Spin-Statistics Theorem

. Observation:

- KG field ϕ: Spin 0 $\rightarrow$ commutator (bosonic particles)
- Dirac field ψ: Spin $\frac{1}{2}$ $\rightarrow$ anticommutators (fermionic particles)

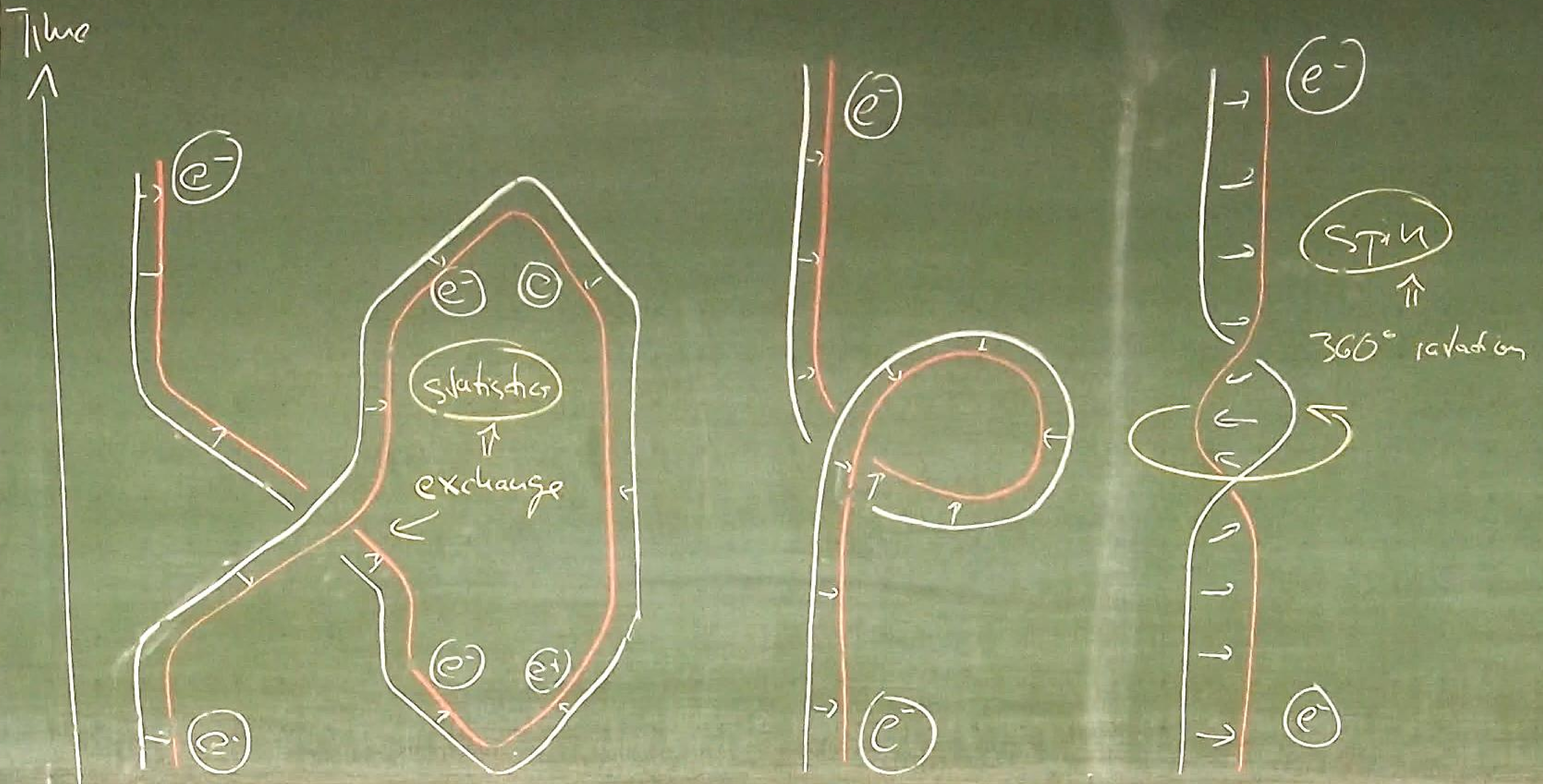
Spin statistics theorem:

- Lorentz invariance
- Causality
- Positive energies
- Positive norms

$\Rightarrow$ Integer Spin $\leftrightarrow$ Bosons

Half-Integer Spin $\leftrightarrow$ Fermions

"Proof by picture"

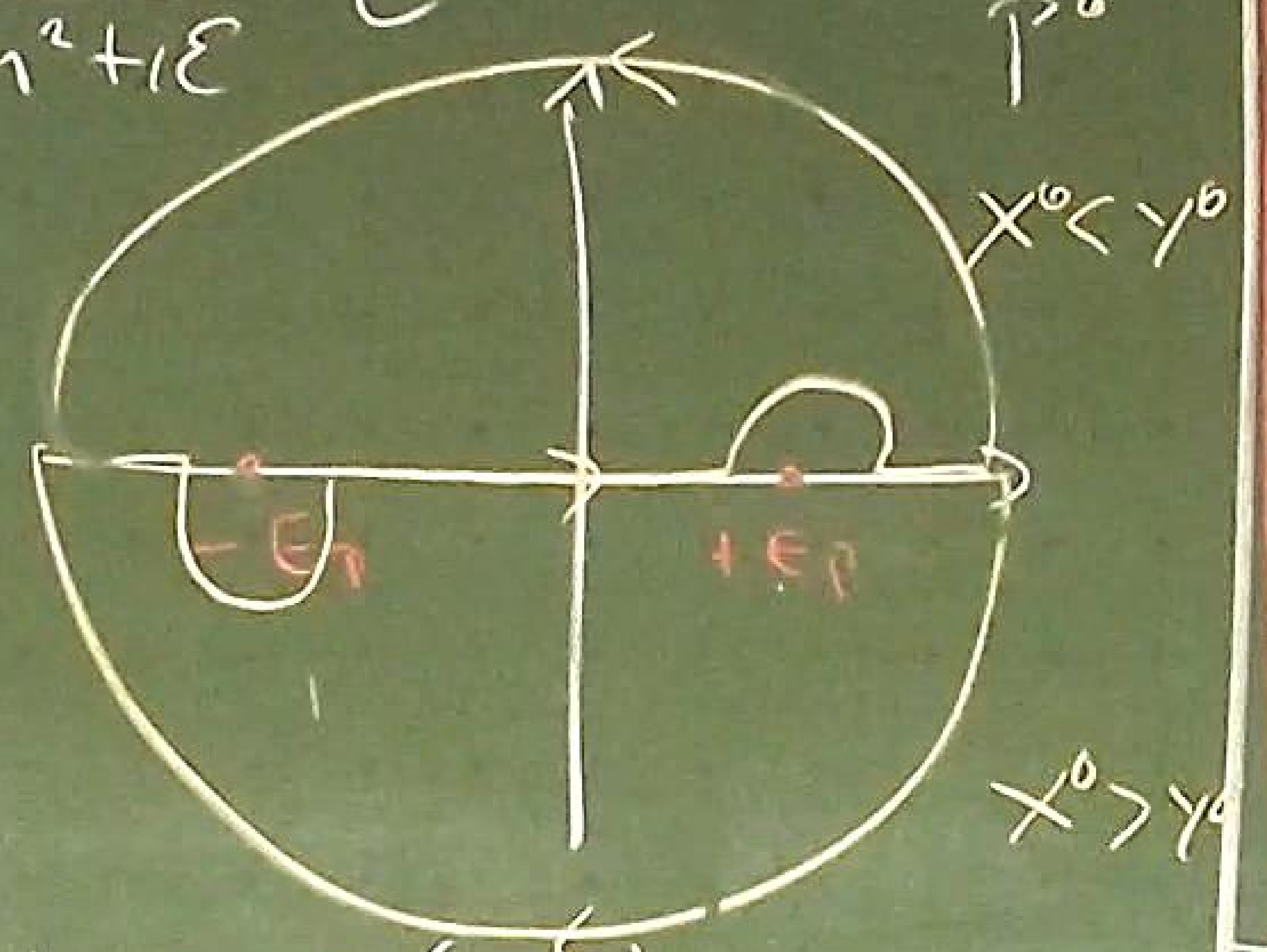


Dirac Propagator

1)

