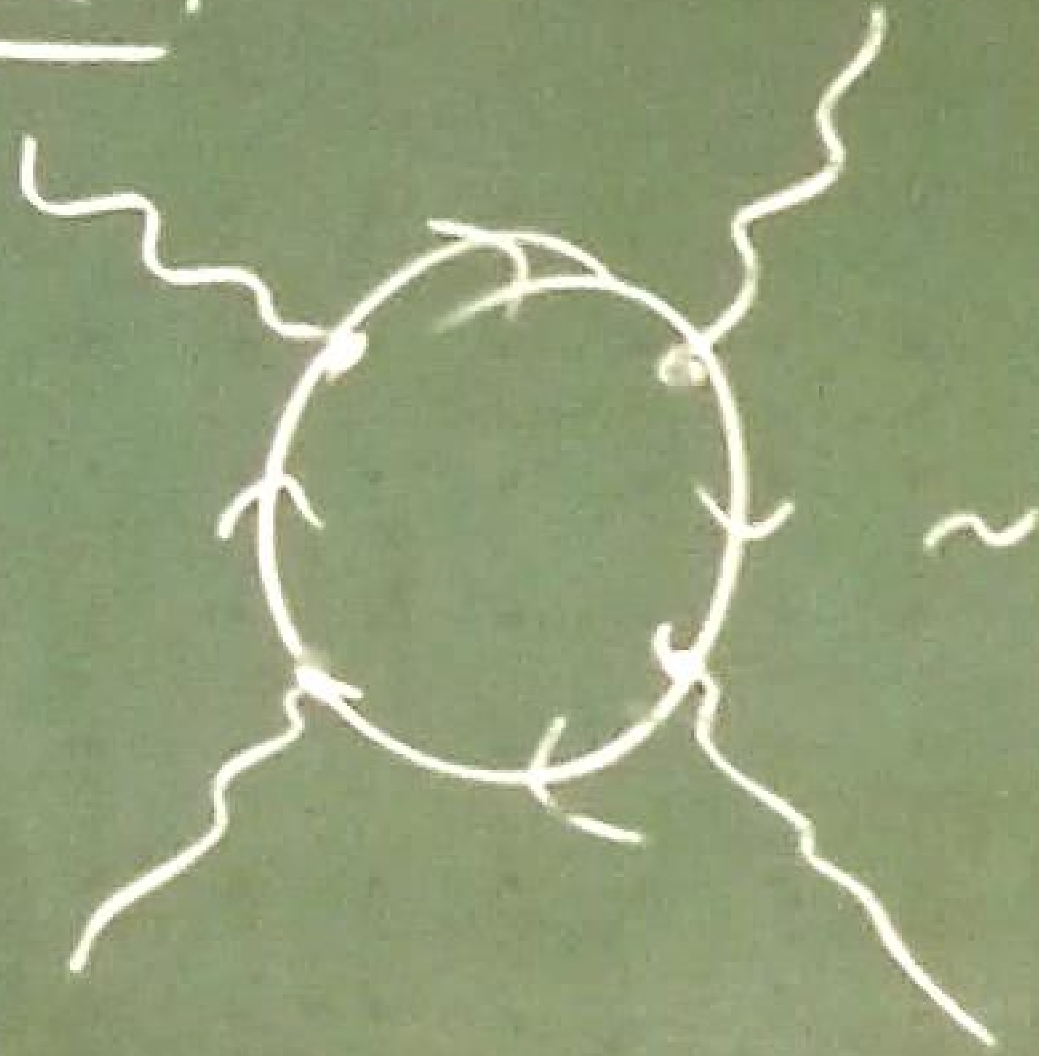
$\overset{\text{Furry}}{=} 0$

e] $\underline{N_\gamma = 4}$ $D = 0$



Ward ~ 1

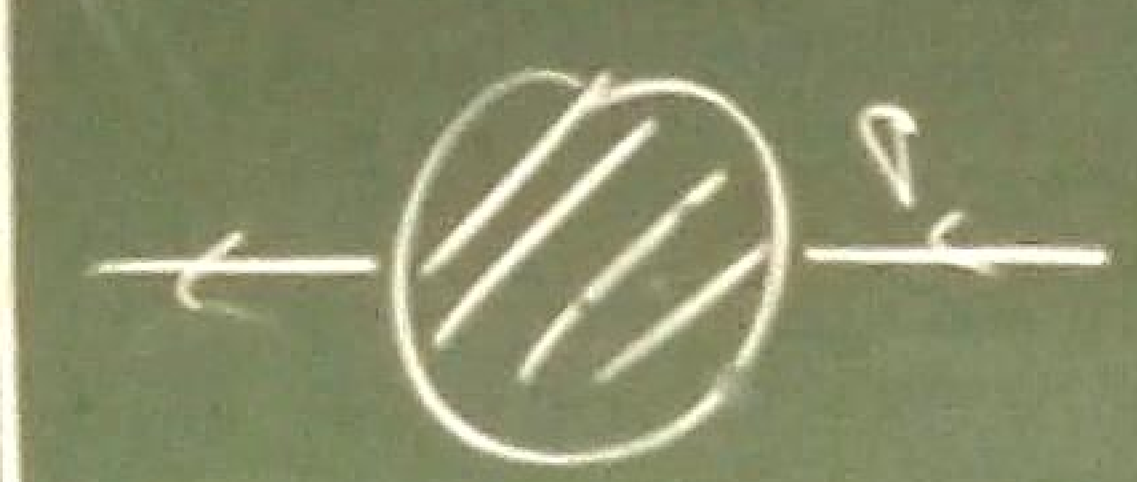
Note:



$\sim \alpha^2$

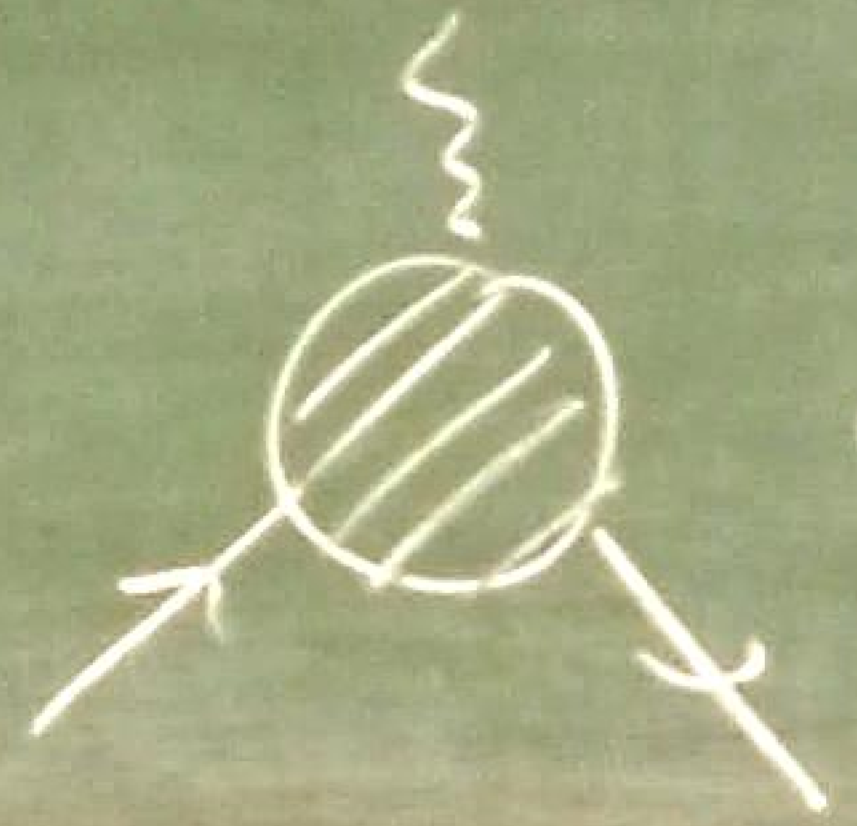
ii] $\underline{N_e = 2}$ $D = 4 - N_\gamma - \frac{3}{2} N_e$

a] $\underline{N_\gamma = 0}$ $D = 1$



$$= \underbrace{\text{const} \cdot \log \Lambda}_{a_1(\Lambda)} + \cancel{p}\, \underbrace{\log \Lambda \cdot \text{const}}_{a_2(\Lambda)}$$

b] $\underline{N_\gamma = 1}$ $D = 0$



$$\sim \underbrace{-ie\gamma^\mu \overbrace{\log(\Lambda)}^{F_1}}_{a_3(\Lambda)}$$

→ Diagrams only diverge if the contain one of the three diverging subdiagrams

→ QED contains only <u>four</u> UV-divergent numbers: a_0, a_1, a_2, a_3

7] <u>Idea</u>: Absorb <u>finite</u> number of div. quantities into <u>finite</u> number of diverging unobservable Lagrangian parameters

8] Generalization: QED in d spacetime dimensions

$$\boxed{\begin{aligned} D_{QED} &= dL - (P_e + 2P_\gamma) \\ &\overset{\circ}{=} d + \left(\frac{d-4}{2}\right)V - \left(\frac{d-2}{2}\right)N_\gamma - \frac{(d-1)}{2}N_e \end{aligned}}$$

→ For $d<4$: diagrams of high-enough order converge

→ For $d=4$: D_{QED} is independent of the order of V

→ For $d>4$: all diagrams of high-enough order diverge

9]

- Super-Renormalizable theory
  (example: QED in $d<4$)
  · Only finite number of divergent diagrams
- Renormalizable theory
  (ex. QED $d=4$)
  · Only finite number of divergent amplitudes
- Non-renormalizable theory
  (ex. QED in $d>4$)
  All amplitudes diverge at sufficiently high order in perturbation theory.

Alternative Approach.

1] ϕ^4-theory

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

2] N_ϕ = # external lines

P_ϕ = # propagators

V = # vertices

L = # indep. loops

3] Superficial degree of divergence

$$\boxed{D_{\phi^4} = dL - 2P_\phi \overset{\circ}{=} d + \left[4\left(\frac{d-2}{2}\right) - d\right]V - \left(\frac{d-2}{2}\right)N_\phi}$$

4) Alternative approach: Dimensional analysis

i) $\hbar = c = 1$

$$\lambda_c = \frac{h}{mc} = \frac{2\pi}{m}$$

$$\rightarrow \quad [\lambda_c] = M^{-1}$$

dimension of mass

ii) $[S] = 1$

iii) $S = \int d^dx \, \mathcal{L}$

$$[d^dx] = M^{-d}$$

$$\Rightarrow [\mathcal{L}] = M^d$$

iv) $\mathcal{L} = \boxed{(\partial_\mu \phi)^2} - \boxed{m^2\phi^2} - \boxed{\lambda \phi^4}$

$$\boxed{[\partial_\mu] = M} \Rightarrow [\phi] = M^{\frac{d-2}{2}}$$

$$[m] = M^1$$

$$\boxed{[\lambda] = M^{d-\frac{n(d-2)}{2}}}$$

v) & Amplitude $\mathcal{M}$ with N_ϕ external lines

$$\rightarrow \boxed{[\mathcal{M}] = M^{d-\frac{N_\phi(d-2)}{2}}}$$

vi) & Diagram with V vertices

$$\mathcal{M} \sim \lambda^V \Lambda^D$$

$$\rightarrow [\lambda]^V [\Lambda]^D = [\mathcal{M}] = M^{d-\frac{N_\phi(d-2)}{2}}$$

$$[\Lambda] = M$$

$$\boxed{D = d - \underbrace{\boxed{\log_M [\lambda]}}_{d-\frac{n(d-2)}{2}} V - \left(\frac{d-2}{2}\right) N_\phi}$$

- Super renormalizable: $\log_\mu[\lambda] > 0$
- Renormalizable: $\log_\mu[\lambda] = 0$
- Non-renormalizable: $\log_\mu[\lambda] < 0$

$\hbar = c = 1$

QED $[e] = 1$
$d=4$

$$G = \frac{\hbar c}{m_p^2} = \frac{1}{m_p^2} \Rightarrow [G] = M^{-2}$$

$$S = \frac{1}{16\pi G}\int d^4(x)\sqrt{-g}\,R$$

Recap:

Ex. ϕ^4-theory:

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{n!}\phi^4$$

Superficial degree of divergence:

$$\boxed{D_{\phi^4} = d - \log_\mu[\lambda]\, V - \left(\frac{d-2}{2}\right) N_\phi}$$

Mass dimension of λ

$$\log_\mu[\lambda] = d - n\,\frac{d-2}{2} \overset{n=4,\, d=4}{=} 0$$

- $\log_\mu[\lambda] > 0 \Rightarrow$ Super-renormalizable (QED in $d<4$)
- $\log_\mu[\lambda] = 0 \rightarrow$ Renormalizable (QED in $d=4$, $\boxed{\phi^4 \text{ in } d=4}$) today
- $\log_\mu[\lambda] < 0 \Rightarrow$ Non-renormalizable (Einstein gravity)

7.2. Renormalized Perturbation Theory

Recipe:

(i) Compute UV-divergent amplitude with UV-regulator Λ:

$$\mathcal{M} = \mathcal{M}(m_0, e_0; \Lambda) + \mathcal{O}(\alpha_0^4)$$

(ii) Compute physical mass, charge, field strength: $Z = Z(m_0, e_0, \Lambda) + \mathcal{O}$

$m = m(m_0, e_0; \Lambda) + \mathcal{O}$, $e = e(m_0, e_0, \Lambda) + \mathcal{O}$

(iii) Renomalization:

Eliminate bare parameters in favour of physical parameters:

$$e_0 = e_0(m, e, \Lambda)\ ,\ m_0 = m_0(m, e, \Lambda)$$

(iv)

$$M(m,e) = \lim_{\Lambda\to\infty} M(m_0(m,e,\Lambda), e_0(m,e,\Lambda), \Lambda)$$

is finite and independent of Λ in all orders of α.

⇒ Bare perturbation theory

Note:

$$\Rightarrow \frac{e_0^2}{q^2(1-\Pi(q))}$$

$$\Pi = \Pi_2 + O(\alpha^4)$$

$$e = \sqrt{Z_3}\, e_0 \qquad Z_3 = \frac{1}{1-\Pi(0)}$$

$$\frac{e^2}{q^2(1-\underbrace{(\Pi_2(q)-\Pi_2(0))}_{\hat{\Pi}_2}}$$

→ Equivalent formalism:

Renormalized perturbation theory

1| ϕ^4-theory.

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4$$

2| $D_{\phi^4} = 4 - N_\phi$

. Vacuum energy shift.

- $N_\phi = 0,\ D_{\phi^4} = 4.$
- $N_\phi = 2,\ D_{\phi^4} = 2.$

$\sim \Lambda^2 + p^2 \log\Lambda$

- $N_\phi = 4,\ D_{\phi^4} = 0.$

$\sim \log\Lambda$

↦ 3 diverging quantities

3| Recall:

$$\int d^4x\, e^{ipx}\langle\Omega|T\phi_r(x)\phi_r(0)|\Omega\rangle = \frac{i\cancel{Z}\,1}{p^2-m^2} + \underbrace{\ldots\ldots}_{}\quad p^2 \gtrsim (2m)^2$$

Absorb Z in the fields:

$$\boxed{\phi_r := \frac{\phi}{\sqrt{Z}}}$$

4|

$$\mathcal{L}_{\phi^4} = \frac{1}{2}Z(\partial_\mu\phi_r)^2 - \frac{1}{2}m_0^2 Z\phi_r^2 - \frac{\lambda_0}{4!}Z^2\phi_r^4$$

5|

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu\phi_r)^2 - \frac{1}{2}m^2\phi_r^2 - \frac{\lambda}{4!}\phi_r^4$$

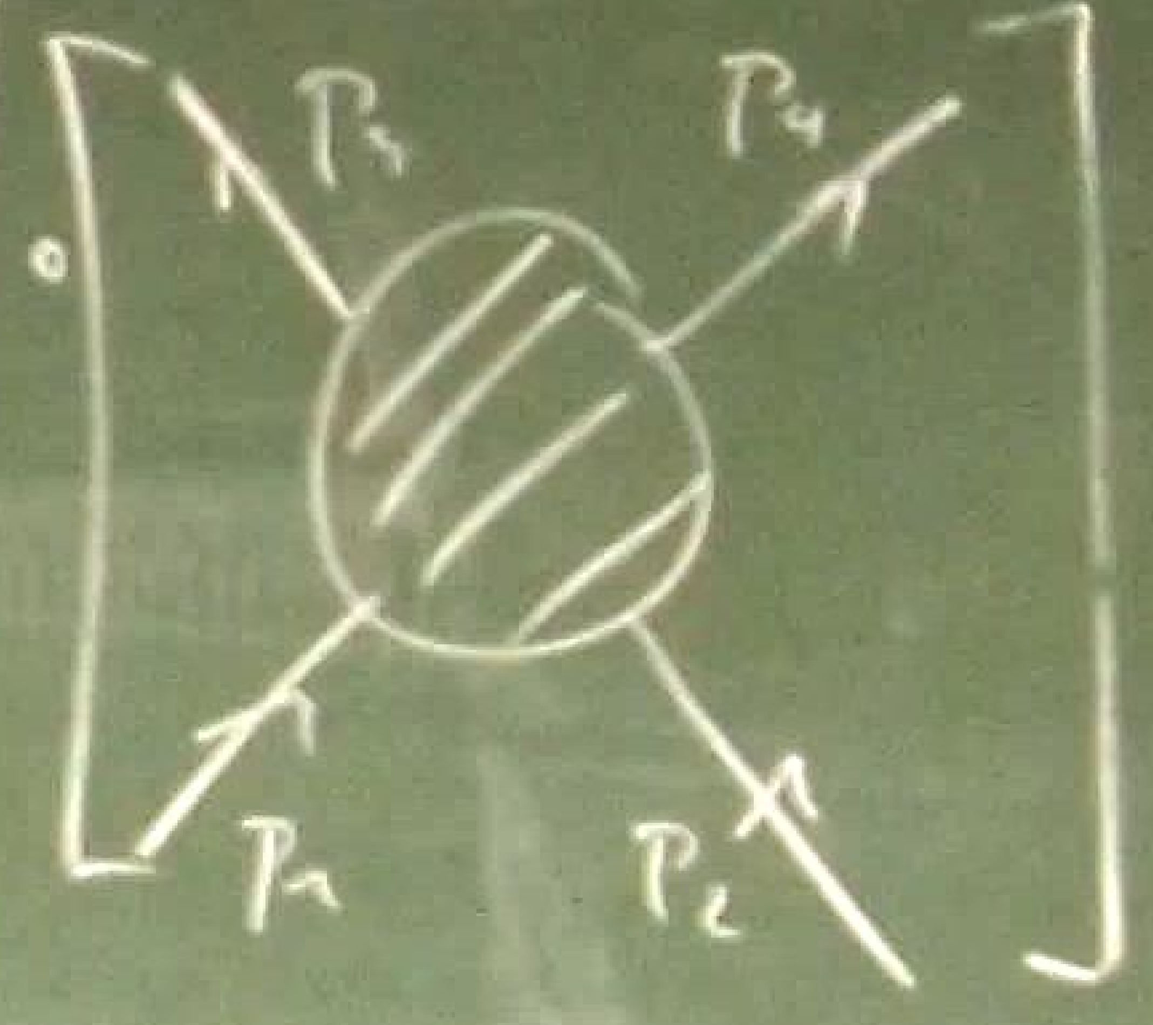
$$\underbrace{+\frac{1}{2}\underbrace{\boxed{(Z-1)}}_{\delta_Z}(\partial_\mu\phi_r)^2 - \frac{1}{2}\underbrace{\boxed{(m_0^2Z-m^2)}}_{\delta_m}\phi_r^2 - \frac{1}{4!}\underbrace{\boxed{(\lambda_0Z^2-\lambda)}}_{\delta_\lambda}\phi_r^4}_{\text{counter terms}}$$

→ $\delta_Z, \delta_m, \delta_\lambda$ absorb unobservable diverging shifts of bare and physical quantities.

6| Renormalization conditions:

•


$$\overset{!}{=} \frac{1}{p^2-m^2} + \ldots$$

•


$$\overset{!}{=} -i\lambda$$

fc+4

$p_i = (m,0)$

7| Perturbation theory

→ Feynman rules for renormalized perturbation theory.

