Recap

① Interactions:

$$\mathcal{L}_{\phi^4} = \boxed{\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2} \boxed{- \frac{\lambda}{4!}\phi^4}$$

free KG field Interaction

$$\Rightarrow H_{int} = \int d^3x \, \frac{\lambda}{4!} \phi^4(x)$$

② Goal: Correlators

$$\langle \Omega | T \, \phi(x)\phi(y) | \Omega \rangle$$

Vacuum of interacting theory

Heisenberg operators of interacting theory

③ Todo:

$\phi(x) = \dots$ free fields $\phi_I(x)$? ✓

$|\Omega\rangle = \dots$ free vacuum $|0\rangle$? ✓

④ Definitions:

Free field: $\phi_I(t,\vec{x}) = e^{iH_0(t-t_0)} \phi(t_0,\vec{x}) e^{-iH_0(t-t_0)}$

free KG Hamiltonian

interaction picture

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ipx} + a_p^\dagger e^{ipx} \right) \Big|_{p^0 = E_p}$$

Interacting field:

interacting Hamiltonian: $H_0 + H_{int}$

$$\phi(t,\vec{x}) = e^{iH(t-t_0)} \phi(t_0,\vec{x}) e^{-iH(t-t_0)}$$

$$= U^\dagger(t,t_0) \, \phi_I(t,\vec{x}) \, U(t,t_0)$$

⑤ Time evolution operator:

$$U(t,t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$$

$\Leftrightarrow$

$$\boxed{\begin{aligned} i\partial_t U &= H_I(t)\, U \\ U(t_0,t_0) &= \mathbb{1} \end{aligned}}$$

"interaction in interaction picture"

$$H_I(t) = \int d^3x \, \frac{\lambda}{4!} \phi_I^4(t,\vec{x})$$

$\rightarrow$ Solution: Dyson series

$$U(t,t_0) = T \exp\left[-i \int_{t_0}^{t} d\tau \, H_I(\tau) \right]$$

time ordering operator

⑥ Ground state:

$T \to \infty(1-i\varepsilon)$ "Trick"

$\Downarrow$

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0(t_0+T)} \underbrace{\langle \Omega | 0 \rangle}_{\neq 0} \right)^{-1} \times U(t_0,-T)|0\rangle$$

⑦ Group property:

$$U(t_1,t_2)\,U(t_2,t_3) = U(t_1,t_3)$$

8. Main result:

$$\langle\Omega|T\,\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T}dt\,H_I(t)\right]\right\}|0\rangle}{\langle 0|T\left\{\exp\left[-i\int_{-T}^{T}dt\,H_I(t)\right]\right\}|0\rangle}$$

$H_I(t)$ depends on Φ_I

9. Perturbative expansion of exp[..] in orders of λ

Efficient evaluation? $\longrightarrow$ Wick's theorem

$$\langle\Omega|T\,\phi(x)\phi(y)|\Omega\rangle = \sum \overbrace{\langle 0|T\{\phi_I(x_1)\ \dots\ \phi_I(x_n)\}|0\rangle}$$

similar terms in denominator!

10. Definitions.

Contraction:

$$\overbrace{\phi(x)\phi(y)} := \begin{cases}[\phi^+(x),\phi^-(y)] & x^0>y^0\\ [\phi^+(y),\phi^-(x)] & x^0<y^0\end{cases} \;\hat{=}\; D_F(x-y)$$

drop the I

interaction picture fields!

Feynman propagator

Normal order: $:a_1^{(\dagger)}\dots a_n^{(\dagger)}: :=$ (creation op.) $\times$ (annihilation op.)

(Example: $:aa^\dagger: = a^\dagger a$)

$$\boxed{\phi_i \equiv \phi(x_i) \equiv \phi_I(x_i)}$$

Note: $:A\overbrace{\phi_i B\phi_j}C: = D_F(x_i-x_j)\cdot\, :ABC:$

products of fields/modes

number

normal ordered operators

11. Wick's theorem:

including the "trivial contraction" $\phi_1\dots\phi_n$

$$\boxed{T\{\phi_1\dots\phi_n\} = :\text{all possible contractions}:}$$

Since $\langle 0|:A:|0\rangle = 0$ if $:A:\neq \mathbb{1}$ $\Rightarrow$ $\langle 0|T\{\phi_1\ \ \phi_n\}|0\rangle =$ all full contractions