$$\cdot\ \langle 0|\psi_a(x)\bar\psi_b(y)|0\rangle = \underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}}_{D(x-y)}\ \underbrace{\sum_s u^s_a(p)\bar u^s_b(p)}_{(\not p+m)_{ab}} \overset{x^0>y^0}{=} \int \frac{d^4p}{(2\pi)^3}\frac{1}{2E_p}\,\frac{i(\not p+m)_{ab}}{p^2-m^2+i\varepsilon}\,e^{-ip(x-y)}$$

$$= (i\not\partial_x+m)_{ab}\,D(x-y)$$



$$\cdot\ \langle 0|\bar\psi_b(y)\psi_a(x)|0\rangle = \int \text{''}\ e^{-ip(y-x)}\ \underbrace{\sum_s v^s_a\bar v^s_b}_{(\not p-m)_{ab}} \overset{x^0<y^0}{=} \ominus\int \frac{d^4p}{(2\pi)^4}\,\frac{i(\not p+m)_{ab}}{p^2-m^2+i\varepsilon}\,e^{-ip(x-y)}$$

$$= -(i\not\partial_x+m)\,D(y-x)$$

2) Feynman propagator:

$$S_F^{ab}(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\frac{i(\not p+m)_{ab}}{p^2-m^2+i\varepsilon}\,e^{-ip(x-y)}$$

$$= \begin{cases} \langle 0|\psi_a(x)\bar\psi_b(y)|0\rangle & x^0>y^0 \\ \ominus\ \langle 0|\bar\psi_b(y)\psi_a(x)|0\rangle & x^0<y^0 \end{cases}$$

$$\equiv \langle 0|T\,\psi_a(x)\bar\psi_b(y)|0\rangle$$

Note. $T\,\psi(t_2)\psi(t_1) \overset{t_1>t_2}{=} -\psi(t_1)\psi(t_2)$

Causality

1) Measurable Operators:

$$\hat{O}(x) = \sum \prod_{\text{even}} \{\psi_i^{(+)}, \partial\psi_i^{(+)}, \dots\}$$

Example: * $j^\mu = \bar{\psi}\gamma^\mu\psi$ ✓

* $\psi_a + \psi_a^+$ ✗

2) Causality $\Leftrightarrow$ $\{\psi_a(x), \bar{\psi}_b(y)\} = 0$ for $(x-y)^2 < 0$

(Superselection)

We find:

$$\{\psi_a(x), \bar{\psi}_b(y)\} \hat{=} (i\partial\!\!\!/_x + m)_{ab}[D(x-y) - D(y-x)]$$