1. Edges:  $= \frac{i}{p^2-m^2+i\varepsilon}$

2. Vertices:  $= -i\lambda$

 $= -i\delta_\lambda$

 $= i(p^2\delta_Z - \delta_m)$

3. External lines:  $= 1$

4. Momentum conservation at vertices

5. Integrate undetermined moments

6. Divide by sym. factor.

8| Procedure:

(i) Sum all relevant Feynman diagrams

(ii) Undiverging integrals ⇒ Regulator Λ, ε

(iii) Result depends on
- λ, m
- $\delta_\lambda, \delta_m, \delta_Z$
- Regulators Λ, ε

(iv) Choose ('renormalize') parameters $\{\delta\}$ to satisfy renorm. conditions.

$$\delta = \delta(m, \lambda, \Lambda)$$

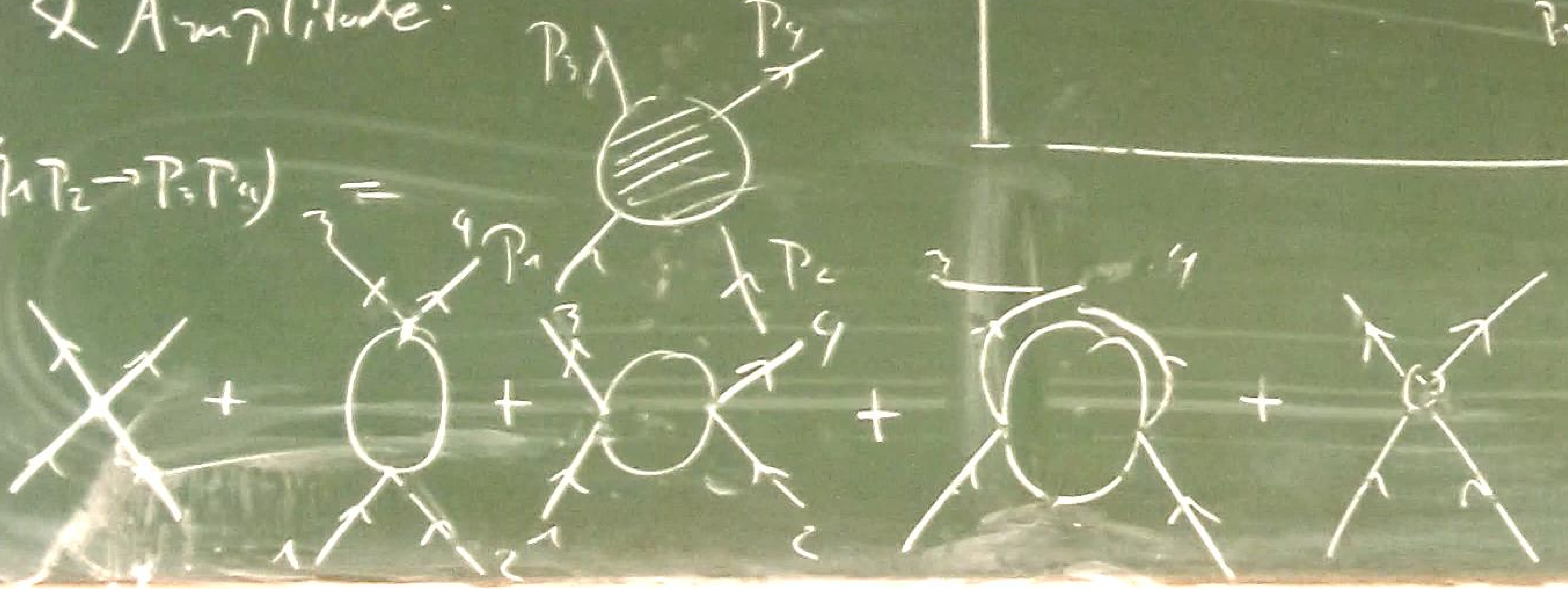
(v) With this δ the amplitude is finite and independent of the regulator

9] Bare and renormalized perturbation theory are equivalent !?

10] Example:

i] 4 Amplitude

$\mathcal{M}(p_1 p_2 \to p_3 p_4) =$



$\simeq$

$\hat{=} -i\lambda + (-i\lambda)^2\left[iV(s) + iV(t) + iV(u)\right]$

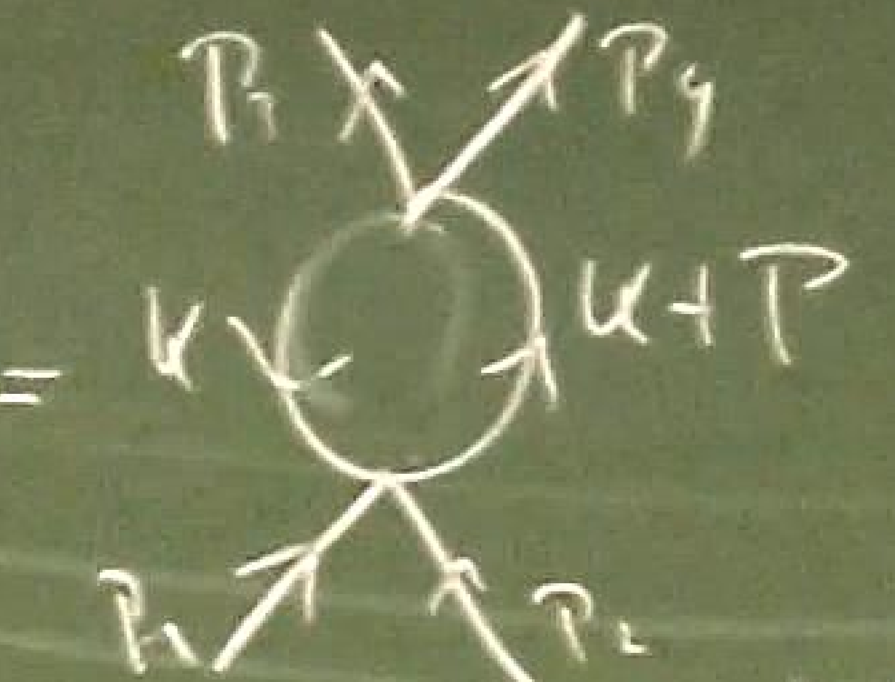
$-i\delta_\lambda$

Mandelstam variables

$s=(p_1+p_2)^2$ $t=(p_3-p_1)^2$

$u=(p_4-p_1)^2$

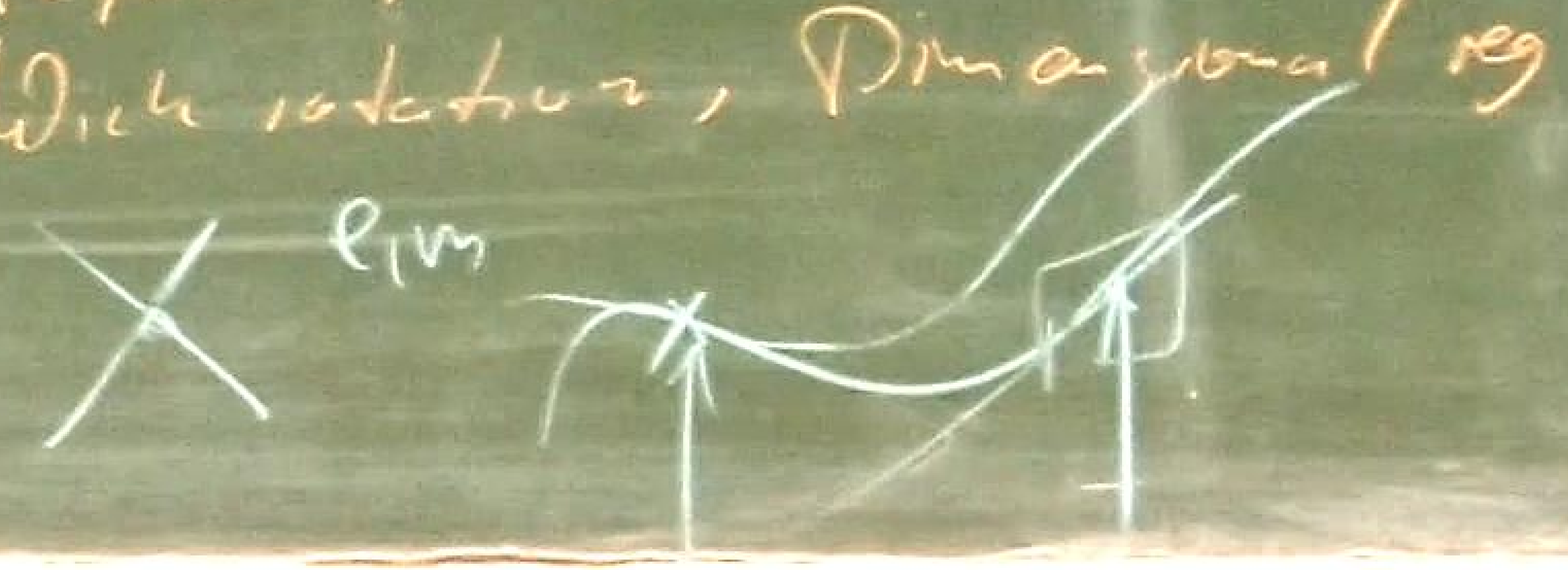
ii] $(-i\lambda)^2 iV(s) =$



$p^2=(p_1+p_2)^2=s$

$$= \frac{(-i\lambda)^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}\frac{1}{(k+p)^2-m^2}$$

Feynman param, Substitution, Wick rotation, Dimensional reg

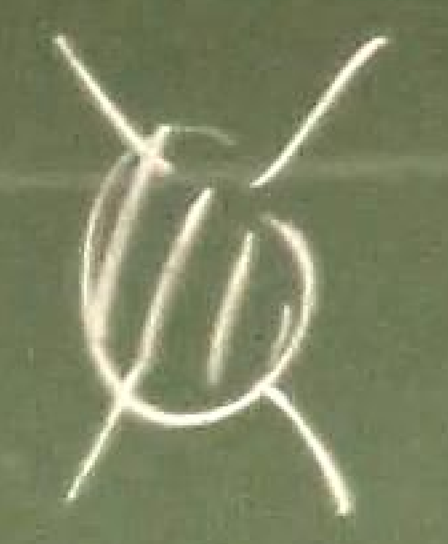


$p_i = \begin{pmatrix} m \\ \vec{0} \end{pmatrix}$

$$\overset{\varepsilon\to 0}{\sim} (-i\lambda)^2\frac{i}{32\pi^2}\int_0^1 dx \left\{\boxed{\frac{2}{\varepsilon}} - \gamma + \log(4\pi) - \log\left[m^2 - x(1-x)p^2\right]\right\}$$

iii] Enforce renormalization condition:

$$\boxed{i\mathcal{M}\Big|_{\substack{s=4m^2\\ t=u=0}} \overset{!}{=} -i\lambda}$$



$$\delta_\lambda = -\lambda^2\left[V(4m^2) + 2V(0)\right]$$

$$\overset{\varepsilon\to 0}{\sim} \frac{\lambda^2}{32\pi^2}\int_0^1 dx \left\{ \boxed{\frac{6}{\varepsilon}} - 3\gamma + 3\log 4\pi - \log\left[m^2-(x1-x)4m^2\right] - 2\log\left[m^2\right] \right\}$$

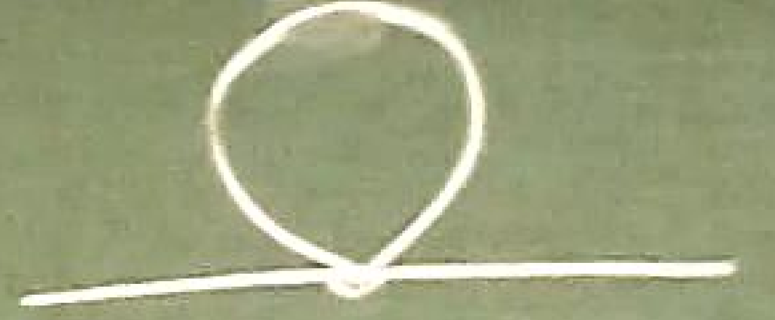
iv] Amplitude:

$$i\mathcal{M} = -i\lambda - i\lambda^2 \mathcal{F}(\{p_i\}, m)$$

finite function of momenta

Regulator ε drops out!

$$-i\mathcal{M}^2(p^2) \approx$$



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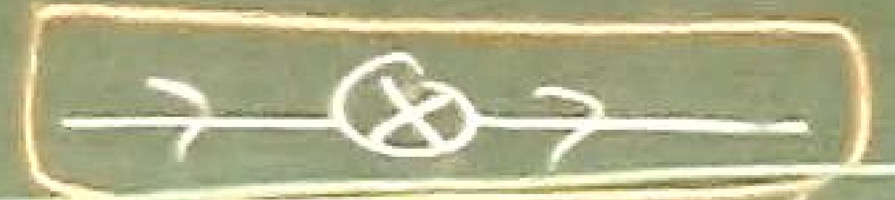


$4-\varepsilon$

$$= \boxed{-\frac{i\lambda}{2}\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(1-\frac{d}{2})}{[illegible]}} + i\left(p^2\delta_Z - \boxed{\delta_m}\right)$$

$$\Rightarrow \boxed{\delta_Z = 0}$$

v]

a)

$$= -i\mathcal{M}^2(p^2)$$

b)

$$= \frac{1}{p^2-m^2-\mathcal{M}^2(p^2)}$$

$$\overset{!}{=} \frac{i\cdot 1}{p^2-m} + \ldots\ldots$$

- $\mathcal{M}^2(m^2) \overset{!}{=} 0$
- $\left.\frac{d\mathcal{M}^2}{dp^2}\right|_{p^2=m^2} \overset{!}{=} 0$

8. Functional Methods

So far:

Hamiltonian → Canonical quant.

→ Feynman rules

Alternative:

Lagrangian → Path integral

→ Feynman rules

8.1. Path integrals in QM

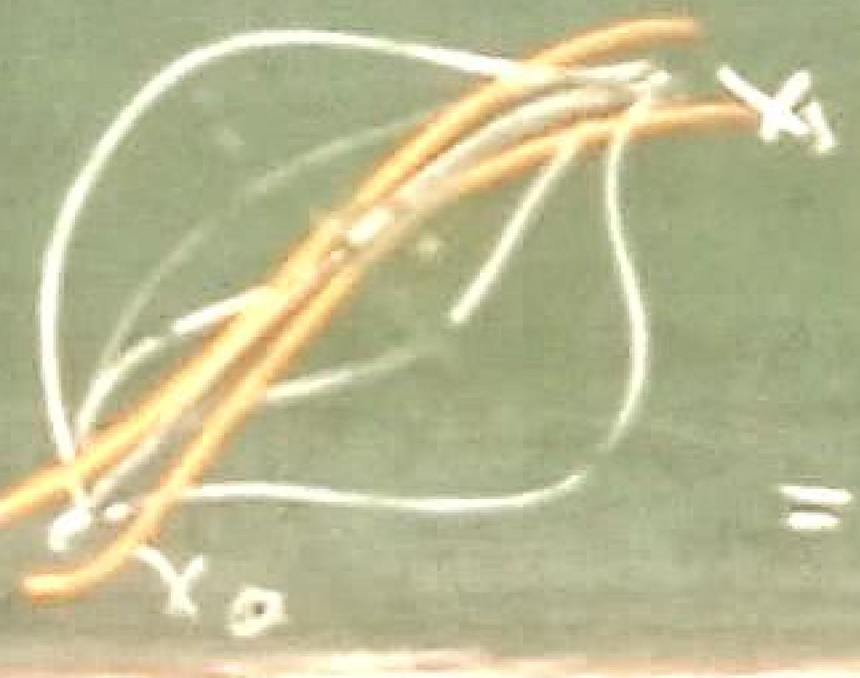
1| ☆ nonrel. particle in 1D: $H=\frac{p^2}{2m}+V(x)$

2| Time evolution operator:

$$U(x_a,x_b,T)=\langle x_b|e^{-\frac{i}{\hbar}\hat{H}T}|x_a\rangle$$

3| PI formalism → Alternative expression for U.

$$U(x_a,x_b,T)=\underbrace{\sum_i}_{\text{All paths }x(t)} e^{i\overset{\text{Functional}}{\overset{\downarrow}{F}}[x(t)]}\,dx$$



Superpos. principle

$$=\int_{x_a=x(0)}^{x_b=x(T)}\mathcal{D}x(t)\,e^{iF[x(t)]}$$

4| F?

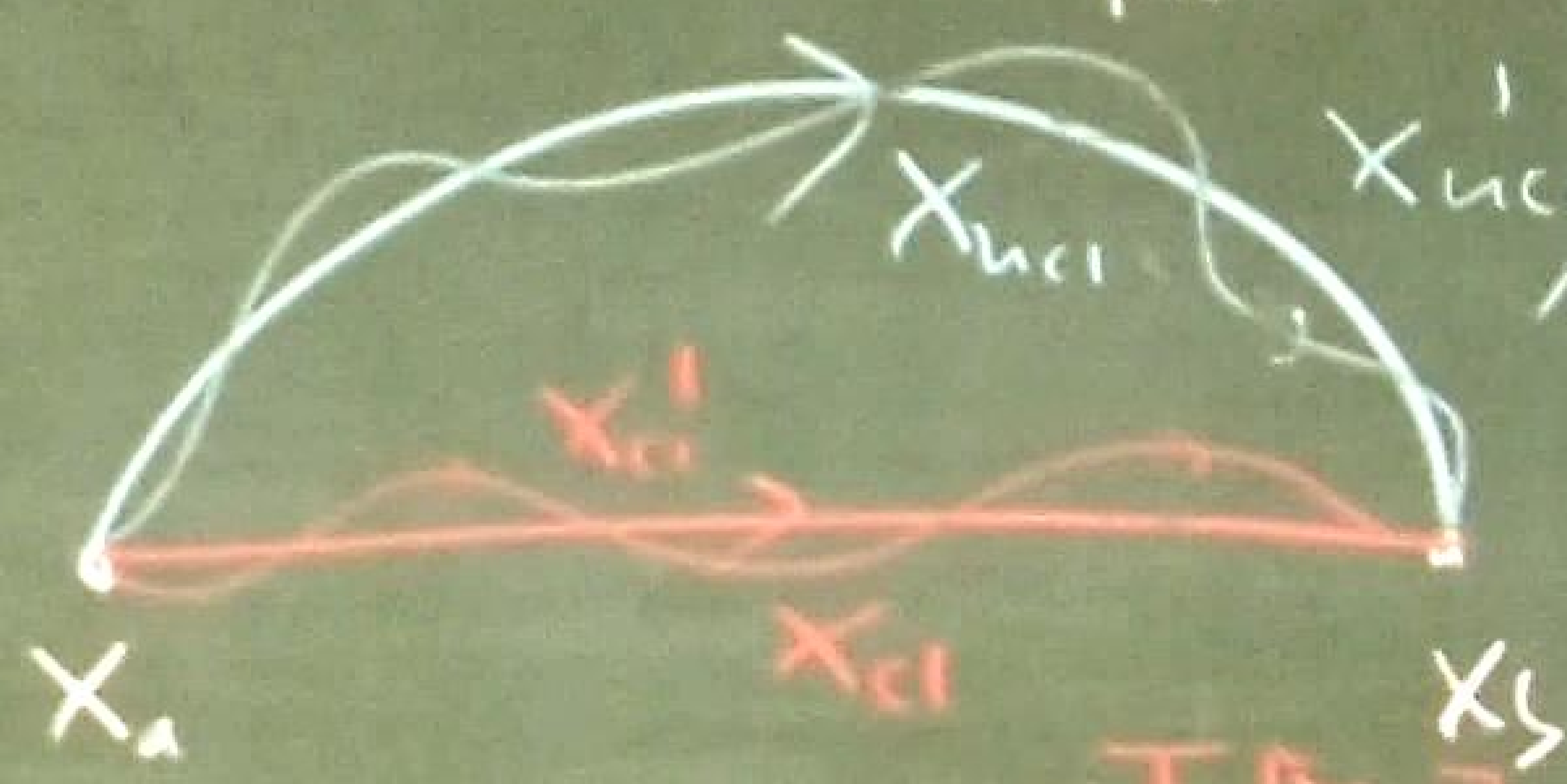
i| Descr. System

ii| Functional of path.