6) Example:

$$T\{\phi_1\phi_2\phi_3\phi_4\} =$$

$$\int :\phi^\dagger\phi + \pi^\dagger\pi:$$

$$:e^{i\phi}:$$

$$:\phi_1\phi_2\cancel{\phi_3}\phi_4: +$$

$$\overline{\phi_1\phi_2}\phi_3\phi_4 + \overline{\phi_1\phi_2\phi_3}\phi_4 + \overline{\phi_1\phi_2\phi_3\phi_4}$$

$$+ \phi_1\overline{\phi_2\phi_3}\phi_4 + \phi_1\overline{\phi_2\phi_3\phi_4} + \phi_1\phi_2\overline{\phi_3\phi_4}$$

$$+ \overline{\phi_1\phi_2}\,\overline{\phi_3\phi_4} + \phi\phi\phi\phi + \phi\phi\phi\phi \,:$$

$$\cancel{a^\dagger a} = :aa^\dagger: \ominus :a^\dagger a + 1:$$

CCR

$\theta = 1$

linear

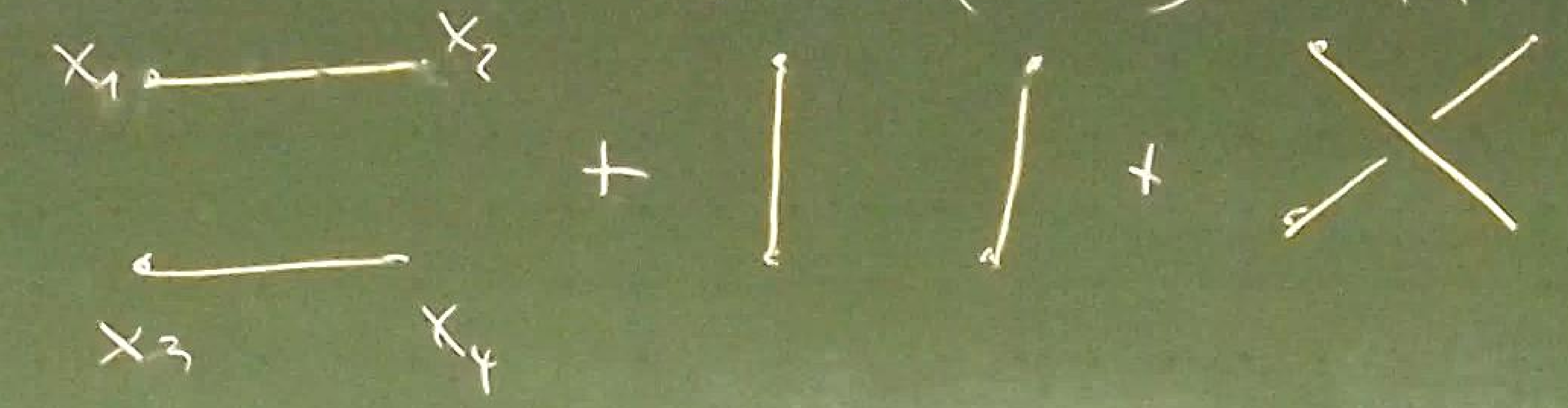
$$\oplus\ :a^\dagger a: + :1: = \cancel{aa^\dagger} + 1$$

$$\langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle$$

$$= \overline{\phi_1\phi_2}\,\overline{\phi_3\phi_4} + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4$$

$$= D_F(x_1-x_2)\,D_F(x_3-x_4) + D_F(x_1-x_3)\,D_F(x_2-x_4)$$

$$+ D_F(x_1-x_4)\cdot D_F(x_2-x_3)$$



4.4 Feynman Diagrams

1) $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$

$$\propto \langle 0|T\{\underbrace{\phi(x)\phi(y)}_{\lambda^0} + \overbrace{\phi(x)\phi(y)\left[-i\int dt\, H_I(t)\right]}^{\lambda^1} + \underbrace{\ldots}_{\lambda^2}\}|0\rangle$$

2) λ^0: $\langle 0|T\phi(x)\phi(y)|0\rangle$

$$= \overline{\phi\phi} = D_F(x-y)$$

$$= x \text{———} y$$

3) λ^1:

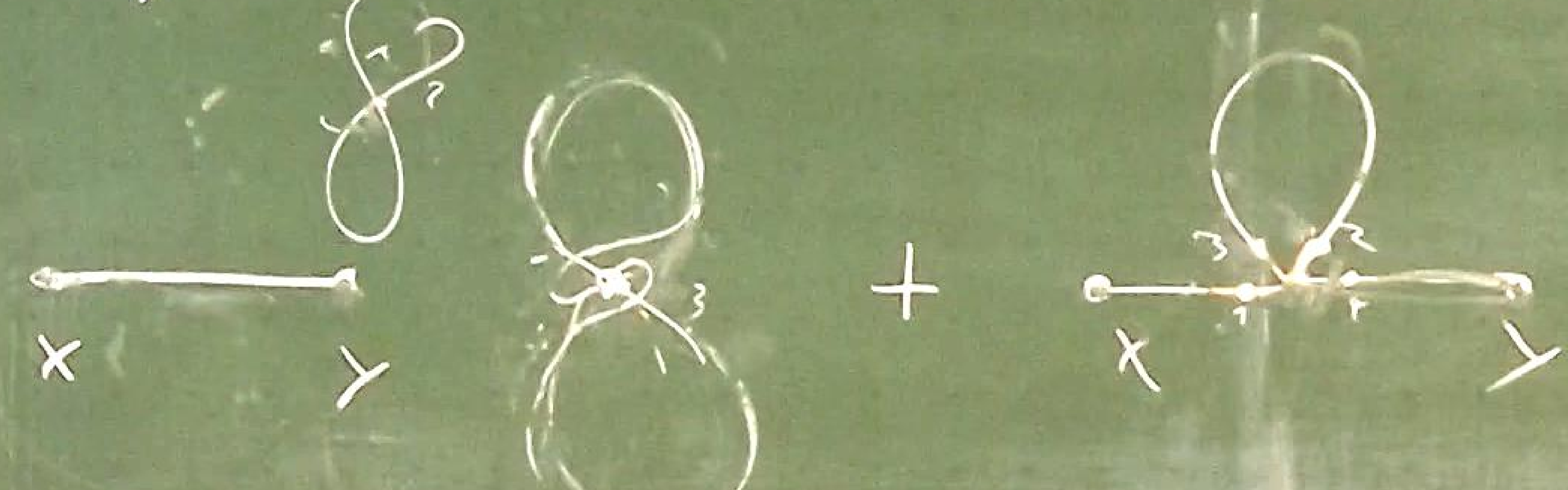
$$\langle 0| T\left\{\phi(x)\phi(y)\frac{-i\lambda}{4!}\int d^4z\,\phi(z)\phi(z)\phi(z)\phi(z)\right\}|0\rangle$$

W-T

$$= 3\,\frac{-i\lambda}{4!}\,D_F(x-y)\int d^4z\, D_F(z-z)\, D_F(z-z)$$

$$+\,12\cdot\frac{-i\lambda}{4!}\int d^4z\, D_F(x-z)\,D(y-z)\,D_F(z-z)$$

=



→ Interpretation:

Feynman diagram:
- edges = propagators ↔ D_F
- Internal points = vertices ↔ $-i\lambda\int d^4z$
- external points = spacetime points ↔ x, y, ...

4) Prefactors:

- T-O. ⟺ sum all identical terms.
- $\propto \lambda^n$

→ $\frac{1}{n!}$, n integrals/vertices

→ n! equivalent permutation

→ ignore the $\frac{1}{n!}$

- 4 contractions at each vertex

Analytic expression → 4! possibilities to interchange contractions

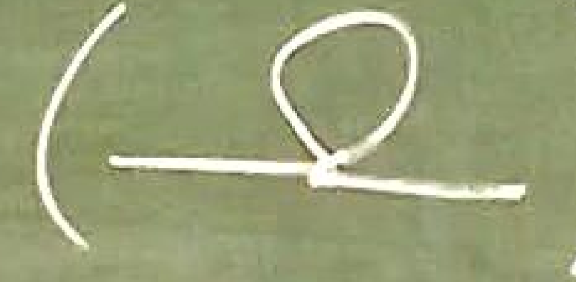
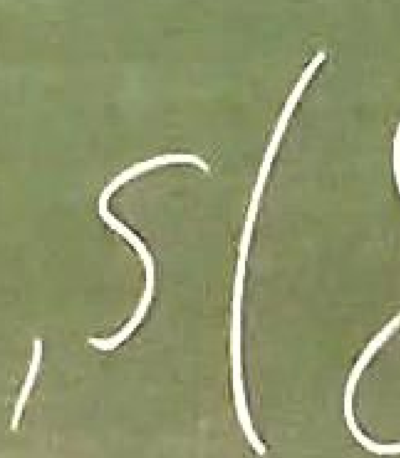
→ $\frac{1}{4!}$ cancels 4!

