4. Interacting Fields and Feynman Diagrams

4.1. Preliminaries

- $H_{int} = \int d^3x\, \mathcal{H}_{int}(\phi(x))$
  $= -\int d^3x\, \mathcal{L}_{int}(\phi)$

- Causality $\rightarrow$ Interactions are local

* Examples:

1) ϕ^4-Theory

dimensionless coupling constant

$$\boxed{\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \underbrace{\frac{\lambda}{4!}\phi^4(x)}_{\text{Interaction}}}$$

EOM $\longrightarrow$ $(\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3$

$\longrightarrow$ Cannot be solved by Fourier modes

2) Yukawa theory

$$\boxed{\mathcal{L}_{Yukawa} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi + \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \underbrace{- g\,\phi\,\bar{\Psi}\Psi}_{\text{Interaction}}}$$

3) QED

$$\mathcal{L}_{QED} = \underbrace{\bar{\Psi}(i\partial\!\!\!/ - m)\Psi}_{\partial_\mu\gamma^\mu} - \underbrace{\frac{1}{4}(F_{\mu\nu})^2} \underbrace{- e\,\bar{\Psi}\gamma^\mu\Psi A_\mu}_{\text{Interaction}}$$

$$\boxed{\begin{array}{l} e = -|e| < 0 \text{ : Electron charge} \\ D_\mu = \partial_\mu + ieA_\mu \end{array}}$$

covariant derivative

$$= \bar{\Psi}(iD\!\!\!\!/ - m)\Psi - \frac{1}{4}(F_{\mu\nu})^2$$

Gauge theory:

- $A'_\mu = A_\mu - \frac{1}{e}\partial_\mu \alpha(x)$
- $\psi'(x) = e^{i\alpha(x)}\psi(x)$

EOM $\longrightarrow$

- $i(\not{\partial} - m)\psi(x) = 0$
- $\partial_\mu F^{\mu\nu} = e j^\nu = \bar{\psi}\gamma^\mu\psi$

- No known exactly solvable interacting QFTs in $D > 1+1$ $\longrightarrow$ $\boxed{\text{CFT (conformal field theories)}}$

$\longrightarrow$ Perturbation Theory

$\hookrightarrow$ Numerics (Lattice gauge theories)

4.2. Perturbation Expansion of Correlation Functions

1) Goal: $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$

- $|0\rangle$: Ground state of free theory
- $|\Omega\rangle$: — " — interacting — " —

2) Remember:

$\langle 0|T\,\phi(x)\phi(y)|0\rangle = D_F(x-y)$

3) Now:

$$H_{\phi^4} = H_0 + \underbrace{\int d^3x\,\frac{\lambda}{4!}\phi^4(\vec{x})}_{H_{int}\ =\ \text{Perturbation}}$$

4) Todo:

Express $\left\{\begin{matrix}\phi(x)\\ |\Omega\rangle\end{matrix}\right\}$ in terms of $\left\{\begin{matrix}\text{free field } \phi_I(x)\\ \text{free vacuum } |0\rangle\end{matrix}\right\}$

5 $\tilde{x}$ Reference time t_0

$$\phi(t_0,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_{\vec{p}}\,e^{i\vec{p}\vec{x}} + a^\dagger\,\boxed{e^{-i\vec{p}\vec{x}}\Big)}\right.$$

6 Definitions:

Heisenberg picture:

$$\phi(x) = \phi(t,\vec{x}) = e^{iH(t-t_0)}\,\phi(t_0,x)\,e^{-iH(t-t_0)}$$

Interaction picture:

$$\phi_I(t,\vec{x}) = e^{iH_0(t-t_0)}\,\phi(t_0,x)\,e^{-iH_0(t-t_0)}$$

$$\Rightarrow\quad \phi_I(t,\vec{x}) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_{\vec{p}}\,e^{-ipx} + a^\dagger_{\vec{p}}\,e^{ipx}\right)$$

$$\phi(t,\vec{x}) = U^\dagger(t,t_0)\,\phi_I(t,\vec{x})\,U(t,t_0)$$

$$U(t,t_0) = e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)}$$

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$$\boxed{\begin{aligned} &\cdot\ i\partial_t U(t,t_0) \doteq H_I(t)\,U(t,t_0) \\ &\cdot\ U(t_0,t_0) = \mathbb{1} \end{aligned}}$$

with

$$H_I(t) = e^{iH_0(t-t_0)}\,H_{int}\,e^{-iH_0(t-t_0)} = \int d^3x\,\frac{\lambda}{4!}\,\phi_I^4(t,\vec{x})$$

8 Solution: Dyson series.

$$\boxed{\begin{aligned} U(t,t_0) &= \mathbb{1} + (-i)\int_{t_0}^{t} dt_1\,H_I(t_1) \\ &\quad + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1\int_{t_0}^{t} dt_2\,\mathcal{T}\{H_I(t_1)H_I(t_2)\} \\ &\quad + \dots\dots \\ &= \mathcal{T}\exp\left[-i\int_{t_0}^{t} d\tau\,H_I(\tau)\right] \end{aligned}}$$

9) Properties:

- $U(t,t') \overset{t>t'}{=} e^{iH_0(t-t_0)} e^{-iH(t-t')} e^{-iH_0(t'-t_0)}$ $\boxed{(*)}$
- $U^{-1}(t,t') = U^{+}(t,t')$
- $U(t_1,t_2)U(t_2,t_3) \overset{!}{=} U(t_1,t_3) \quad t_1>t_2>t_3$

10) Grand state $|\Omega\rangle$?