iii| Classical paths $x_{cl}(t)$ should dominate $\hbar\to0$

$$U(x_a,x_b,T)\overset{\hbar\to0}{\approx}\sum_i e^{iF[x(t)]}$$

paths close to x_{cl}



$$e^{iF[x_{ucl}]}+e^{iF[x'_{ucl}]}\approx0$$

$$e^{iF[x_{cl}]}+e^{iF[x'_{cl}]}\neq0$$

Therefore

$$\left.\frac{\delta F}{\delta x}\right|_{x=x_{cl}} \overset{!}{=} 0$$

$$\Rightarrow F = \frac{S}{\hbar} = \frac{1}{\hbar}\int dt\, L(x(t))$$

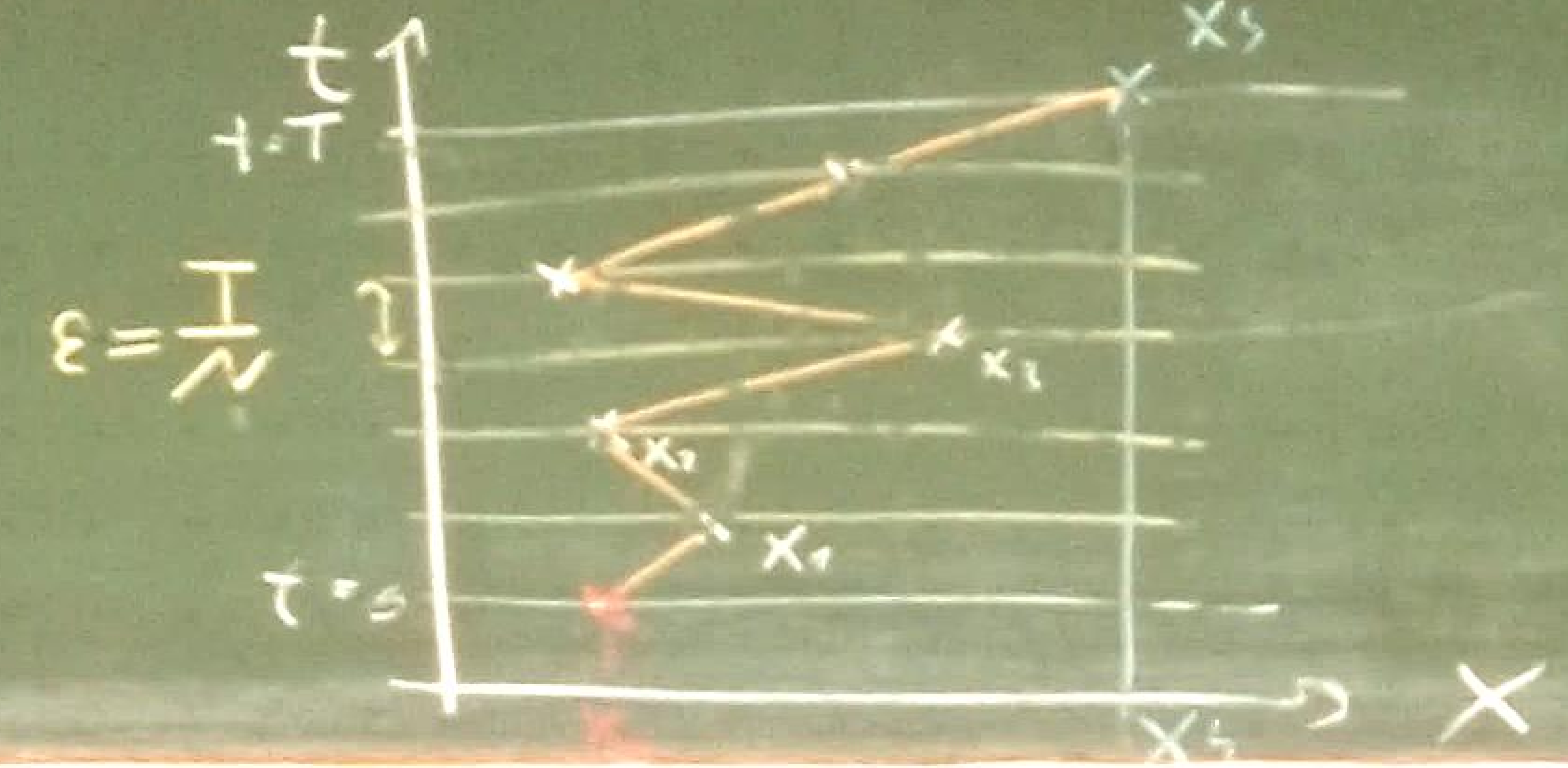
<u>Note</u> $\hbar \to 0$

$$U(x_a,x_b,T) \sim e^{\frac{i}{\hbar}\underbrace{S[x_{cl}(t)]}_{S(x_a,x_b;T)}}$$

5| Propagation amplitude:

$$\boxed{\begin{aligned} U(x_a,x_b,T) &= \int_{x_a}^{x_b} \mathcal{D}x(t)\, e^{\frac{i}{\hbar}S[x(t)]} \\ &\overset{?}{=} \langle x_b|\, e^{-\frac{i}{\hbar}\hat{H}T}\,|x_a\rangle \end{aligned}}$$

6| Def. PI via timeslices:



$$\int \mathcal{D}x(t) = \lim_{N\to\infty}\frac{1}{C_\varepsilon}\int\frac{dx_1}{C_\varepsilon}\cdots\int\frac{dx_{N-1}}{C_\varepsilon}$$

$$= \lim_{N\to\infty}\frac{1}{C_\varepsilon}\prod_{k=1}^{N-1}\int\frac{dx_k}{C_\varepsilon}$$

$\varepsilon = \frac{T}{N}$, C_ε constant

Example 8.1: Particle in potential $V(x)$

1| Lagrangian: $L = \frac{m}{2}\dot{x}^2 - V(x)$

2| Action:

$$S = \int_0^T dt\, L \simeq \sum_{k=0}^{N-1}\left[\frac{m}{2}\left(\frac{x_{k+1}-x_k}{\varepsilon}\right)^2 - \varepsilon V\left(\frac{x_{k+1}+x_k}{2}\right)\right]$$

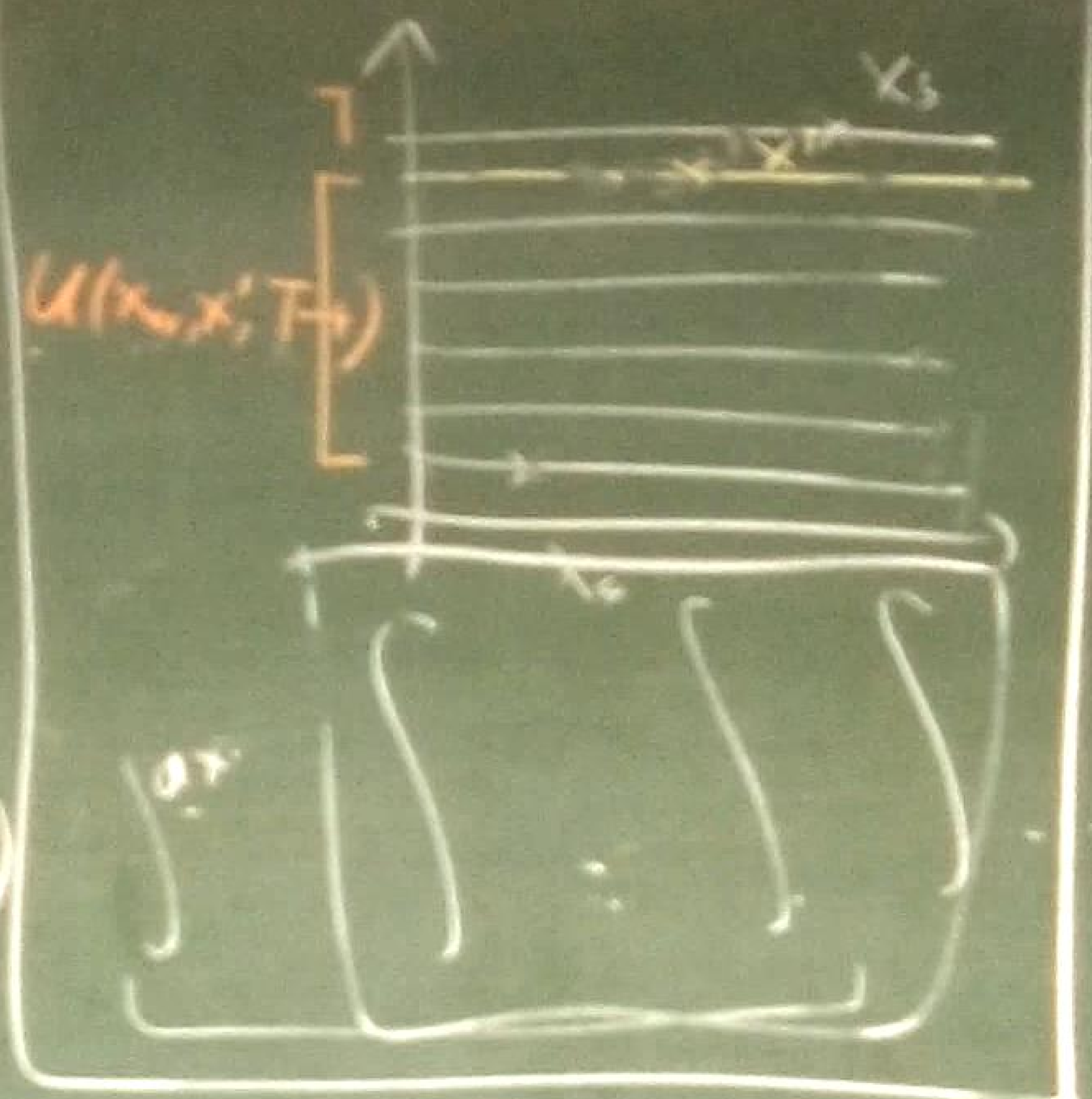
$$U(x_a, x_s, T) = \int_{-\infty}^{\infty} \frac{dx'}{C_\varepsilon} \exp\left[\frac{i}{\hbar} \frac{m(x_s - x')^2}{\varepsilon} - \frac{i}{\hbar}\varepsilon V\left(\frac{x_s + x'}{2}\right)\right]$$

$$\times\, U(x_a, x', T-\varepsilon) \qquad x_s + \boxed{\frac{x' - x_s}{2}} \sim \varepsilon$$

$$= \int \frac{dx'}{C_\varepsilon} e^{\frac{i}{\hbar} m \frac{(x_s - x')^2}{\varepsilon}} \Big[1 - \frac{i}{\hbar}\varepsilon \underbrace{V\left(\frac{x_s + \cancel{x'}}{\cancel{2}}\right)}_{V(x_s) + O(\varepsilon)} + O(\varepsilon^2)\Big]$$

$$\times \left[1 + (x' - x_s)\partial_{x_s} + \frac{(x' - x_s)^2}{2}\partial^2_{x_s} + \dots\right] U(x_a, x_s, T-\varepsilon)$$

$$\int dx\, e^{-x^2} \begin{Bmatrix} 1 \\ x \\ x^2 \end{Bmatrix}; \quad \overset{\circ}{=} \underbrace{\frac{1}{C_\varepsilon}\sqrt{\frac{2\pi\hbar\varepsilon}{-im}}}_{1} \left[1 - \frac{i}{\hbar}\varepsilon V(x_s) + \frac{i\hbar}{2m}\varepsilon \partial^2_{x_s} + O(\varepsilon^2)\right] \underbrace{U(x_a, x_s, T-\varepsilon)}_{U(x_a, x_s, T) - \varepsilon \partial_T U(x_a, x_s, T) + \dots}$$



4] & $\varepsilon \to 0$

$$\boxed{C_\varepsilon = \sqrt{\frac{2\pi\hbar\varepsilon}{-im}}}$$

5] & lin. Order in ε:

$$\boxed{i\hbar\,\partial_T U = \left[-\frac{\hbar^2}{2m} + V(x_s)\right] U}$$

Schrödinger equation

6] Check initial condition:

$$U(x_a, x_s, \varepsilon) \xrightarrow{\varepsilon \to 0} \delta(x_a - x_s)$$

$$\underset{\wedge=:}{\uparrow} \qquad = \langle x_s | \overset{\uparrow}{x_a} \rangle$$

$$\mathbb{1} = e^{-\frac{i}{\hbar} H\, 0}$$

7] $U_{PI} = U_{CQ}$

Generalization: (PSet 12)

1] ∀ coordinates q_i, momenta p_i,
Hamiltonian $H(\vec{q}, \vec{p})$

2] Canonical quant.: $[q_i, p_i] = i\hbar\delta_{ij}$

$$U(\vec{q}_a, \vec{q}_b, T) = \langle \vec{q}_b | e^{-i\hat{H}T} | \vec{q}_a \rangle$$

3] Time slicing:

$$e^{-i\hat{H}T} = \underbrace{e^{-i\hat{H}\varepsilon} \cdots\cdots e^{-i\hat{H}\varepsilon}}_{N}, \quad \varepsilon = \frac{T}{N}$$

$$\uparrow \; \mathbb{1}_n = \int d\vec{q}_n |\vec{q}_n\rangle\langle\vec{q}_n|$$

$$e^{-iH\varepsilon} \int d\vec{q}_n |\vec{q}_n\rangle \boxed{\boxed{\langle \vec{q}_n | e^{-iH\varepsilon} \int d\vec{q}_{n+1} |\vec{q}_{n+1}\rangle} \langle \vec{q}_{n+1}|]} \ldots$$

$$\langle \vec{q}_{n+1} | e^{-i\hat{H}\varepsilon} | \vec{q}_n \rangle = \langle q_{n+1} | \left[\mathbb{1} - i\hat{H}\varepsilon + O(\varepsilon^2) \right] | \vec{q}_n \rangle$$

$$\int dp_n |\vec{p}_n\rangle\langle\vec{p}_n|$$

5] $\hat{H} = \hat{H}_1(\vec{q}) + \hat{H}_2(\vec{p})$

$$\langle \vec{q}_{n+1} | \hat{H} | \vec{q}_n \rangle \doteq \int \frac{d\vec{p}_n}{2\pi} H\left(\frac{\vec{q}_{n+1} + \vec{q}_n}{2}, \vec{p}_n \right) \times e^{i\vec{p}_n(\vec{q}_{n+1} - \vec{q}_n)}$$

Weyl transform/ quantization

6) Hamiltonian phase-space path integral:

$$e^{i p_k \frac{(q_{k+1}-q_k)}{\varepsilon}\varepsilon} \qquad \dot q \qquad \left(p = \frac{\partial \mathcal{L}}{\partial \dot q}\right)$$

$$\boxed{U(\vec q_a, \vec q_b, T) \doteq \underbrace{\int_{\vec q_a}^{\vec q_b} \mathcal{D}\vec q(t) \int \mathcal{D}p(t)}_{\lim\limits_{N\to\infty}\prod\limits_k \int \frac{d\vec q_k\, d\vec p_k}{2\pi\hbar}} \exp\left[\frac{i}{\hbar}\underbrace{\int_0^T dt\, \overbrace{\left(\vec p\cdot\dot{\vec q} - H(\vec q,\vec p)\right)}^{\text{"}\mathcal{L}(q,\dot q)\text{"}}}_{S[\vec q,\vec p]}\right]}$$

$$\boxed{\begin{aligned}\frac{\partial S}{\partial \vec p} &= \dot q - \frac{\partial H}{\partial \vec p} = 0\\ \dot q &= \frac{\partial H}{\partial \vec p}\end{aligned}}$$

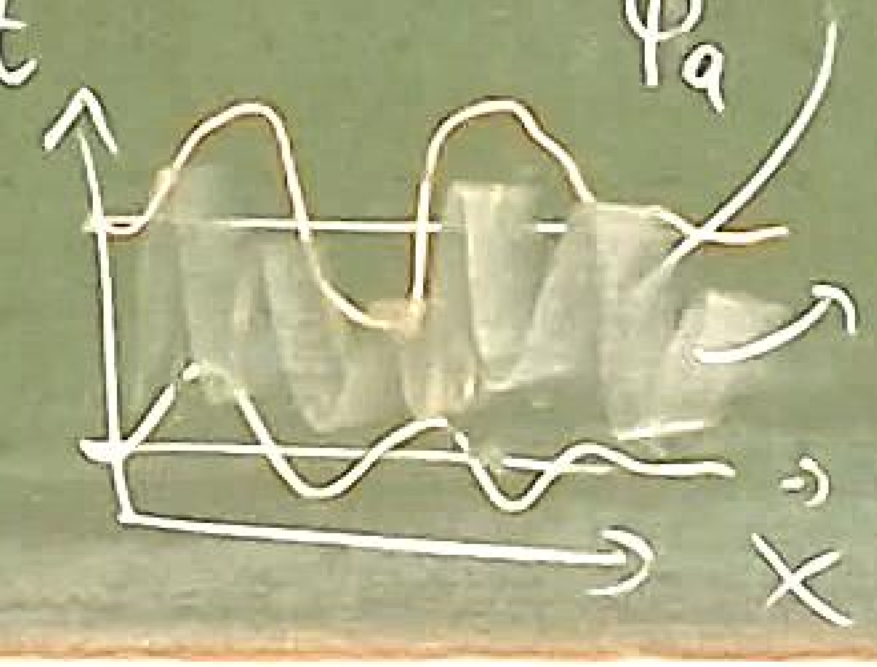
8.2. Path Integrals for scalar fields

Identification: $q_i \longleftrightarrow \phi(x)$

Example 8.2: Real scalar field:

$$\langle\phi_b| e^{-iHT}|\phi_a\rangle$$

$$= \int_{\phi_a}^{\phi_b} \mathcal{D}\phi\, \mathcal{D}\pi \exp\left[\frac{i}{\hbar}\int_0^T d^4x \left[\pi\dot\phi - \underbrace{\frac{1}{2}\pi^2}_{\dot\phi^2} - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right]\right]$$



$\phi(x) \longleftrightarrow q_i(t)$

$\phi_a(\vec x) \longleftrightarrow q_a$

$$= \int_{\phi_a}^{\phi_b} \mathcal{D}\phi \exp\left[\frac{i}{\hbar}\int_0^T d^4x\, \mathcal{L}(\phi, \partial\phi)\right]$$

- Lagrangian: $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$
- Boundaries: $\phi(\vec x, 0) \equiv \phi_a(\vec x)$
  $\phi(\vec x, T) \equiv \phi_b(\vec x)$

$\partial_\mu\phi\, \partial^\mu\phi$

$F_{\mu\nu}F^{\mu\nu}$

ϕ^2

Recap:

8. Functional Methods

8.2. Path integrals for scalar fields

$$U(\phi_a,\phi_b;T) = \langle \phi_b | e^{-i\hat{H}T} | \phi_a \rangle$$

$$= \int_{\phi(\vec{x},0)\equiv\phi_a(\vec{x})}^{\phi(\vec{x},T)\equiv\phi_b(x)} \boxed{\mathcal{D}\phi}\, \exp\Big[\frac{i}{\hbar} \underbrace{\int_0^T d^4x\, \mathcal{L}(\phi,\partial\phi)}_{\text{Action } S[\phi]}\Big]$$

Functional/Pathintegral over all field trajectories $\phi(x) = \phi(\vec{x},t)$

Note on $\mathcal{D}\phi$:

Fourier transform: $\phi(x) = \sum_k \phi_k e^{ikx}$

Fourier coefficients $\in \mathbb{C}$

$k = \frac{2\pi}{L}\cdot n \quad n\in\mathbb{Z}$

∢ finite volume LT

→ Use $\{\phi_k\}$ to parametrize ϕ

→ Integrate over all $\{\phi_k\}$:

$$\int \mathcal{D}\phi := \lim_{L\to\infty} \prod_k \int d\phi_k^I \int d\phi_k^R$$

imaginary part

real part

$$\phi_k = \phi_k^R + i\,\phi_k^I \in \mathbb{C}$$

- If ϕ is real: $\phi_k^* = \phi_{-k}$ ⟹ restrict k to half space: $k^0 > 0$

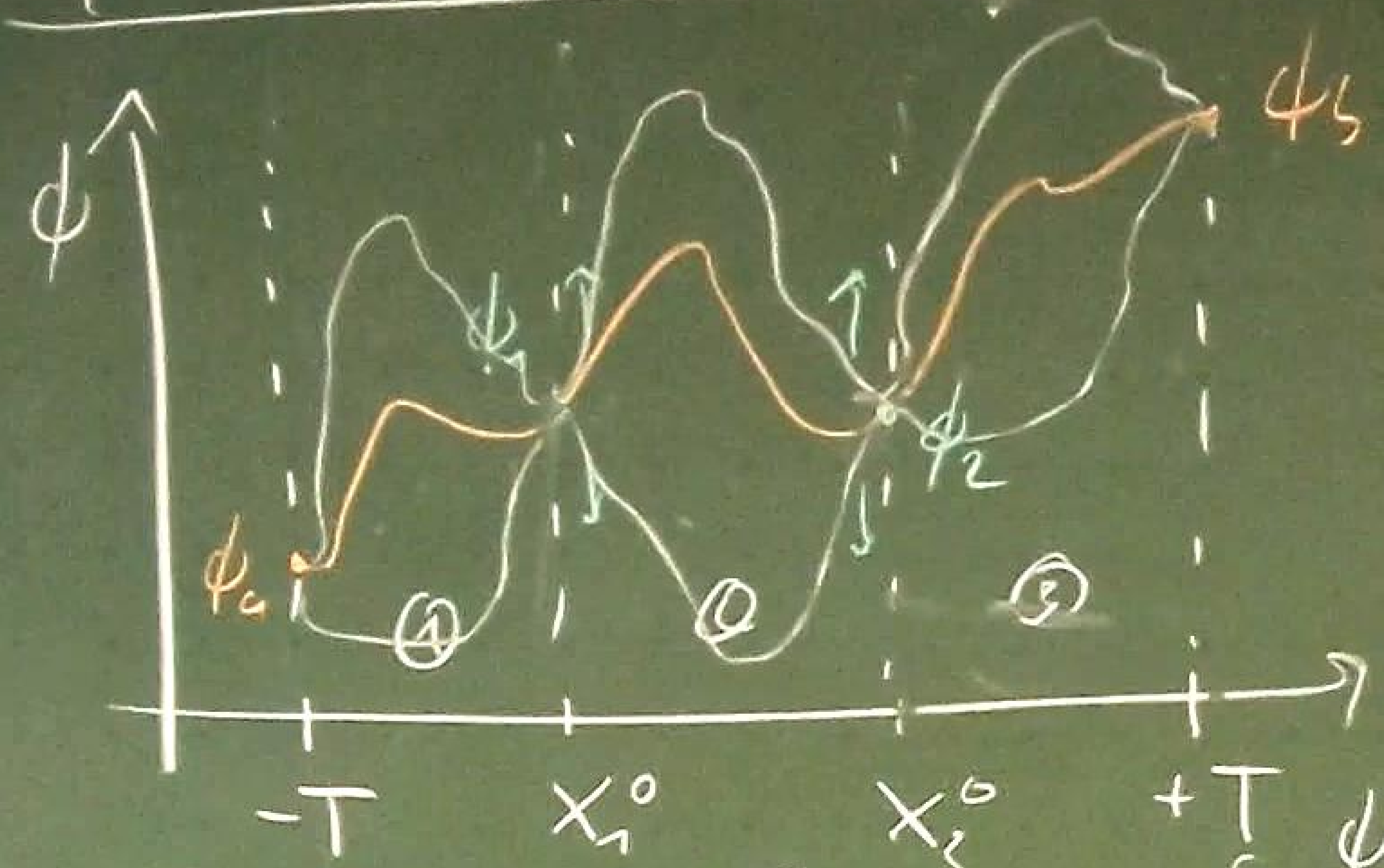
Correlation functions:

1 Goal:

$$\langle \Omega | T\, \phi_H(x_1)\phi_H(x_2) | \Omega \rangle$$

$$\overset{?}{\longleftrightarrow} \int_{\phi(-T)=\phi_a}^{\phi(+T)=\phi_b} \mathcal{D}\phi\ \phi(x_1)\phi(x_2)\, e^{i\int_{-T}^{T} d^4x\, \mathcal{L}} = (*)$$

2| Split functional integral.



$$\int \mathcal{D}\phi = \int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \int_{\phi(\vec{x},x_1^0)=\phi_1(\vec{x})}^{\phi(\vec{x};x_2^0)=\phi_2(\vec{x})} \mathcal{D}\phi$$

3|

$$(*) = \int \mathcal{D}\phi_1 \int \mathcal{D}\phi_2 \, \phi_1(\vec{x})\phi_2(\vec{x}) \underbrace{\langle \phi_b | e^{-iH(T-x_2^0)} | \phi_2 \rangle}_{③} \underbrace{\langle \phi_2 | e^{-iH(x_2^0-x_1^0)} | \phi_1 \rangle}_{②} \underbrace{\langle \phi_1 | e^{-iH(x_1^0+T)} | \phi_a \rangle}_{①} = (**)$$

4|

- $\hat{\phi}_S(\vec{x}_1) | \phi_1 \rangle = \phi_1(\vec{x}_1) | \phi_1 \rangle$
- $\int \mathcal{D}\phi_1(\vec{x}) \, |\phi_1\rangle\langle\phi_1| = \mathbb{1}$ $\quad \left\{ \int dx \, |x\rangle\langle x| = \mathbb{1} \right.$

$$(**) \overset{x_2^0 > x_1^0}{=} \langle \phi_b | \underbrace{e^{-iH(T-x_2^0)}}_{\to |\Omega\rangle\langle\Omega|} \underbrace{\left(\phi_S(\vec{x}_2) \, e^{-iH(x_2^0-x_1^0)} \right.}_{\phi_H(x_2)} \underbrace{\left. \phi_S(\vec{x}_1) \, e^{-iH(x_1^0} \right)}_{\phi_H(x_1)} \underbrace{{}^{+T)}}_{\to |\Omega\rangle\langle\Omega|} | \phi_a \rangle$$

$T \to \infty(1-i\epsilon)$

$$\longrightarrow C \, \langle \Omega | T \{ \phi_H(x_1) \, \phi_H(x_2) \} | \Omega \rangle$$

5|

$$\boxed{\langle\Omega|T\phi_H(x_1)\phi_H(x_2)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int\mathcal{D}\phi\,\phi(x_1)\phi(x_2)\,e^{i\int_{-T}^{T}d^4x\,\mathcal{L}}}{\int\mathcal{D}\phi\,e^{i\int_{-T}^{T}d^4x\,\mathcal{L}}}}$$

8.3. Application: Quantization of the Electromagnetic Field

Goal: $\text{PI} \to \frac{-ig_{\mu\nu}}{k^2+i\epsilon}$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

1| $S[A] = \int d^4x \left[-\frac{1}{4}(F_{\mu\nu})^2\right]$

Partial integration

$$\overset{\circ}{=} \frac{1}{2}\int d^4x\, A_\mu(x)\left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right)A_\nu(x)$$

Fourier transform

$$\overset{\circ}{=} \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,\underbrace{\tilde{A}_\mu(k)\left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right)\tilde{A}_\nu(-k)}_{\square}$$

2| $\forall\ \tilde{A}_\mu(k) = k_\mu\alpha(k)$

$\square = 0 \to S[A] = 0$

$\to \int\mathcal{D}A\,\underbrace{e^{i0}}_{1} = \infty$

3| Problem: Gauge invariance

$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha$

$A_\mu = 0 \to A_\mu \propto \partial_\mu\alpha$

$\to \tilde{A}_\mu \sim k_\mu\alpha$

4] **Solution**: Count each physical configuration only once: Fadeev & Popov procedure

i] Gauge fixing. $G(A) = 0$ $\quad \{ G(A) = \partial_\mu A^\mu$

ii] $A_\mu^\alpha = A_\mu + \frac{1}{e}\partial_\mu \alpha$

Note:

$$1 = \prod_1^N \int dg_i \, \delta^{(n)}(\vec{g}) \underset{\substack{\uparrow \\ \vec{g} = \vec{g}(\vec{\alpha})}}{=} \left[\prod_i \int da_i\right] \delta^{(n)}(g(\vec{a})) \det\left(\frac{\partial \vec{g}}{\partial \vec{a}}\right)$$

$$1 = \int \mathcal{D}\alpha \, \delta(\underbrace{G(A^\alpha)}_{\sim g(\alpha)}) \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right)$$

iii] Assumption: $\frac{\delta G(A^\alpha)}{\delta \alpha}$ independent of α and A ~ does not hold non-abelian → Ghost fields

iv] $$\int \mathcal{D}A \, e^{iS[A]} \cdot 1 = \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right) \int \mathcal{D}\alpha \int \mathcal{D}A \, e^{iS[A]} \, \delta(G(A^\alpha))$$

$$\left[\begin{array}{l} \bullet\ \tilde{A} = A^\alpha = A + \frac{1}{e}\partial\alpha \longrightarrow \mathcal{D}A = \mathcal{D}\tilde{A} \\ \bullet\ S[A] = S[\tilde{A}] \quad \text{Gauge invariance} \end{array}\right]$$

$$= \det\left(\frac{\delta(G(A^\alpha))}{\delta \alpha}\right) \Big(\underbrace{\int \mathcal{D}\alpha}_{\infty} \Big) \underbrace{\int \mathcal{D}\tilde{A} \, e^{iS[\tilde{A}]} \, \delta(G(\tilde{A}))}_{\text{Physically distinct configurations}}$$

v] Choose $G(A^\alpha) = \partial^\mu A^\alpha_\mu - \omega(x)$ $\qquad A^\alpha_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha \qquad \frac{\delta\alpha}{\delta\alpha} = 1$

$$\det\left(\frac{\delta G}{\delta\alpha}\right) = \det\left(\frac{1}{e}\partial^2\right)$$

$$(*) = \det\left(\frac{1}{e}\partial^2\right)\left(\int\mathcal{D}\alpha\right)\int\mathcal{D}A\, e^{iS[A]}\,\delta(\partial^\mu A_\mu - \omega(x))$$

vi]

$$(*)\ \overbrace{N(\xi)\int\mathcal{D}\omega\, e^{-i\int d^4x\,\frac{\omega^2}{2\xi}}}^{1}$$

$$\times\det\left(\frac{1}{e}\partial^2\right)\left(\int\mathcal{D}\alpha\right)\int\mathcal{D}A\, e^{iS[A]}\,\delta(\partial^\mu A_\mu - \omega(x))$$

$$\boxed{\underbrace{N(\,)\int\mathcal{D}\omega\,(\omega)}_{1}\int\mathcal{D}A\,e^{iS[A]} = N(\,)\int\mathcal{D}\omega}$$

$$= N(\xi)\det\left(\frac{1}{e}\partial^2\right)\left(\int\mathcal{D}\alpha\right)\int\mathcal{D}A\, e^{iS[A]}\exp\left[-i\int d^4x\,\frac{(\partial^\mu A_\mu)^2}{2\xi}\right]$$

$\tilde{A} = A^\alpha \qquad S[\tilde{A}] = S[A],\ O(\tilde{A}) = O(A)$

vii] $\langle\Omega|T\,O(\hat{A})|\Omega\rangle$

gauge invariant

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\int\mathcal{D}A\,O(A)\exp\left\{i\int_{-T}^{T}d^4x\left[\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right]\right\}}{\int\mathcal{D}A\exp\{\cdots\}}$$

5] $\tilde{S}[A] = \int d^4x\left[-\frac{1}{4}(F_{\mu\nu})^2 - \boxed{\frac{1}{2\xi}(\partial^\mu A_\mu)^2}\right]$

Part. int. / FT

$$= \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\tilde{A}_\mu(k)\left[-k^2g^{\mu\nu} + (1-\xi^{-1})k^\mu k^\nu\right]\tilde{A}_\nu(-k)$$

Argument of 1] no longer works!

6| Propagator:

$$D_F^{\mu\nu}(x-y) = \langle\Omega| T A^\mu(x) A^\nu(y) |\Omega\rangle$$

$$\rightarrow \langle\Omega| \tilde{A}^\mu(k) \tilde{A}^\nu(q) |\Omega\rangle = 0 \quad \text{whenever } k \neq -q$$

$$\tilde{D}_F^{\mu\nu}(q) = \langle\Omega| \tilde{A}(q) \tilde{A}^\nu(-q) |\Omega\rangle$$

Add $+i\varepsilon$ for regularization

$$= \frac{\int \mathcal{D}A\, \tilde{A}^\mu(q) \tilde{A}^\nu(-q) \exp\left\{\frac{i}{2}\int\frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(k) \overbrace{\left[-k^2 g^{\mu\nu} + (1-\xi^{-1}) k^\mu k^\nu\right]}^{M^{\mu\nu}(k)} \tilde{A}_\nu(-k)\right\}}{\int \mathcal{D}A \exp\{-\,,-\}}$$

$$= i\left[M^{-1}(q)\right]^{\mu\nu}$$

↳ Problemset 12

$$\xrightarrow{\text{Finally}} \boxed{\tilde{D}_F^{\mu\nu}(q) \overset{6}{=} \frac{-i}{q^2+i\varepsilon}\left[g^{\mu\nu} - (1-\xi)\frac{q^\mu q^\nu}{q^2}\right]}$$

- $\nabla\cdot\vec{A}=0$
- $\partial_\mu A^\mu=0$

7| Gauges:

- Set $\xi=1$: Feynman gauge

$$\boxed{\tilde{D}_F^{\mu\nu}(q) = -\frac{i g^{\mu\nu}}{q^2+i\varepsilon}}$$

- Set $\xi=0$: Landau gauge

$$\tilde{D}_F(q) = \frac{-i}{q^2+i\varepsilon}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)$$

Vote:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ipx} + a_p^\dagger e^{ipx}\right)\Big|_{p^0=E_p}$$

$$\phi(\vec{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-i\vec{p}\vec{x}} |\vec{p}\rangle \equiv |\vec{x}\rangle$$

$$\uparrow$$

$$\frac{1}{m} = \lambda$$

$$)_{\mp}(x-y) = \langle 0|T\{\phi(x)\phi(y)\}|0\rangle$$

$$\overset{x^0>y^0}{=} \langle 0|\phi(x)\phi(y)|0\rangle$$

$$= \underset{\langle 0|}{\langle 0| e^{iHx^0}} \phi(\vec{x}) \underbrace{e^{-iHx_0} e^{iHy^0}}_{U = e^{-iH(x^0-y^0)}} \phi(\vec{y}) \underset{|0\rangle}{e^{-iHy^0} |0\rangle}$$

$$= \langle \vec{x}|U(y^0; x^0)|\vec{y}\rangle$$

$$= U(\vec{y}, \vec{x}, T = x^0 - y^0)$$

9. Non-Abelian Gauge Theories

Motivation.

* Interactions? $\bar{\psi}\gamma^\mu\psi A_\mu$ others?

A^4, $(\partial A)A^2$?

* Massless particle + Vector field A_μ

$$\downarrow$$

$$ISO(2) = \underset{\substack{\downarrow \\ \text{Helicity } \pm 1}}{SO(2)} \times \text{Trans.}$$

$$A^\mu = \int d\vec{p} \sum_{\sigma=\pm} a_{\vec{p}\sigma} + a^\dagger_{\vec{p},\vec{\sigma}}$$

$$\longrightarrow$$

$$U(\Lambda) A^\mu(x) U^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu A^\nu(\Lambda x) + \partial_\mu \Omega(\Lambda, x)$$

CC

$$U(\Lambda) \psi(x) U^{-1}(\Lambda) = \Lambda_{\frac{1}{2}}^{-1} \psi(\Lambda x)$$

* 't Hooft. Gauge symmetry

⇓

Renormalizability

9.1. The Geometry of Gauge Invariance

1| ✗ Local $U(1)$ symmetry G of Dirac field:

$$\tilde{\psi}(x) = e^{i\alpha(x)}\psi(x)$$

arbitrary $\alpha(x): \mathbb{R}^{1,3} \to \mathbb{R}$

2| Goal: Construct invariant Lagrangians

3| No problem without derivatives:

global $U(1)$ invariance → local $U(1)$ inv.

Example: $\bar{\psi}\psi$

4| ✗ Directional derivative along $n \in \mathbb{R}^{1,3}$:

$$n^\mu \partial_\mu \psi := \lim_{\varepsilon \to 0} \frac{\psi(x+\varepsilon n) - \psi(x)}{\varepsilon}$$

↳ $n^\mu \partial_\mu \psi$ no simple transformation law under G

5| ✗ "Comparator" $U: \mathbb{R}^{1,3} \times \mathbb{R}^{1,3} \longrightarrow \mathbb{C}$

$$\tilde{U}(y,x) = e^{i\alpha(y)}\underset{e^{i\phi(y,x)}}{U(y,x)}\, e^{-i\alpha(x)}$$

$$U(y,y) = 1$$

↳ $\psi(y)$, $U(y,x)\cdot\psi(x)$

Differential geometry
- ∗ fiber bundles
- ∗ principal bundles
- ∗ connection / parallel transport.

6| Covariant derivative:

$$\boxed{n^\mu D_\mu \psi := \lim_{\varepsilon \to 0} \frac{\psi(x+\varepsilon n) - U(x+\varepsilon n, x)\psi(x)}{\varepsilon} \qquad (1)}$$

7|

$$U(x+\varepsilon n, x) = 1 - i\underset{\text{arbitrary constant}}{e}\,\varepsilon\, n^\mu \underset{\substack{\text{new vector field}\\ \text{gauge connection}}}{A_\mu(x)} + O(\varepsilon^2) \qquad (2)$$

8| (2) in (1)

$$\boxed{D_\mu \psi(x) \overset{\circ}{=} \partial_\mu \psi(x) + ie A_\mu \psi(x)}$$

9] $\overset{\circ}{\longrightarrow}$

$$\boxed{\tilde{A}_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)}$$

10]

$$\boxed{\tilde{D}_\mu \tilde{\Psi}(x) = e^{i\alpha(x)} D_\mu \Psi(x)}$$

→ $D_\mu \Psi$ transforms like Ψ

→ All globally U(1) invariant terms are allowed if $\partial \to D$

$$\bar{\Psi} \not\partial \psi \longrightarrow \bar{\Psi} \not D \psi$$

11] Conclusion:

Local symmetry → Gauge field A_μ needed for covariant derivatives

12] Kinetic energy for A_μ?