→ internal vertex → $-i\lambda\int d^4z$

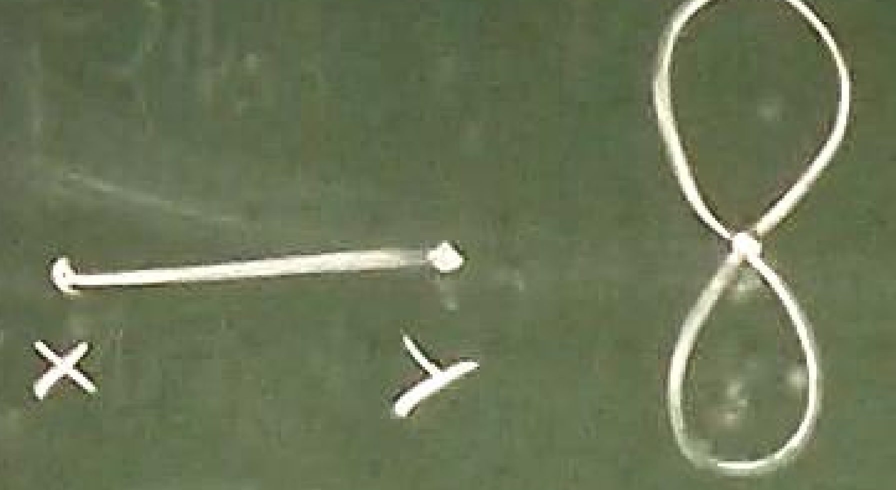
- Symmetries of diagrams reduce number of different contractions.

→ divide expression by symmetry factor S

Examples:

S() = 2, S() = 2·2·2 = 8

4) Therefore:

 $= \frac{1}{8} \cdot D_F(x-y)\,(-i\lambda)\int d^4z\, D_F(z-z)\, D_F(z-z)$

 $= \frac{1}{2}(-i\lambda)\int d^4z\, D_F(x-z)\, D_F(z-y)\, D_F(z-z)$

$\langle\Omega| T\, \phi(x)\phi(x)\phi(z)|\Omega\rangle = 0$

5) Therefore

$\langle 0| T\left\{\phi(x)\phi(y)\, e^{-i\int dt\, H_I(t)}\right\}|0\rangle$

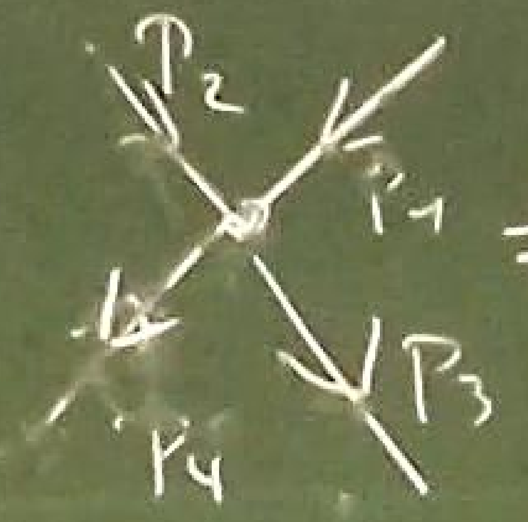
$= \sum \left\{ \text{Feynman diagrams with two external points} \right\}$

with position/real-space Feynman rules for ϕ^4-theory:

1. $x \text{ --- } y = D_F(x-y)$
2.  $= (-i\lambda)\int d^4 z_i$
3. $x|\text{———} = 1$
4. Divide by sym factor: $\frac{1}{S}$.

6) → Momentum space:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, \frac{i}{p^2-m^2+i\varepsilon}\, e^{-ip(x-y)}$$

 $= -i\lambda \int d^4 z$

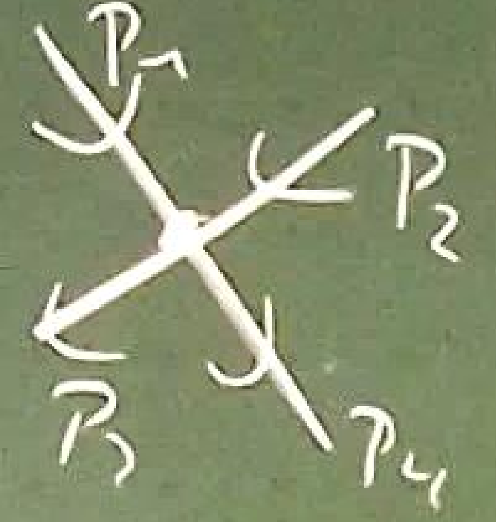
$= (-i\lambda)(2\pi)^4\, \delta^{(4)}(p_1+p_2-p_3-p_4)$

→ Momentum conservation at vertices

→ <u>Momentum-space Feynman rules.</u>

1. 

$= \dfrac{i}{p^2 - m^2 + i\varepsilon}$

2. 

$= (-i\lambda)(2\pi)^4 \, \delta(p_1 + p_2 - p_3 - p_4)$

3. 

$= e^{-ipx}$

4. Integrate: $\prod_i \int \dfrac{d^4 p_i}{(2\pi)^4}$

5. Divide by sym. fac. $\dfrac{1}{S}$