$\lambda \ll 1 \rightarrow \langle\Omega|0\rangle \neq 0$

$$e^{-iHT}|0\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle$$

$$= e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle$$

$E_n > E_0 \quad n\neq 0$

$$\boxed{|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} \boxed{e^{-iHT}}|0\rangle \overset{(*)}{=} \lim_{T\to} \left(e^{-iE_0(t_0+T)}\langle\Omega|0\rangle\right)^{-1} \boxed{U(t_0,-T)}|0\rangle}$$

$$\langle\Omega| = \lim_{T\to\infty(1-i\varepsilon)} \langle 0|U(T,t_0)\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}$$

11] ☆ Two point correlator

$x^0 > y^0 > t_0$

$\langle\Omega|\cancel{T}(\phi(x)\phi(y))|\Omega\rangle$

$= \lim_{T\to\infty(1-i\varepsilon)} N_T^{-1} \langle 0| U(T,x^0)\,\phi_I(x)\,U(x^0,y^0)\,\phi_I(y)\,\boxed{U(y^0,-T)|0\rangle}$

$\langle\Omega|\Omega\rangle = 1$

$N_T \overset{!}{=} \langle 0| U(T,t_0)\,U(t_0,-T)|0\rangle$

$= \langle 0| U(T,-T)|0\rangle$

$$\boxed{\langle\Omega|T\,\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T}dt\ H_I(t)\right]\right\}|0\rangle}{\langle 0|T\left\{\exp\left[-i\int_{-T}^{T}dt\ H_I(t)\right]\right\}|0\rangle}}$$

4.3. Wick's Theorem

Goal: Evaluate

$\langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle = \sum_i \langle 0|T\{\phi_I(x_1)\,\phi_I(x_2)\,\phi_I \ldots\}|0\rangle$

1) $\phi_I(x) = \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p e^{-ipx}}_{\phi_I^+(x)} + \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p^\dagger e^{+ipx}}_{\phi_I^-(x)}$

$\phi^+|0\rangle = 0$

$\langle 0|\phi^- = 0$

2) Observation. $\boxed{x^0 > y^0}$ $\boxed{x^0 < y^0}$

$T(\phi_I(x)\phi_I(y)) = \phi_x^+\phi_y^+ + \phi_x^+\phi_y^- + \phi_x^-\phi_y^+ + \phi_x^-\phi_y^-$

$= \text{--''--} \quad \phi_y^-\phi_x^+ \quad \text{--''--} \quad \text{--''--} + [\phi_x^+, \phi_y^-]$

$\phi_x^-\phi_y^+ \quad + [\phi_y^+, \phi_x^-]$

3) Definition

• Contraction

$\boxed{\overline{\phi(x)\phi(y)} := \begin{cases} [\phi^+(x), \phi^-(y)] & x^0 > y^0 \\ [\phi^+(y), \phi^-(x)] & x^0 < y^0 \end{cases}}$

$\boxed{\hat{=} D_F(x-y)}$

$T\{\phi(x)\phi(y)\} = :\phi(x)\phi(y): + \overline{\phi(x)\phi(y)}$

$\to \langle 0|T\phi(x)\phi(y)|0\rangle$

$= D_F(x-y)$

• Normal order:

$\boxed{:a_1^{(\dagger)} \dots a_n^{(\dagger)}: := \text{creation op} \times \text{annihilation op}}$

4) Wick's theorem:

$$\boxed{\begin{aligned}&T\{\phi(x_1)\dots\phi(x_n)\} = :\phi(x_1)\dots\phi(x_n) + \text{all possible contractions}: \\ &\text{where} \quad :\wick{A\c\phi_i B\c\phi_j} C: \;:=\; D_F(x_i-x_j)\,:ABC:\end{aligned}}$$

5) $\langle 0|T(\phi_1\dots\phi_n)|0\rangle$ = all full contractions

$$\langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle = \wick{\c1\phi_1\c2\phi\c2\phi\c1\phi} \quad \wick{\c1\phi\c1\phi\c2\phi\c2\phi} \quad \wick{\c1\phi\c2\phi\c1\phi\c2\phi}$$