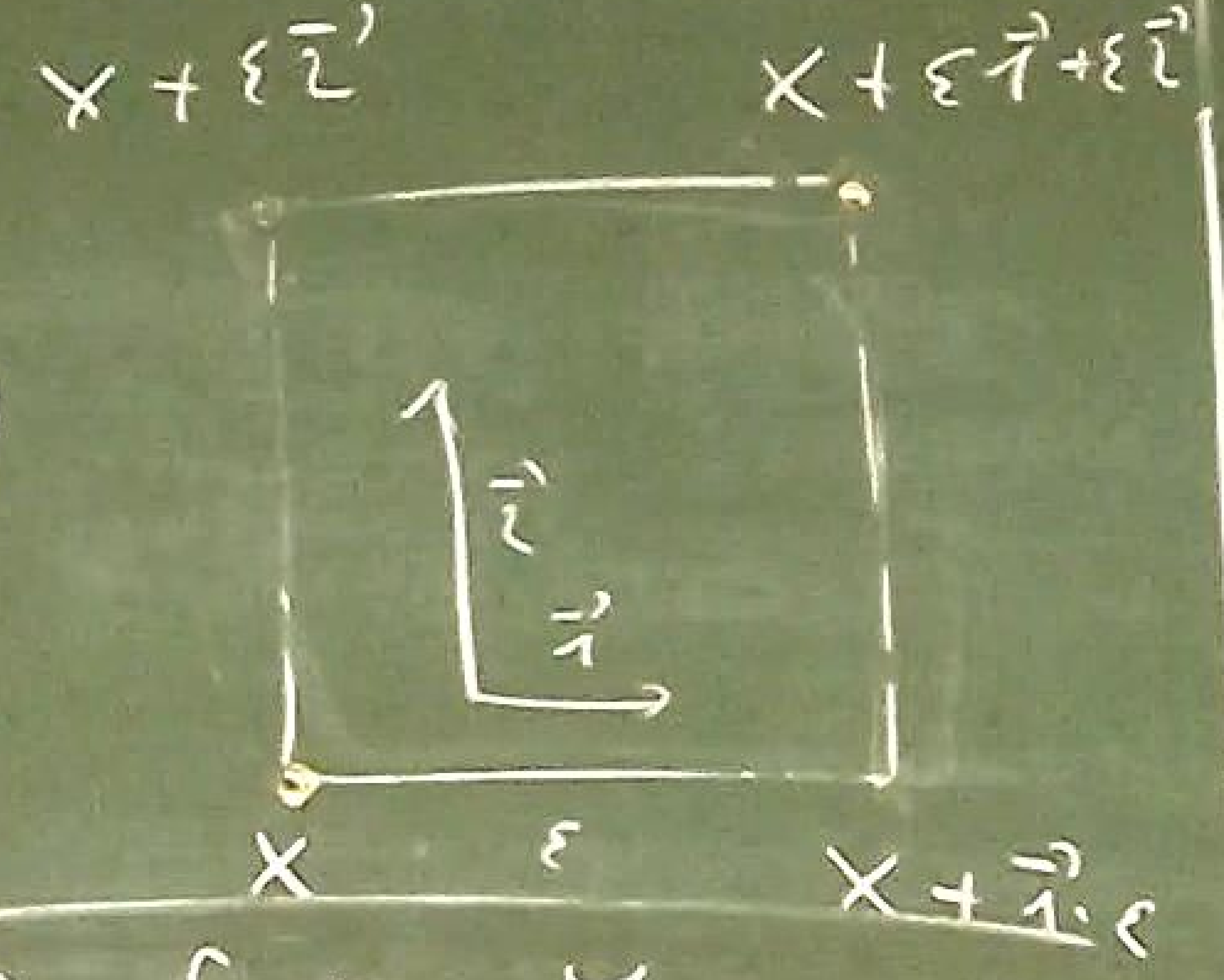
i] ∡ Locally invariant loop.

e^{ieA_μ}

$U(1) = \mathbb{R}$

$$U(x) := U(x, x+\varepsilon\vec{2})$$
$$\times U(x+\varepsilon\vec{2}, x+\varepsilon\vec{1}+\varepsilon\vec{2})$$
$$\times U(x+\varepsilon\vec{1}+\varepsilon\vec{2}, x+\varepsilon\vec{1})$$
$$\times U(x+\varepsilon 1, x)$$

$$\rightarrow \tilde{U} = U$$



$$S = \int d^4x\, \mathcal{L}$$

$$U(x) \overset{\circ}{=} 1 - i\varepsilon^2 e(\partial_1 A_2 - \partial_2 A_1) + O(\varepsilon^3)$$

$$\rightarrow \boxed{F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu}$$ Field strength tensor

is gauge invariant by construction.

13] Most general Lagrangian in $D = 1+3$.

- * Gauge invariant → ψ, $D_\mu\psi$, $F_{\mu\nu}$, $\partial_\mu F_{\mu\nu}$
- * Relativistic → Lorentz scalar
- * Renormalizable

→ Terms of mass dimension at most 4

$\longrightarrow$

$$\boxed{\begin{aligned}\mathcal{L} &= \bar{\psi}\, i \not{D} \psi - m \bar{\psi}\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ &\quad -\underset{\uparrow}{c_1 \varepsilon^{\alpha\beta\mu\nu}} F_{\mu\nu} F_{\alpha\beta} + \underset{\downarrow}{\cancel{c_2 (\bar{\psi}\psi)^2}} + \ldots\end{aligned}}$$

pseudo tensor
invariant under $SO^+(1,3)$
ut not P, T $(O(1,3))$

not
renormalizable

$\rightarrow$ Most general P/T symmetric
Lagrangian : QED

9.2. The Yang-Mills Lagrangian

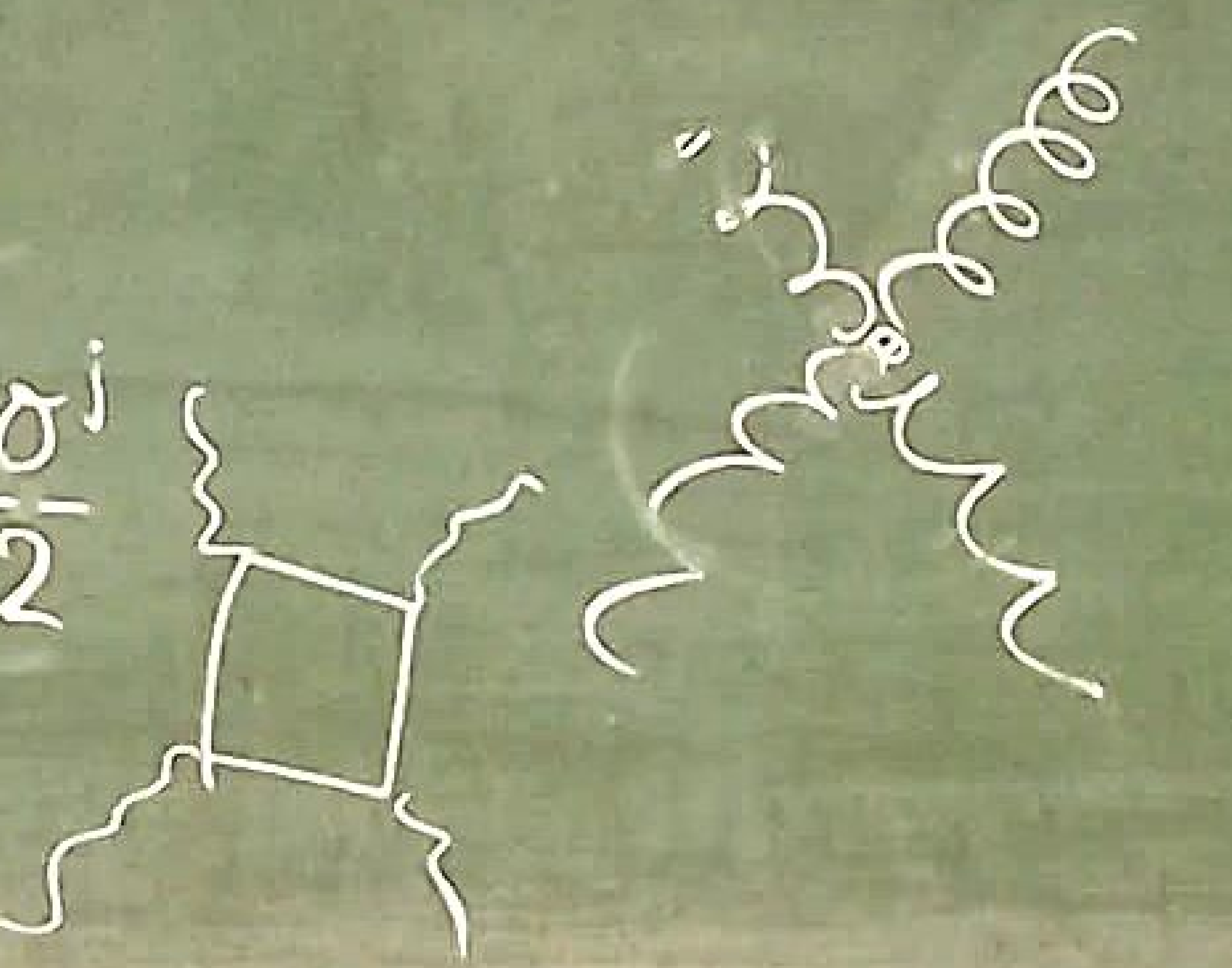
Goal: Replace local symmetry group $U(1)$ by non-abelian Lie group G

Examples: $SO(3)$, $SU(2)$, $SU(3)$

1| $\forall$ Lie group G represented $n \times n$ unitary matrices V

Example: $G = SU(2)$

$$V = e^{i\omega_j \frac{\sigma^j}{2}}$$



2| Fields $\psi = (\psi_1, \ldots, \psi_n)$ n-plets of Dirac fields ψ_i

$$\tilde{\psi}(x) = V(x)\psi(x)$$

$$\psi : \mathbb{R}^{1,3} \longrightarrow \underset{\substack{\uparrow \\ \text{Lorentz group}}}{\mathbb{C}^4} \otimes \underset{\substack{\uparrow \\ \text{gauge group}}}{\mathbb{C}^n} = \mathbb{C}^{4n}$$

3| G Lie group $\longrightarrow$ Lie algebra $\mathfrak{g}$
with N Hermitian generator t^a

$$[t^a, t^b] = i \underset{\substack{\uparrow \\ \text{structure symbol}}}{f^{abc}} t^c$$

($n \times n$ matrices $a = 1 \ldots N$)

Example: $SU(2)$: $N=3$, $[L^i, L^j] = i \varepsilon^{ijk} L_k$

Recap:

9.2. The Yang-Mills Lagrangian

Goal: Gauge theories for non-abelian Lie groups $G = SU(2), SU(3), \ldots$

1] Lie group G with defining representation:

V: unitary $n \times n$ matrix

↑ dimension of the representation

2] V acts on n-plets of Dirac spinor fields.

$$\tilde{\psi}(x) = V(x)\,\psi(x)$$

$V: \mathbb{R}^{3,1} \to G$

$(\psi_1, \ldots, \psi_n)^T$ ← Dirac bispinor

local gauge transformation

3] Lie algebra $\mathfrak{g}$ describes Lie group G:

$t^a \in \mathfrak{g}$: $n \times n$ Hermitian matrix

↑ generators $(a = 1, \ldots, N)$

↑ Dimension of Lie group

$$\boxed{[t^a, t^b] = i f^{abc} t^c}$$

Lie bracket (here commutator)

$f^{abc} \in \mathbb{C}$: structure constants

$$\rightarrow V(x) = \underbrace{\exp[i\underbrace{\alpha^a(x)\,t^a}_{\in \mathfrak{g}}]}_{\in G} = 1 + i\alpha^a(x)\,t^a + O(\alpha^2)$$

$\alpha^a: \mathbb{R}^{1,3} \to \mathbb{R}$

"Real Lie algebra"

Example: $G = SU(2)$

$= \left\{ \begin{array}{l} 2\times 2 \text{ unitary matrices} \\ \text{with determinant } 1 \end{array} \right\}$

- $n = 2$, $N = 3$ → $i = 1, 2, 3$ (x y z)
- $t^i = \frac{\sigma^i}{2}$ ← Pauli matrices
- $[t^i, t^j] = i\,\boxed{\varepsilon^{ijk}}\,t^k$

$\shortparallel$

f^{ijk} for $SU(2)$

Representation with $\underline{n = 3}$

⇓

Spin 1

4] "Comparator".

$$\tilde{U}(x,y) = V(y)\,\underset{\uparrow}{U}(y,x)\,V^\dagger(x)$$

$U(y,y) = \mathbb{1}$ n×n unitary matrix

$$\rightarrow U(x+\varepsilon n, x) = 1 + ig\varepsilon n^\mu A^a_\mu t^a + O(\varepsilon^2)$$

g: arbitrary constant

A^a_μ: N vector fields

5] Covariant derivative:

$$\boxed{D_\mu \overset{\circ}{=} \partial_\mu - ig\,A^a_\mu t^a}$$

$A^a_\mu t^a = A_\mu$ ← n×n matrix

6] Transformation of A^a_μ:

i] $$\boxed{\tilde{A}^a_\mu t^a \overset{\circ}{=} V(x)\left[A^a_\mu t^a + \frac{i}{g}\partial_\mu\right]V^\dagger(x)}$$

only

Valid for all $V \in G$

$D_\mu\psi$ transforms like ψ

$\Downarrow$

$\bar{\psi}\,\not{D}\,\psi$

ii] ✗ Infinitesimal gauge trafo.

$V \approx \mathbb{1} + \alpha \ldots + O(\alpha^2)$

$$\boxed{\tilde{A}^a_\mu \overset{\circ}{\approx} A^a_\mu + \frac{1}{g}\partial_\mu\alpha^a + \boxed{f^{abc}A^b_\mu\alpha^c}}$$

Infinit. transformation is $O(\alpha)$

new for non-abelian groups

7]

$\Leftarrow$ $$\boxed{\tilde{D}_\mu\tilde{\psi} \overset{\circ}{=} V D_\mu \psi} \quad (\partial_\mu V^\dagger)\cdot V = -V^\dagger(\partial_\mu V)$$

$$0 = \partial_\mu\mathbb{1} = \partial_\mu(V^\dagger V) = (\partial_\mu V^\dagger)V + V^\dagger(\partial_\mu V)$$

8) Kinetic term for A^a_μ

i) $$\tilde{D}_\mu \underbrace{\tilde{D}_\nu \tilde{\psi}}_{\tilde{\phi}} = V D_\mu D_\nu \psi$$

$$\Rightarrow \underbrace{[\tilde{D}_\mu, \tilde{D}_\nu]}\tilde{\psi} = V[D_\mu, D_\nu]\psi$$

$$= \underbrace{V[D_\mu, D_\nu]V^\dagger}\tilde{\psi}$$

$=$

ii) On the other hand:

$$-ig F^a_{\mu\nu} t^a := [D_\mu, D_\nu]$$

with $$\boxed{F^a_{\mu\nu} \hat{=} \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu}$$

$F^a_{\mu\nu}$: N field-strength tensors

$F_{\mu\nu} = F^a_{\mu\nu} t^a$. $n\times n$ matrix

iii) $$\boxed{\tilde{F}_{\mu\nu} = \tilde{F}^a_{\mu\nu} t^a = V F_{\mu\nu} V^\dagger}$$

→ $F_{\mu\nu}$ is no longer gauge invariant

iv)

$$\boxed{\begin{aligned}\mathcal{L}_{YM} &= -\tfrac{1}{2}\,\mathrm{Tr}[F^2] \\ &= -\tfrac{1}{2}\,\mathrm{Tr}[F^a_{\mu\nu} t^a F^{\mu\nu b} t^b] \\ &= -\tfrac{1}{2}(F^a_{\mu\nu} F^{\mu\nu b})\,\underbrace{\mathrm{Tr}[t^a t^b]}_{\frac{1}{2}\delta^{ab}} \\ &= -\tfrac{1}{4}(F^a_{\mu\nu} F^{\mu\nu a}) \\ &= -\tfrac{1}{4}(F^a_{\mu\nu})^2\end{aligned}}$$

Yang-Mills Lagrangian

Note 9.1

$$F^2 \sim (\partial A)^2 + f(\partial A)AA + f^2 AAAA$$

→ Interacting QFT for $f \neq 0$ ↑ non-abelian

→ Gauge bosons can interact

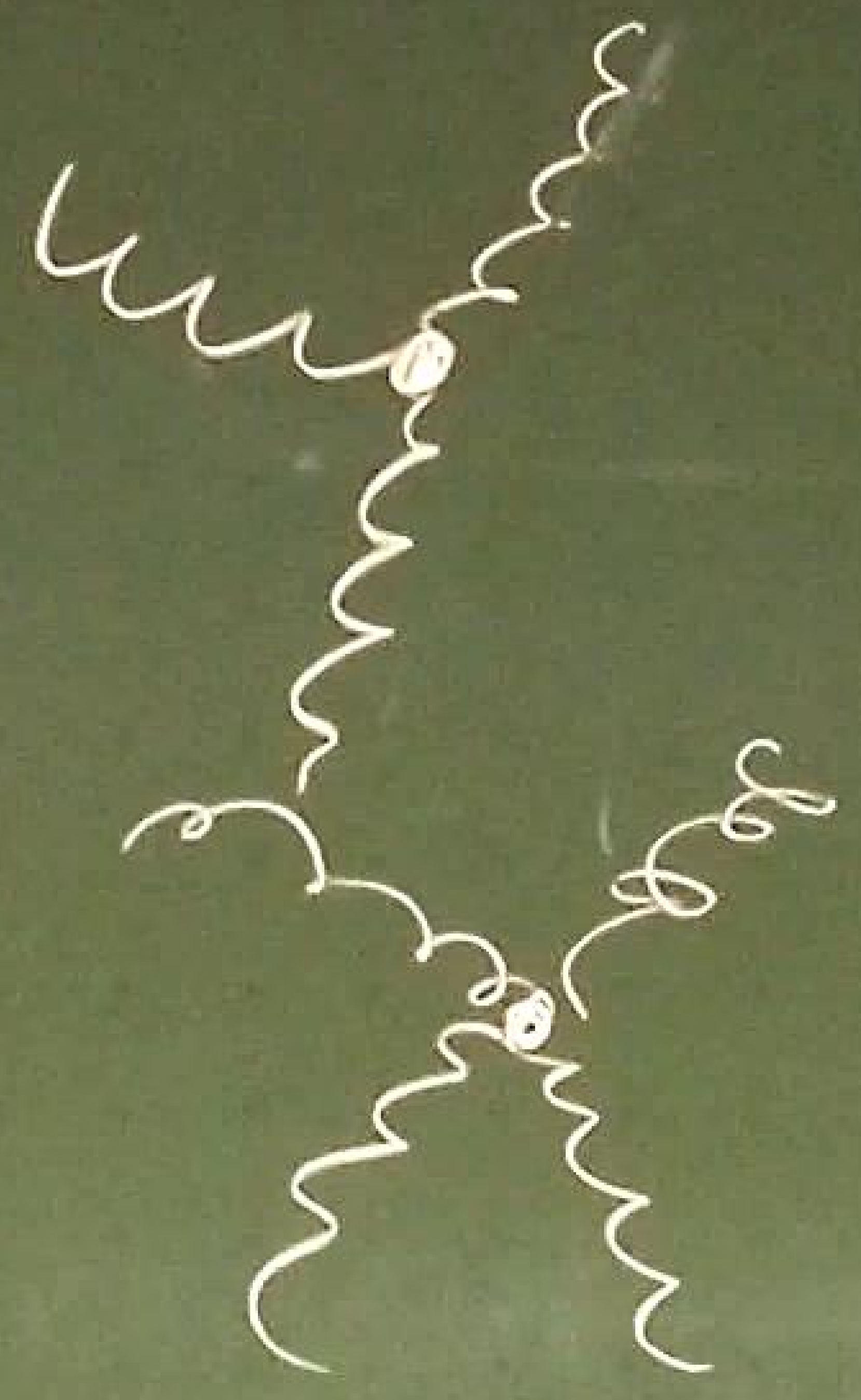
Example: $G = SU(3)$
(Quantum Chromodynamics)

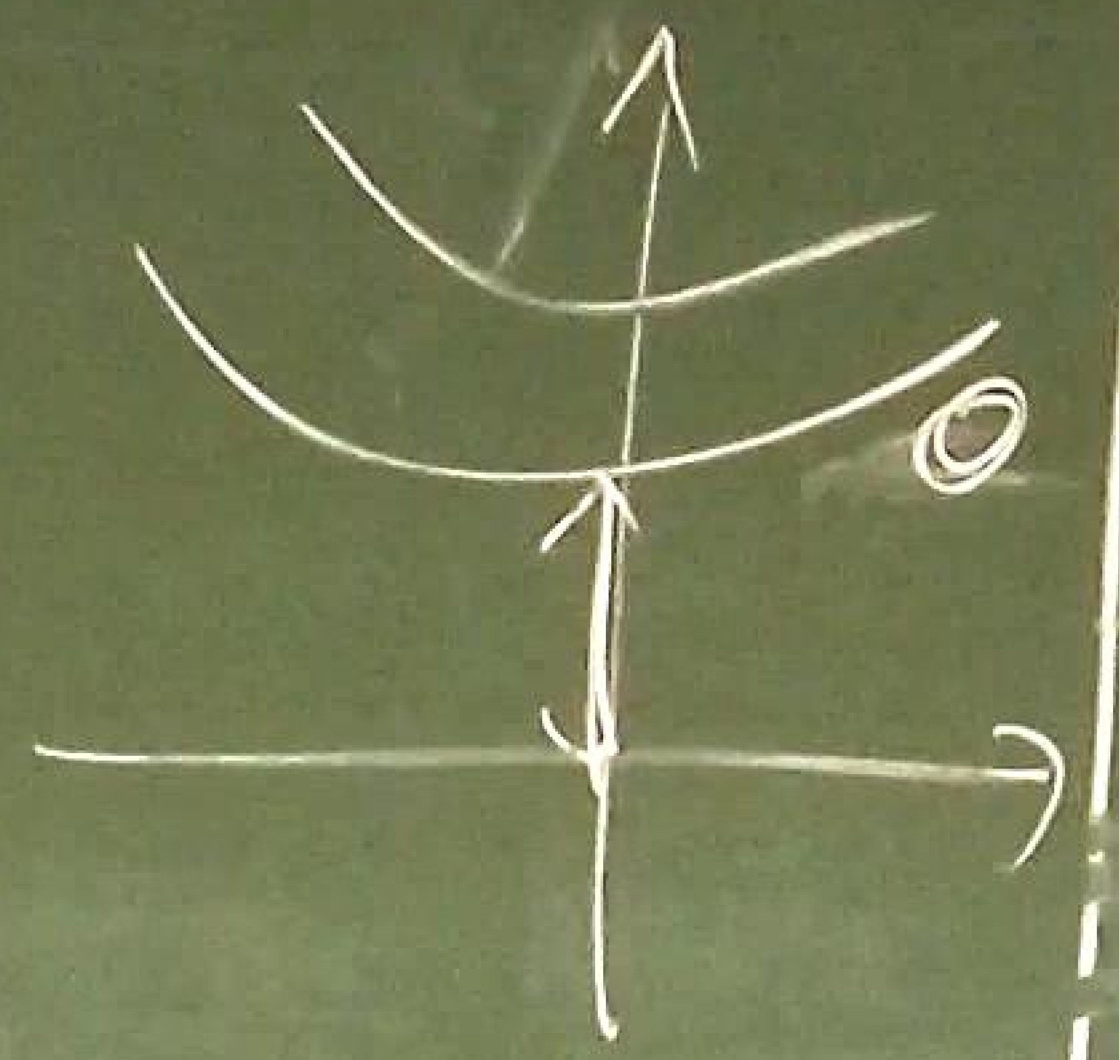
Gauge bosons = Gluons ($8 \times A^a_\mu$)

→ Pure gluon vertices:

$(\partial A)AA \;\hat{=}$

$A^4 \;\hat{=}$





→ Bound states of gluons

→ Glueballs

9|

$$\boxed{\mathcal{L}_{YMD} = \overline{\Psi}(i\cancel{D} - m)\Psi - \frac{1}{4}(F^a_{\mu\nu})^2}$$

Two parameters:

- m: mass of Fermions
- g: coupling constant

- gauge invariant
- Lorentz invariant
- renormalizable
- P- and T-symmetric

Note 9.2

- The mass term A^2 is not allowed because it is not gauge invariant

→ Gauge bosons of YM are massless

Problem: Weak interaction is short ranged

⇒ gauge bosons $W^\pm, Z$ are massiv

Solution: Higgs mechanism

10. Excursion

10.1 The Higgs Mechanism

1| $\mathcal{L}$ Maxwell theory

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + |D_\mu\phi|^2 - V(\phi)$$

ϕ ↑ complex scalar field

- $V(\phi) = \mu^2|\phi|^2 + \lambda|\phi|^4$
- $D_\mu = \partial_\mu + ieA_\mu$

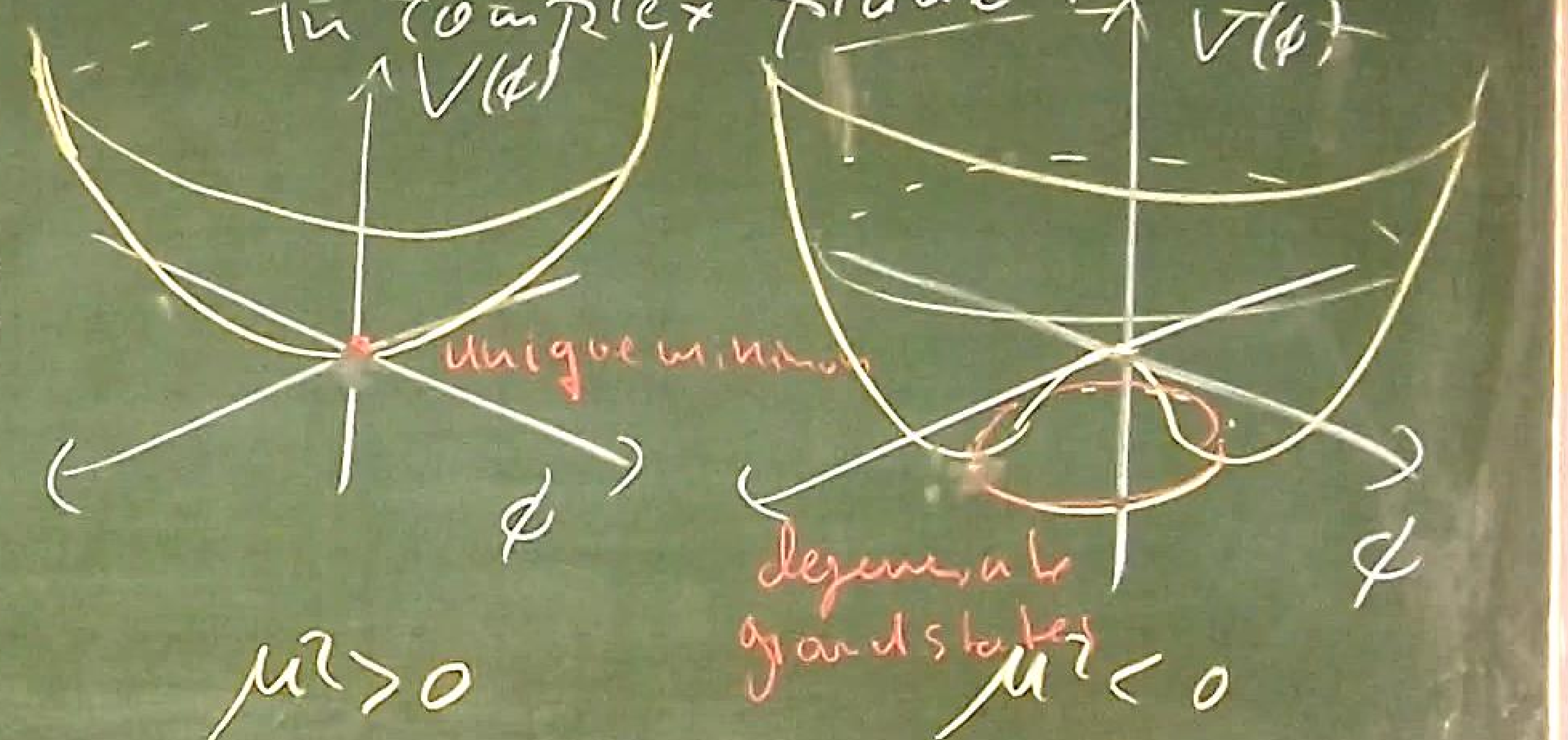
2| U(1) gauge transformations:

$$\tilde{\phi}(x) = e^{i\alpha(x)}\phi(x)$$

$$\tilde{A}_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x)$$

3| $\mathcal{L}$ $V(\phi)$ for $\lambda > 0$

in complex plane:



• $\mu^2 < 0$: Mexican hat potential:

Degenerate minima with non-zero vacuum expectation vale (VEV)

$$\phi_0 := \langle \phi \rangle \quad \text{and} \quad v = |\phi_0| = \sqrt{\frac{-\mu^2}{2\lambda}} \neq 0$$

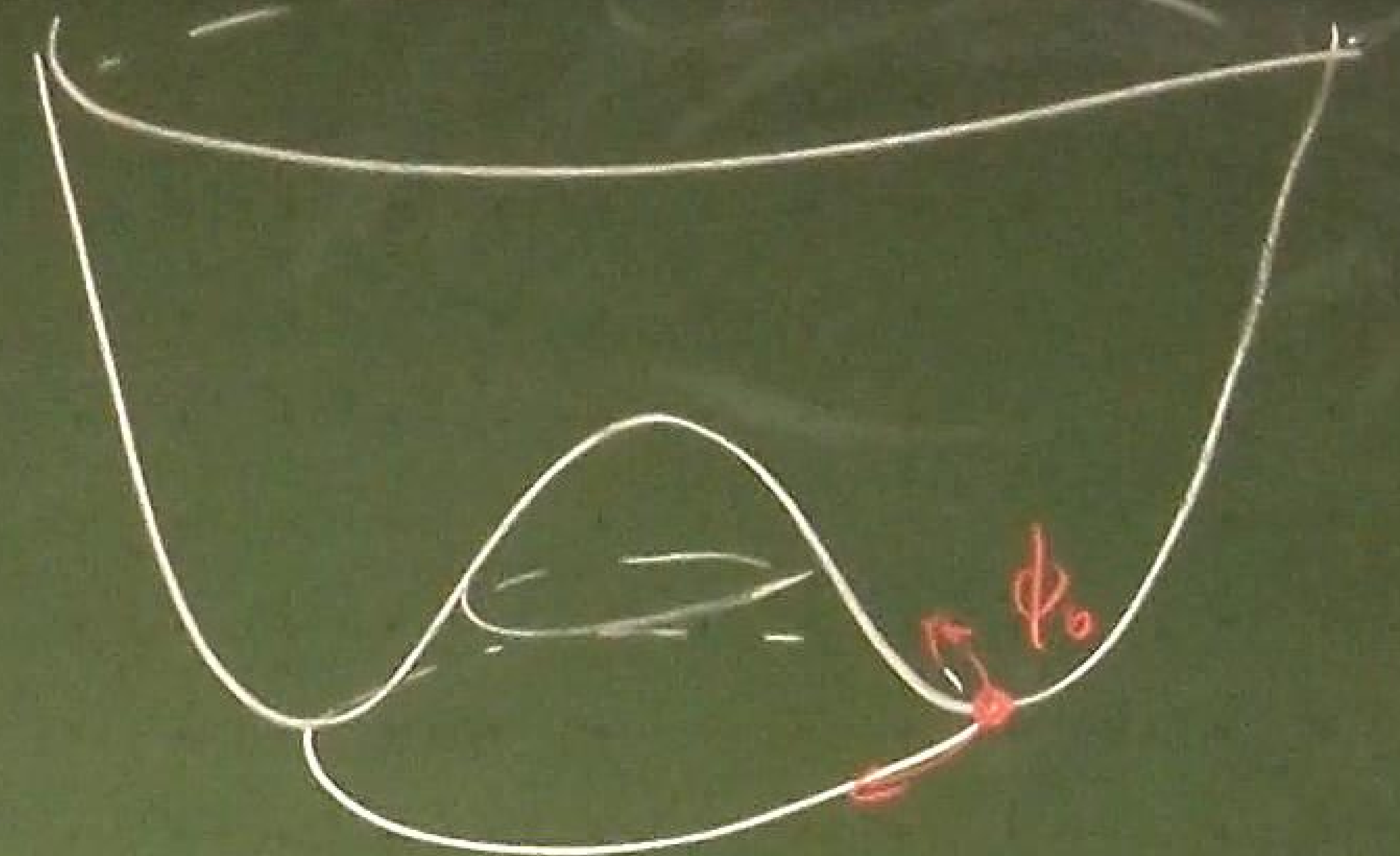
→ Ground states are not symmetrix under global phase rotations

→ Spontaneous Symmetry Breaking (SSB)

4) Aside: The Goldstone theorem

If a global, continuous symmetry is spontaneously broken, there is an massless scalar (spin-0) particle for each broken symmetry generator; these are called (Nambu-) Goldstone bosons

Proof by picture:



Example:

- Breaking translation invariance in crystals → Transversal and longitudinal phonons

Recap:

0.1 The Higgs Mechanism

Complex scalar field ϕ + Maxwell:

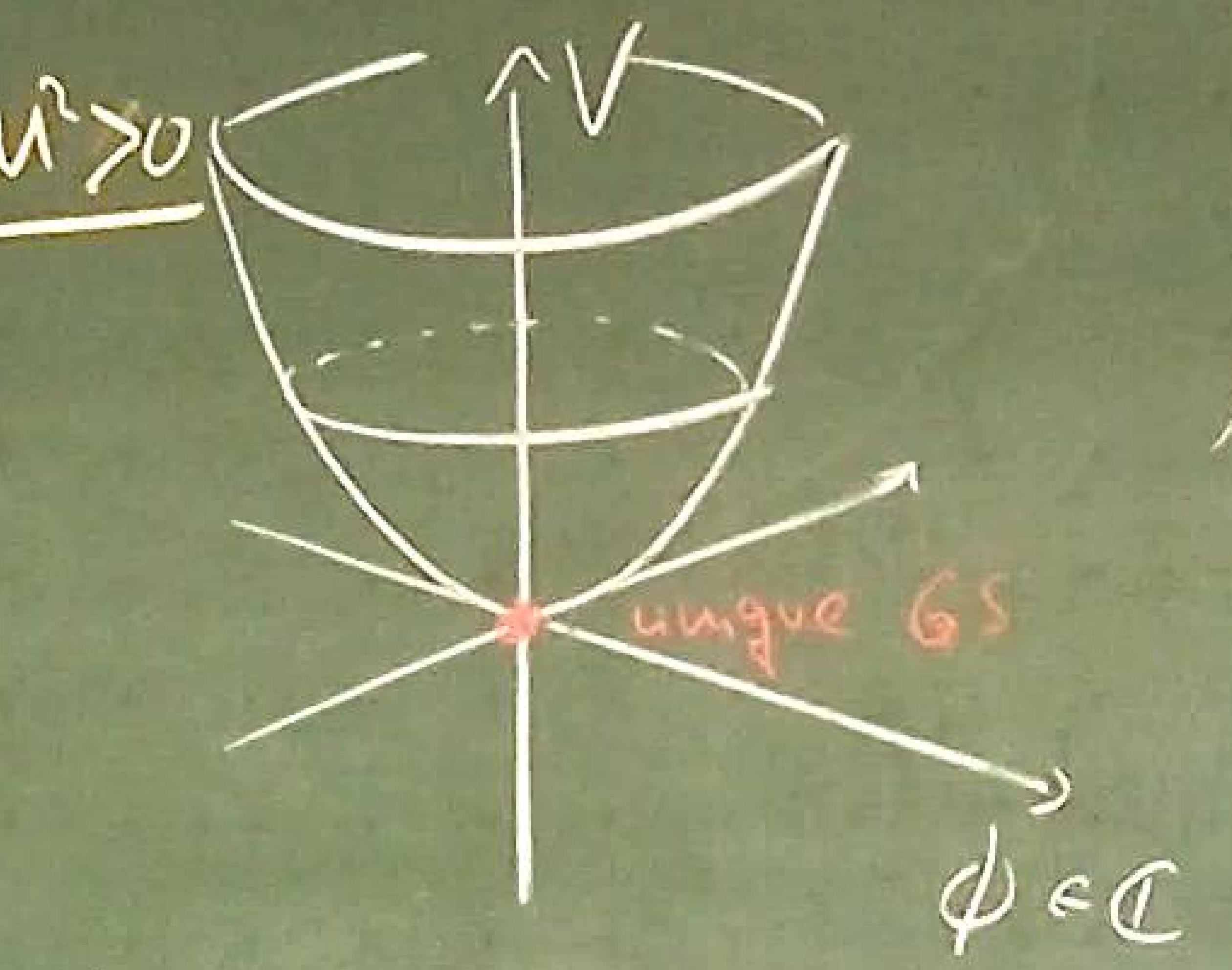
$$\boxed{\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + |D_\mu \phi|^2 - V(\phi)}$$

- $V(\phi) = \mu^2 |\phi|^2 + \lambda |\phi|^4$
- $D_\mu = \partial_\mu + ieA_\mu$

2) U(1) gauge transformations:

$$\tilde{\phi}(x) = e^{i\alpha(x)}\phi(x), \quad \hat{A}_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$$

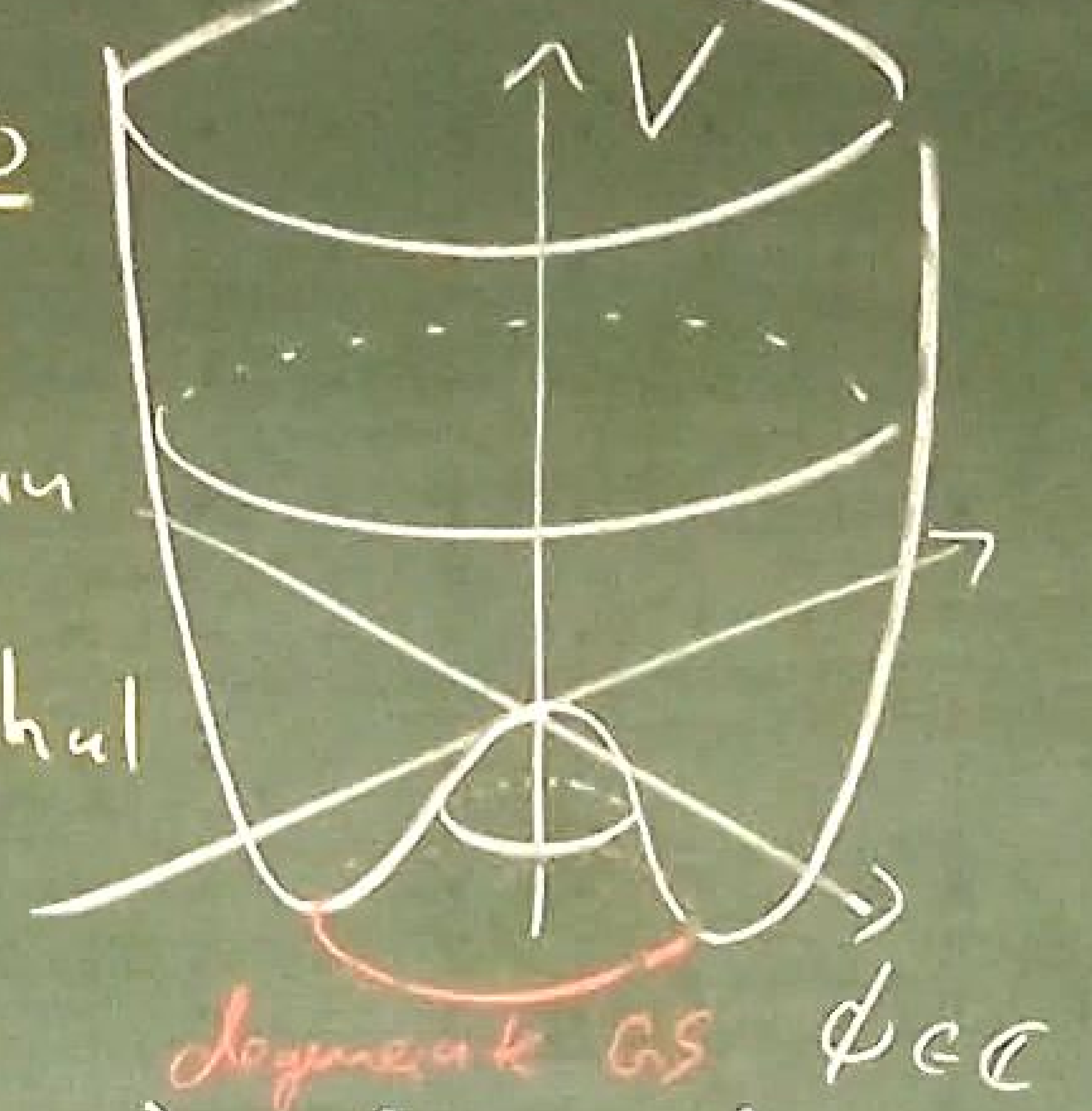
3) Spontaneous U(1) Symmetry Breaking:



$\mu^2 < 0$

$\Downarrow$

Mexican hat potential



Vacuum expectation value (VEV) for $\mu^2 < 0$.

$$\phi_0 = \langle \phi \rangle \text{ with } v = |\phi_0| = \sqrt{\frac{-\mu^2}{2\lambda}} \neq 0$$

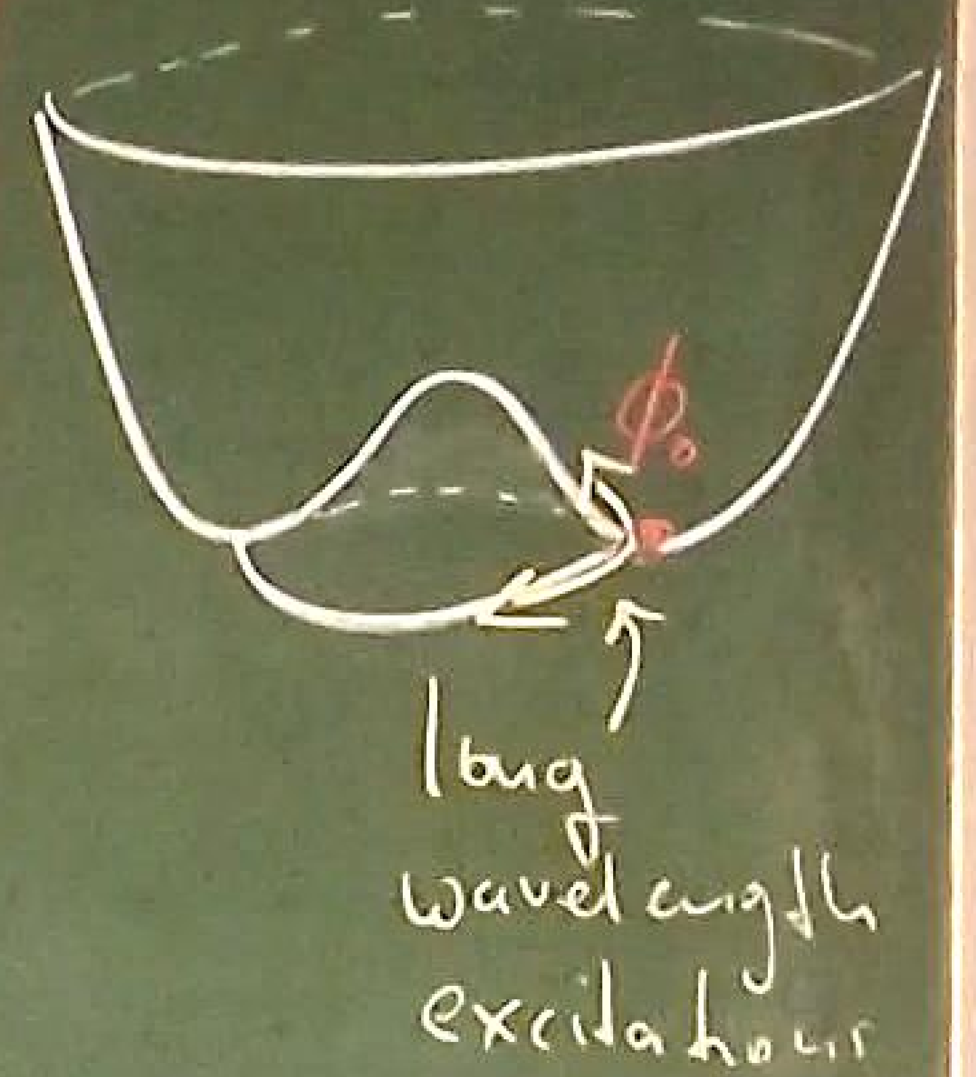
picks a random phase

4) Goldstone Theorem:

Spontaneous breaking of global + continuous symmetry

$\Downarrow$

Massless Spin-0 excitation (Nambu-) Goldstone Boson (one for each broken generator)

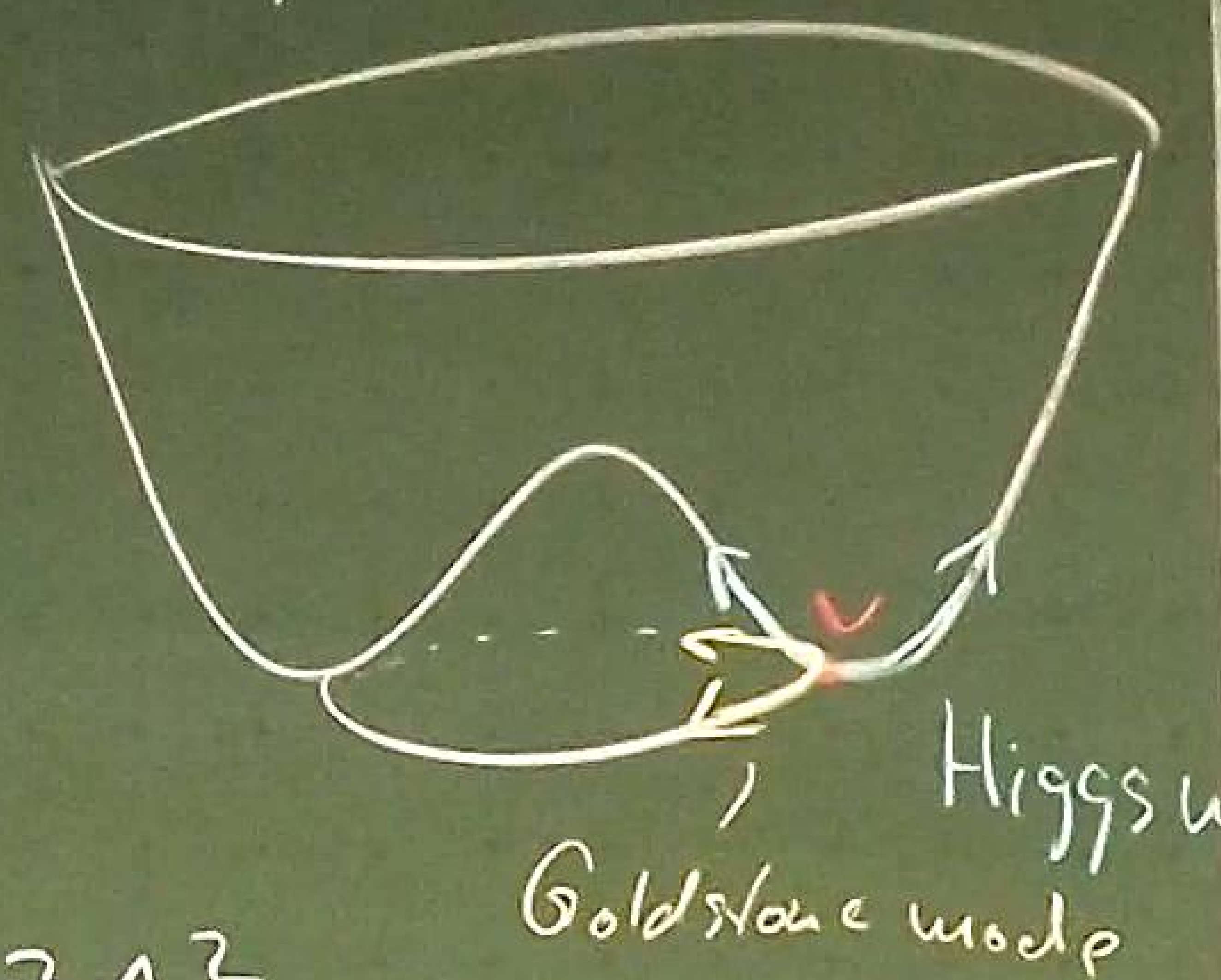


How can the Goldstone Theorem fail? Example: Superconductivity (broken U(1) symmetry, no Goldstone Mode)

5) $\phi_0 = \langle \phi \rangle = v$ breaks global $U(1)$ symmetry.

$$\phi(x) = (v + \underline{h(x)})\, e^{i\underline{\varphi(x)}}$$

$$\downarrow$$



$$\mathcal{L} = \underbrace{-\frac{1}{4}(F_{\mu\nu})^2 + e^2 v^2 A_\mu^2}_{\text{Massive gauge field}}$$

$$+ \underbrace{(\partial_\mu h)^2 - \overset{4\lambda v^2}{m_h^2} h^2}_{\text{Higgs field}}$$

$$+ \underbrace{v^2(\partial_\mu \varphi)^2}_{\text{Massless Goldstone mode}} + \underbrace{2ev^2(\partial_\mu \varphi)A^\mu}_{\text{Quadratic coupling}} + \text{Interactions}$$

Use: $-\mu^2 = 2\lambda v^2$

Note: $\mathcal{L}$ still gauge invariant:

- $\tilde{\varphi} = \varphi + \alpha$
- $\tilde{h} = h$
- $\tilde{A}_\mu = A_\mu - \frac{1}{e}\partial_\mu \alpha$

6) Fix gauge in unitary gauge:

$$\phi \overset{!}{=} \phi^* \iff \varphi \equiv 0$$

⌜ Gauge trafo. $\alpha(x) = -\varphi(x)$

$$\tilde{\phi} = \phi\, e^{i\alpha} \in \mathbb{R}$$ ⌟

$$\rightarrow \boxed{\tilde{\mathcal{L}} = \underbrace{-\frac{1}{4}(\tilde{F}_{\mu\nu})^2 + e^2 v^2 \tilde{A}_\mu^2}_{\text{Massive gauge field}} + \underbrace{(\partial \tilde{h})^2 - m_h^2 \tilde{h}^2}_{\text{Massive Higgs field}} + \text{Interactions}}$$

→ Goldstone mod φ disappeared!

Reason: φ is pure gauge d.o.f. and therefore unphysical!

7) Consistency checks:

#(DOF) before SSB

= 2 (massless vector boson)

+ 2 (complex scalar field) = $\underline{\underline{4}}$

#(DOF) after SSB

= 3 (massive vector boson)

+ 1 (real scalar Higgs field) = $\underline{\underline{4}}$

10.2. The Standard Model

10.2.1. Preliminaries

1) Chiral projectors:

$$P_R = \frac{1}{2}(\mathbb{1}_4 + \gamma^5) \overset{\text{Weyl}}{=} \begin{pmatrix} \mathbb{1}_2 & \\ & 0 \end{pmatrix}$$

$$P_L = \frac{1}{2}(\mathbb{1}_4 - \gamma^5) \overset{\text{Weyl}}{=} \begin{pmatrix} 0 & \\ & \mathbb{1}_2 \end{pmatrix}$$

Chiral fermion fields:

$$\psi_R := P_R \psi, \quad \psi_L = P_L \psi$$

$$\psi = \underbrace{(P_R + P_L)}_{\mathbb{1}} \psi = \psi_R + \psi_L$$

2) Use $\bar{\psi} P_{\substack{R\\L}} = \overline{\psi_{\substack{L\\R}}}$ to show:

$$\left.\begin{aligned}\bar{\psi}(i\partial\!\!\!/ - m)\psi \overset{\circ}{=} & \ \overline{\psi_R}\, i\partial\!\!\!/\, \psi_R + \overline{\psi_L}(i\partial\!\!\!/)\psi_L \\ & - m\,\overline{\psi_L}\psi_R - m\,\overline{\psi_R}\psi_L\end{aligned}\right\} (*)$$

3)

$$[P_{R/L}, \Lambda_{\frac{1}{2}}] = 0$$

$$\sim [\gamma^\mu, \gamma^\nu]$$

SO$^+(1,3)$

→ Terms in (*) are Lorentz invariant

10.2.2. Overview

1) Field content:

- **Fermions** (= Spin-$\frac{1}{2}$)

Generation	I	II	III
Leptons	e_L e_R ν_{eL} (ν_{eR})	μ_L μ_R $\nu_{\mu L}$ ($\nu_{\mu R}$)	τ_L τ_R $\nu_{\tau L}$ ($\nu_{\tau R}$)
Quarks	u_L u_R d_L d_R	c_L c_R s_L s_R	t_L t_R b_L b_R

- **Vector bosons** (= Spin 1)

Force	Electroweak	Strong
Gauge group	$SU(2)_L \times U(1)_Y$	$SU(3)_C$
# Generators	$3 + 1 = 4$	8
Gauge fields	$\underbrace{W_\mu^i \ (i=1,2,3) \quad B_\mu}_{\text{Before SSB}}$	$G_\mu^a \ (a=1 \dots 8)$
Gauge bosons	$\underbrace{\gamma, W^+, W^-, Z}_{\text{After SSB}}$	8 Gluons

- **Scalar boson** (Spin 0)

$$\boxed{\begin{array}{c} \overbrace{2\times \underline{\text{Complex}} \text{ Higgs fields: } \phi^+, \phi^0}^{\text{Before Higgs SSB}} \\ \Big\downarrow \; 3\times \text{SSB} \\ \underbrace{1\times \underline{\text{Real}} \text{ Higgs field } h}_{\text{After Higgs SSB}} \end{array}}$$

How to build a consistent QFT?

$$\mathcal{L}_{SM} = \mathcal{L}_{EWS} + \mathcal{L}_{QCD}$$

$$\left(+ \mathcal{L}_{GF} + \mathcal{L}_{Ghost} \right)$$

$$\uparrow \qquad\qquad \uparrow$$

$$-\frac{(\partial_\mu A)^2}{2\xi} \qquad \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right)$$

10.2.3 The Glashow-Weinberg-Salam Theory

Goal: Generalize the Higgs mechanism to the Standard model

1) $$\mathcal{L}_{EWS} = \mathcal{L}_{Fermion} + \mathcal{L}_{Yang\text{-}Mills} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yukawa}$$

2) Gauge symmetry:

$$\underbrace{SU(2)_L}_{\text{Weak isospin}} \times \underbrace{U(1)_Y}_{\text{Weak hypercharge}}$$

- $SU(2)_L \rightarrow 3$ generators

$$T^i \quad i = 1,2,3$$

$$[T^i, T^j] = i\,\varepsilon^{ijk}\, T^k$$

$\rightarrow$ Irreducible representations:

- 1D: Trivial rep $\hat{T}^i = 0$
  (singlet rep)
- 2D: Pauli matrices $\hat{T}^i = \frac{\sigma^i}{2}$
  (doublet rep)

in the following

$\rightarrow$ Eigenvalues of $\hat{T}^3$

= The weak isospin T^3

$$\begin{cases} T^3 = \pm\frac{1}{2} & \text{doublet} \\ T^3 = 0 & \text{singlet} \end{cases}$$

- $U(1)_Y \rightarrow 1$ generator Y

$[Y, T^i] = 0$

Schur's lemma $\rightarrow$ $\hat{Y} = \text{Hypercharge } Y \times \mathbb{1}$

Recap:

10.2. The Standard Model

0.21. Preliminaries

hiral projectors: $P_R = \frac{1}{2}(1+\gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & 1\!\!1_2 \end{pmatrix}$

hiral fermion fields: $\psi_R = P_R \psi$

$\psi_L = P_L \psi$

Dirac Lagrangian:

$$\bar{\Psi}(i\partial\!\!\!/ - m)\psi = \underset{\uparrow}{\bar{\psi}_L i\partial\!\!\!/ \psi_L} + \bar{\psi}_R i\partial\!\!\!/ \psi_R - \underset{\uparrow}{m}\bar{\psi}_L\psi_R - m\bar{\psi}_R\psi_L$$

all terms $SO^+(1,3)$ inv. mixes L and R

10.22. Overview

Fields:

Generations

	I	II	III	
Fermions:	e_L e_R ν_{eL} (ν_{eR})	μ_L μ_R $\nu_{\mu L}$ $(\nu_{\mu R})$	τ_L τ_R $\nu_{\tau L}$ $(\nu_{\tau R})$	Leptons
Flavours	u_L u_R d_L d_R	c_L c_R s_L s_R	t_L t_R b_L b_R	Quarks

Vector bosons (Spin 1):

Gauge fields

Electroweak force: $\underset{\uparrow 3}{SU(2)_L} \times \underset{\uparrow 1}{U(1)_Y} \rightarrow \begin{cases} W^i_\mu & (i=1,2,3) \\ B_\mu \end{cases}$ 3 + 1 generators

Strong force: $SU(3)_c \rightarrow G^a_\mu \quad (a=1, \ldots, 8)$

8 generators

Scalar bosons (Spin 0):

Higgs fields: $\underbrace{\phi^+, \phi^0}_{4 \text{ real d.o.f.}} \in \mathbb{C}$

Lagrangian:

$$\boxed{\mathcal{L}_{SM} = \mathcal{L}_{EWS} + \mathcal{L}_{QCD}}$$

← ignoring ghosts!

Electroweak Unification (QED, Weak force, Higgs)

⇓

Glashow-Weinberg-Salam theory

Quantum Chromodynamics (strong force)

0.2.3. The Glashow-Weinberg-Salam Theory

1] Lagrangian:

$$\boxed{\mathcal{L}_{EWS} = \mathcal{L}_{\text{Fermion}} + \mathcal{L}_{\text{Yang-Mills}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}}}$$

2] Gauge symmetry:

$$\text{Weak isospin} \rightarrow \boxed{SU(2)_L \times U(1)_Y} \leftarrow \text{Weak hypercharge}$$

Lie A.: $[T^i, T^j] = i\varepsilon^{ijk} T^k$, $[Y, T^i] = 0$

Irreps:
- $\hat{T}^i = 0$ (1D: Singlet)
- $\hat{T}^i = \frac{\sigma^i}{2}$ (2D: Doublet)

$\hat{Y} = Y \cdot \mathbb{1}$

Hypercharge ($\mathbb{R}$)

EV of $\hat{T}^3$: Weak isospin ($\frac{1}{2}\mathbb{N}$)

3] $SU(2)_L$ Representations:

- Left-handed field = Isospin doublets

$$\Psi_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} t_L \\ b_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}$$

→ Weak isospin: $T^3(\nu_{eL}) = +\frac{1}{2}$

$T^3(e_L) = -\frac{1}{2}$

$$\nu_{eL}(x) = \underbrace{\psi_L(x)} \otimes \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$\in L^2(\mathbb{R}^{1,3}) \otimes \mathbb{C}^4 \otimes \mathbb{C}^2$$

$$T^3(\nu_{eL}) = +\frac{1}{2} \Leftrightarrow \hat{T}^3 \nu_{eL}(x) = \psi_L(x) \frac{\sigma^3}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underline{+\frac{1}{2}} \nu_{eL}(x)$$

- Right-handed field = Isospin singlet

$\psi_R = u_R, d_R, e_R, c_R, s_R, \mu_R, t_R, b_R, \tau_R$

→ $T^3(e_R) = 0$

- Higgs field = Isospin doublet:

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \rightarrow T^3(\phi^+) = +\frac{1}{2}$$

$T^3(\phi^0) = -\frac{1}{2}$

Gauge transformations on fields:

LH doublets: $\tilde{\Psi}_L = e^{i\hat{Y}_L\alpha(x)} \underbrace{e^{i\hat{T}^i\beta^i(x)}}_{\to V_L} \Psi_L$

RH singlets: $\tilde{\psi}_R = e^{i\hat{Y}_R\alpha(x)}\psi_R$

Higgs doublet: $\tilde{\Phi} = e^{i\hat{Y}_H\alpha(x)} e^{i\hat{T}^i\beta^i(x)}\Phi$

$\hat{Y}_H = Y\cdot\mathbb{1}$

↑ not yet fixed

4] Kinetic energy for fermions + Minimal coupling:

$$\boxed{\mathcal{L}_{\text{Fermion}} = \sum_{\Psi_L} \bar{\Psi}_L(i\not{D}_L)\Psi_L + \sum_{\psi_R}\bar{\psi}_R(i\not{D}_R)\psi_R}$$

- $D_{L\mu} = \partial_\mu - ig W_\mu^i \hat{T}^i - ig' B_\mu \hat{Y}_L$

↖ coupling constants ↑

$D_{R\mu} = \partial_\mu \quad \underset{\text{singlet}}{\overset{0}{\nearrow}} \quad - ig' B_\mu \hat{Y}_R$

→ Transformation of gauge fields:

$\tilde{B}_\mu = B_\mu + \frac{1}{g'}\partial_\mu\alpha$, $\quad \overset{W_\mu = W_\mu^i\hat{T}^i}{\tilde{W}_\mu} = V_L\left[W_\mu + \frac{i}{g}\partial_\mu\right]V_L^\dagger$

Example:

$$= (\bar{u}_L\,\bar{d}_L)(i\not{D}_L)\begin{pmatrix}u_L\\ d_L\end{pmatrix} + (\bar{\nu}_{eL}\,\bar{e}_L)(i\not{D}_L)\begin{pmatrix}\nu_{eL}\\ e_L\end{pmatrix} + \dots \; ⊜$$

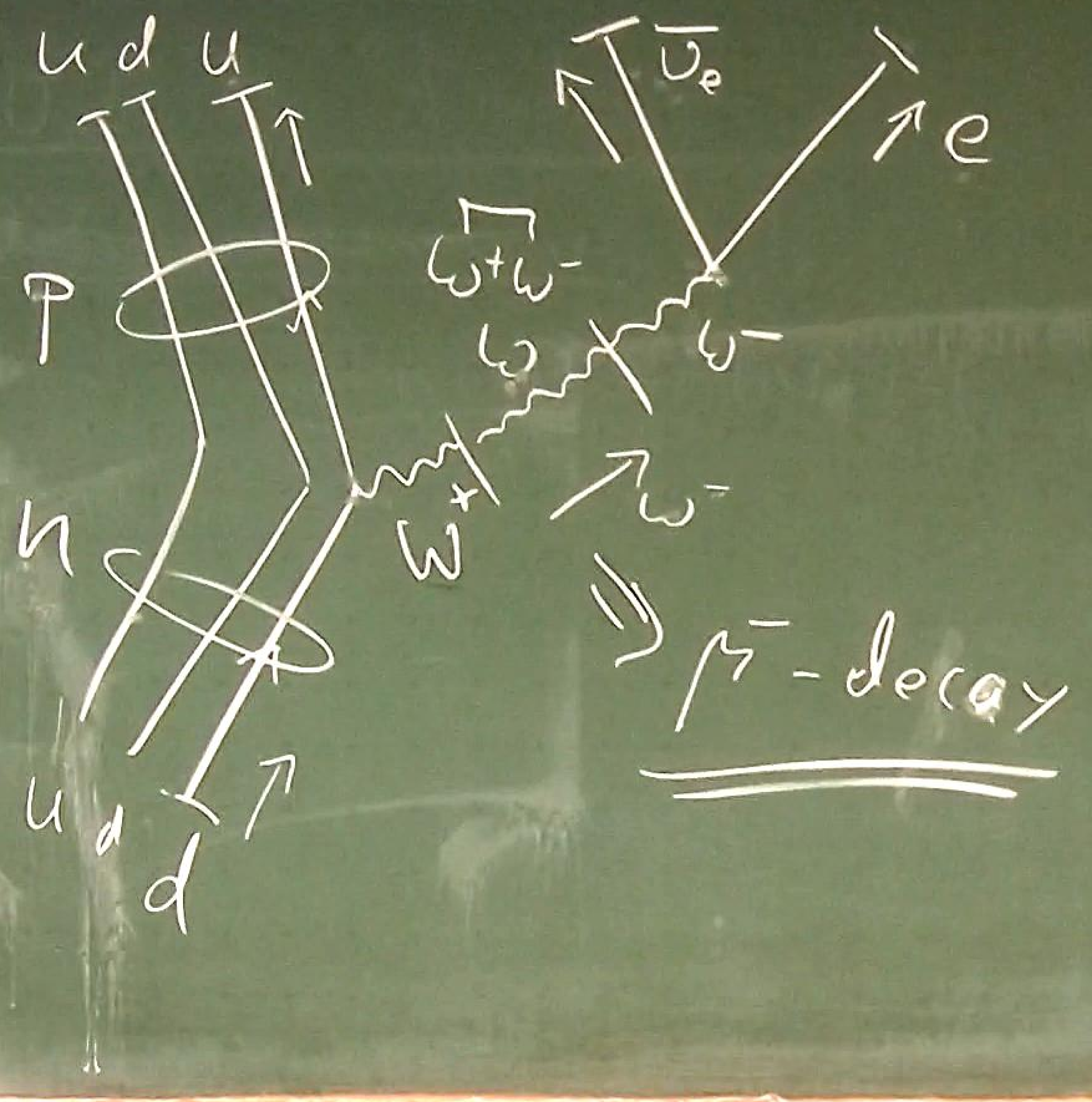
- $D_{L\mu} = -ig(W_\mu^1\hat{T}^1 + W_\mu^2\hat{T}^2) + \dots$

$$= -i\frac{g}{\sqrt{2}}\begin{pmatrix}0 & W_\mu^+\\ W_\mu^- & 0\end{pmatrix} + \dots$$

$$W_\mu^\pm := \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2)$$

$$⊜\; W_\mu^+\bar{u}_L\gamma^\mu d_L + W_\mu^-\bar{e}_L\gamma^\mu\nu_{eL} + h.c. + \dots$$

$$\int a + b^\dagger$$



$\Rightarrow$ β^--decay

5| Dirac mass?

$m(\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L) \rightarrow$ Undefined!

$\rightarrow$ Not $SU(2)_L$ gauge invariant

because X_L is component of doublet

but Y_R is $SU(2)$ singlet.

$Ep^2 \rightarrow$ not Lorentz invariant

$\rightarrow$ Cannot add mass term for fermions!

6| Kinetic energy for gauge bosons:

$\rightarrow$ Yang-Mills-Lagrangian

$$\boxed{\mathcal{L}_{\text{Yang-Mills}} = -\frac{1}{4}(B^{\mu\nu})^2 - \frac{1}{4}(W^i_{\mu\nu})^2}$$

- $B^{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$
- $W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + \underbrace{g\,\varepsilon^{ijk} W^j_\mu W^k_\nu}_{\text{interaction between gauge bosons.}}$

] Higgs field:

$$ {}_{\text{Higgs}} = (D^\mu_H \Phi)^\dagger (D_{H\mu} \Phi) - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2 \quad (*)$$

$$D_{H\mu} = \partial_\mu - i g W^i_\mu \hat{T}^i - i g' B_\mu \hat{Y}_H$$

$$(\Phi^\dagger \Phi)^2 \neq |\phi^+|^4 + |\phi^0|^4$$

8] Higgs mechanism (Part 1).

i] $\mu^2 < 0 \rightarrow$ Non-zero VEV of Higgs field

Wlog. $\langle \Phi \rangle = \Phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$ with $v = \sqrt{\frac{-\mu^2}{\lambda}}$

ii] Define electric charge operator.

$$\boxed{Q = T^3 + Y \in su(2)_L \oplus u(1)_Y}$$

→ Choose: $Y(\Phi) = +\frac{1}{2}$

$$\hat{Q} \Phi_0 = \underbrace{\hat{T}^3 \Phi_0}_{-\frac{1}{2}\phi_0} + \underbrace{\hat{Y} \Phi_0}_{\frac{1}{2}\Phi_0} = \left(-\frac{1}{2} + \frac{1}{2}\right) \Phi_0 = 0$$

$\Rightarrow e^{i \hat{Q} \alpha(x)} \Phi_0 = \Phi_0$

→ Gauge symmetry $U(1)_Q$ is unbroken:

$$\boxed{SU(2)_L \times U(1)_Y \xrightarrow{3 \times SSB} \underbrace{U(1)_Q}_{}}$$

Unbroken gauge symmetry of QED

iii] $\not\alpha$ Fluctuations of Φ around Φ_0 in the unitary gauge.

$$\Phi(x) \underset{\uparrow}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

unitary gauge

real scalar Higgs field

V] $\Phi(x)$ in $(*)$.

$$(D_H^\mu \Phi^\dagger)(D_{H\mu}\Phi) \overset{\circ}{=} \frac{v^2}{8}\left\{ g^2\left[(W_\mu^1)^2+(W_\mu^2)^2\right] + (-gW_\mu^3+g'B_\mu)^2\right\}+\dots$$

] Define new fields

$$\boxed{\begin{aligned} W_\mu^\pm &:= \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp i W_\mu^2\right) \\ Z_\mu &:= \frac{1}{\sqrt{g^2+g'^2}}\left(gW_\mu^3 - g'B_\mu\right) \\ A_\mu &:= \frac{1}{\sqrt{g^2+g'^2}}\left(g'W_\mu^3 + gB_\mu\right) \end{aligned}}$$

$$= \underbrace{\left(\frac{gv^2}{2}\right)^2}_{m_W^2} W_\mu^+ W^{-\mu} + \frac{1}{2}\underbrace{\left(\frac{v}{2}\right)^2(g^2+g'^2)}_{m_Z^2} Z_\mu^2 + \dots$$

- A_μ: massless neutral gauge field of QED
- $W_\mu^\pm$: massive charged gauge bosons
- Z_μ: massive neutral gauge bosons

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ weak interaction

$$\boxed{D_{H\mu} = \partial_\mu - (\dots) - i\underbrace{\boxed{\frac{gg'}{\sqrt{g^2+g'^2}}}}_{e \text{ electric charge}} A_\mu Q}$$

g]

- $Y(e_L) = Q(e_L) - T^3(e_L)$
  $= \underbrace{\boxed{-1}}_{\text{observation}} - \left(-\frac{1}{2}\right) = \underline{\underline{-\frac{1}{2}}}$
- $Y(e_R) = Q(e_R) - T^3(e_R)$
  $= -1 - 0 = \underline{\underline{-1}}$

Recap: $\boxed{\mathcal{L}_{SM} = \mathcal{L}_{EWS} + \mathcal{L}_{QCD}}$

10.2.3. The Glashow Weinberg Salam Theory

$$\boxed{\mathcal{L}_{EWS} = \mathcal{L}_{\text{Fermion}} + \mathcal{L}_{\text{Yang-Mills}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}}}$$

$\mathcal{L}_{\text{Fermion}}$ ↓ Kinetic energy of fermions + Electroweak interaction (e.g. β-decay)

$\mathcal{L}_{\text{Yang-Mills}}$ ↓ Kinetic energy of gauge bosons + Interactions between gauge bosons

$\mathcal{L}_{\text{Higgs}}$ ↓ SSB of Higgs field +Masses of $W^\pm, Z$

$\mathcal{L}_{\text{Yukawa}}$ ↑ today

Remember:

- $\Psi_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \dots\dots$ Isospin doublets
- $\psi_R = u_R, d_R, e_R, \dots$ Isospin singlets
- $\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ Isospin doublet

↓ Higgs mechanism

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

VEV ↗ ↖ real, scalar Higgs field

- Electric charge:

$$\boxed{SU(2)_L \times U(1)_Y \xrightarrow{7x\ SSB} U(1)_Q}$$

with $\boxed{Q = T^3 + Y}$ $\in su(2)_L \oplus u(1)_Y$

⇓

fix Hyperchange.

$$Y(e_L) = Q(e_L) - T^3(e_L) = -1 - \left(\tfrac{1}{2}\right) = -\tfrac{1}{2}$$

$$Y(e_R) = Q(e_R) - T^3(e_R) = -1 - 0 = -1$$

$$Y(\phi) \underset{\uparrow}{=} \frac{1}{2}$$

chosen

10| Higgs mechanism Part II:

Masses for the fermions

i| How to form gauge invariant terms include left- and right handed fermions?

$$\boxed{-\gamma_e \cdot \underset{\vdots}{\bar\Psi_L} \cdot \Phi \; e_R + h.c.} \quad (*) \qquad \Psi_L = \begin{pmatrix}\nu_{eL}\\ e_L\end{pmatrix}$$

$$(\bar\nu_{eL} \; \bar e_L) \cdot \begin{pmatrix}\phi^+\\ \phi_0\end{pmatrix} e_R$$

- $SU(2)_L$ invariant

$$Y(\Phi) + Y(e_R) - Y(\Psi_L) = \frac{1}{2} - 1 - \left(-\frac{1}{2}\right) = 0 \Rightarrow U(1)_Y \text{ invariant}$$

- $$(\bar\Psi_L \Phi)\, e_R = \underset{\displaystyle 0}{\underset{\downarrow}{\phi^+}} \bar\nu_{eL} e_R + \underset{\displaystyle \frac{1}{\sqrt{2}}(v + h(x))}{\underset{\downarrow}{\phi_0}} \bar e_L e_R$$

Higgs SSB

$$(*) = \boxed{-\underset{\downarrow}{\boxed{\gamma_e \frac{v}{\sqrt{2}}}} (\bar e_L e_R + \bar e_R e_L)}$$

Dirac mass

$$m_e = \gamma_e \frac{v}{\sqrt{2}}$$

Yukawa interaction: $\bar\Psi \psi \phi$

$$\bar\Psi \gamma^\mu \psi A_\mu$$

ii|

$$\boxed{\mathcal{L}_{Yukawa} = -\Gamma^u_{mn} \bar Q^m_L \hat\Phi \underset{x}{u^n_R} - \Gamma^d_{mn} \bar Q^m_L \Phi d^n_R - \Gamma^l_{mn} \bar L^m_L \Phi l^n_R}$$

- $m, n \in I, II, III$: fermion generations (summation implied)
- $x \in \{u, d, l, \nu\}$: fermion types

$l^I_R = e_R$, $l^{II}_R = \mu_R$

$u^I_R = u_R$, $u^{II}_R = c_R$

$$\boxed{\begin{aligned} {}_{\text{Yukawa}} = & -\Gamma^u_{mn}\,\bar{Q}^m_L\,\hat{\Phi}\,u^n_R - \Gamma^d_{mn}\,\bar{Q}^m_L\,\Phi\,d^n_R \\ & -\Gamma^\ell_{mn}\,\bar{L}^m_L\,\Phi\,\ell^n_R - \Gamma^\nu_{mn}\,\bar{L}^m_L\,\hat{\Phi}\,\nu^n_R + h.c. \end{aligned}}$$

→ Neutrino masses

Γ^x_{mn}: coupling constants $(\Gamma^\ell_{I,I} = \gamma_e)$

Q^m_L, L^m_L: left-handed quark resp. lepton doublets of gen. m

$$\Gamma \quad L^I_L = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \qquad \bar{\nu}_{eL}\,\nu_{eR}$$

· $\hat{\Phi}_i \equiv \varepsilon^{ij}\,\Phi^*_j$ Higgs doublet with opposite Hypercharge. $Y(\hat{\Phi}) = -\frac{1}{2}$

---

iii] · **Yukawa term** generates fermion masses (but not for neutrinos if right-handed neutrinos are missing)

- leads to generation changing transitions of quarks (CKM matrix)
- Neutrino oscillations (if ν_R exist) (PNMS matrix)

10.2.4 Quantum Chromodynamics

1] Gauge Symmetry

$$\boxed{SU(3)_c}$$

color charge

→ 8 generators K^a $\quad a = 1 \dots 8$

$$[K^a, K^b] = i f^{abc} K^c$$

→ Irreducible representations:

- 1D: Trivial (Singlet). $\hat{K}^a = 0$
- 3D: Defining (Triplet)

$$\hat{\kappa}^a = \frac{\lambda_a}{2}$$

wit λ_a 3x3 Hermitian matrices
(Gell-Mann matrices)

2| Field representations:

- Quarks = $SU(3)_c$ triplets "colors"

$$q = \begin{pmatrix} q_r \\ q_g \\ q_b \end{pmatrix}$$ red green blue for $q \in \{u, d, c, s, t, b\}$

- Leptons + Higgs = $SU(3)_c$ singlets

→ Gauge transformation of fields.

$$\boxed{\text{Quark triplet} \quad \tilde{q} = \underbrace{e^{i\hat{\kappa}^a \beta^a(x)}}_{U_c(x)} q}$$

3| Lagrangian

$$\boxed{\begin{aligned} \mathcal{L}_{QCD} &= \sum_q \bar{q}(i\not{D}_c)q - \frac{1}{4}(G^a_{\mu\nu})^2 \\ D_{c\mu} &= \partial_\mu - i g_s G^a_\mu \hat{\kappa}^a \end{aligned}}$$

g_s: coupling constant of strong force

G^a_μ: 8 gauge fields → 8 gauge bosons
→ 8 Gluons

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu$$

Renormalization:

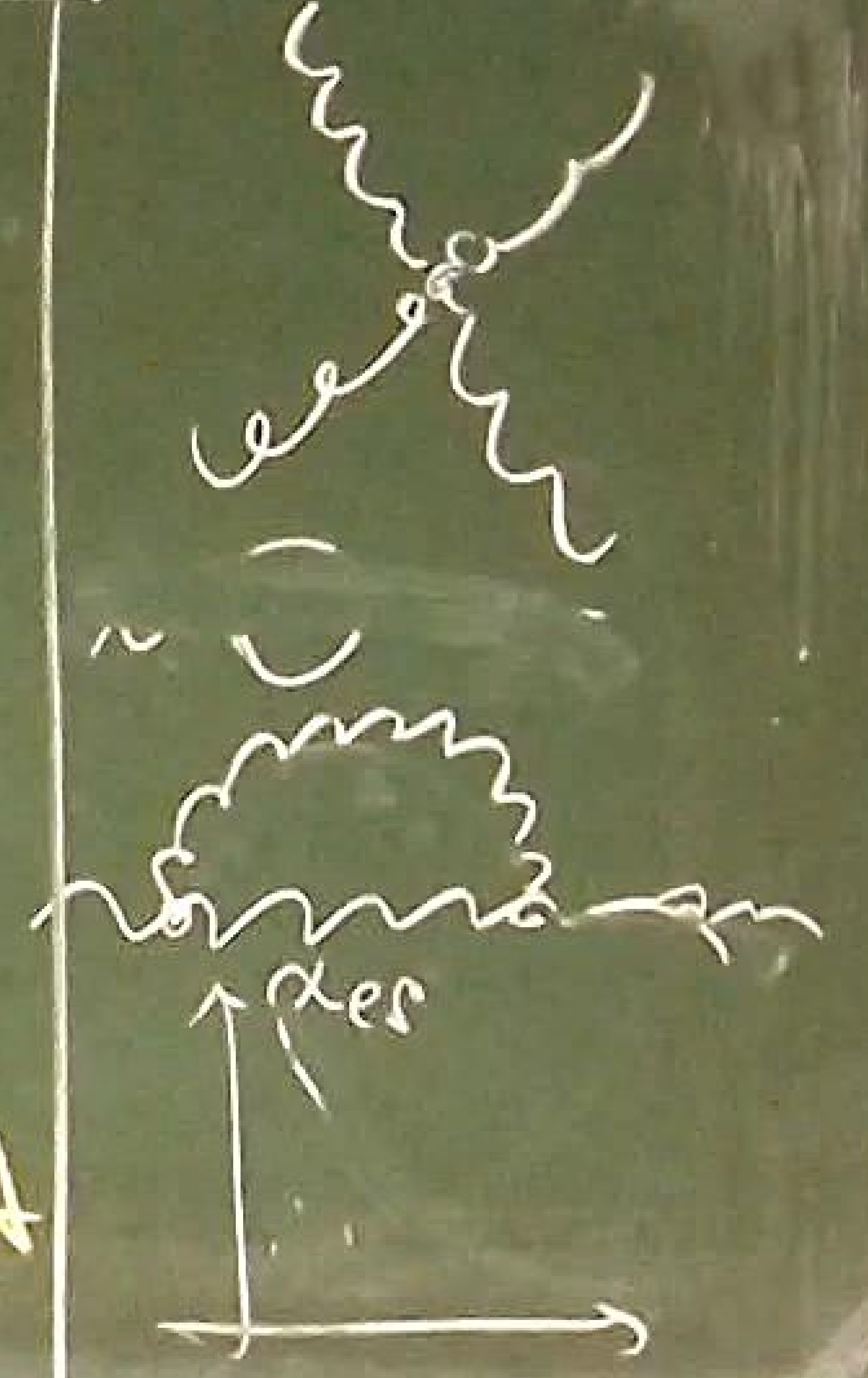
$$\alpha_s = \frac{g_s^2}{4\pi}$$

$$\alpha^{eff}(q^2) \xrightarrow{q^2 \to \infty} 0$$

Asymptotic freedom

$$\alpha^{ef}(q^2) \xrightarrow{q^2 \to 0} \infty$$

(Confinement



10.5.7. Summary

$$\underbrace{SU(2)_L \times U(1)_Y} \times SU(3)_C$$

$$\big\downarrow \text{ SSB, Higgs}$$

$$U(1)_Q$$

(electric charge)

Fields (include ν_R)

$[2$ Leptons + 2 Quarks × 3 Colours$]$ × 3 Generations

$= 24 \underbrace{\text{Dirac bispinors}}_{4 \text{ complex fields}} = 96$ Complex fields

Parameters: (without ν_R)

- × 9 × Fermion masses
- × 1 m_h : Higgs mass
- × 1 Higgs field VEV v
- × 3 Gauge field couplings g, g', g_s
- × 4 × CKM matrix $\theta_{12}, \ldots$

---

18 parameters.