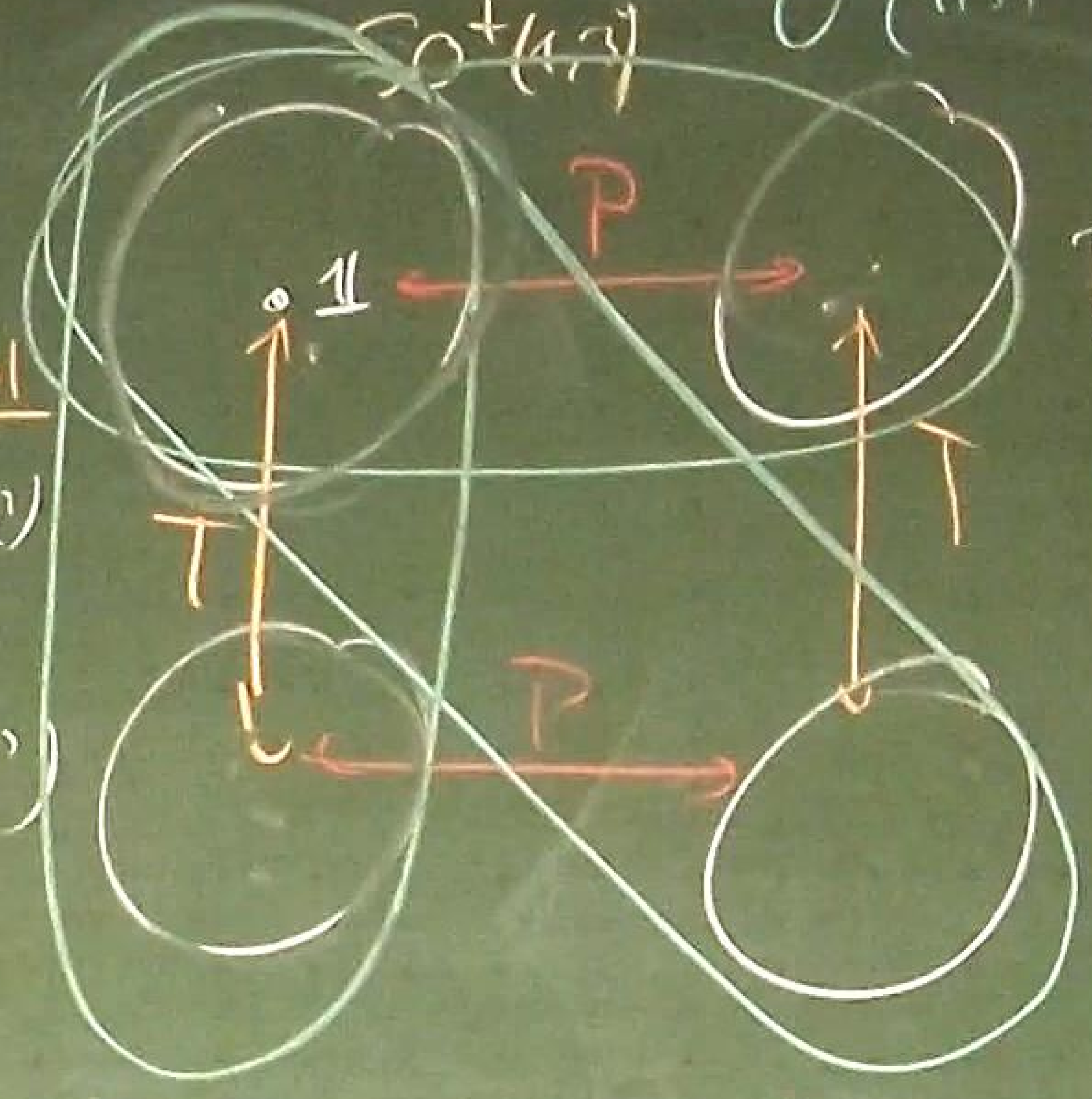
$$\overset{(x-y)^2<0}{=} (i\partial_x + m)_{ab}[\underbrace{D(x-y) - D(x-y)}_{=0}]$$

$$= 0$$

3.5. Discrete Symmetries of the Dirac Theory

Review of the Lorentz group

$O(1,3)$ — orthochronous LT — $O^+(1,3)$



Parity: $P: (t, \vec{x}) \to (t, -\vec{x})$

Time reversal: $T: (t, \vec{x}) \to (-t, \vec{x})$

orthochronous LT

$SO(1,3)$

Parity

1| Unitary rep. on Fock space:

$$U(P)\, a^s_{\vec p}\, U^{-1}(P) = \eta_a^{+1}\, a^s_{-\vec p}$$

$$U(P)\, b^s_{\vec p}\, U^{-1}(P) = \eta_b^{-1}\, b^s_{-\vec p}$$

2|

$$U(P)\,\psi(t,\vec x)\,U^{-1}(P) = \underbrace{\gamma^0}_{P_{\frac12}}\,\psi(t,-\vec x)$$

3| Examples:

$$U(P)\,\bar\psi\psi\,U^{-1}(P) \stackrel{!}{=} +\bar\psi\psi(t,-\vec x)$$

$$U(P)\,\bar\psi\gamma^5\psi\,U^{-1}(P) \stackrel{!}{=} -\bar\psi\psi(t,-\vec x)$$

Time Reversal

$$U(T)\,\psi(t,\vec x)\,U^{-1}(T) = T_{\frac12}\,\psi(-t,\vec x)$$

1|
- $U(T)\,\psi(t,\vec x)\,U^{-1}(T)$
- $U(T)\,a^s_{\vec p}\,U^{-1}(T) = a^{-s}_{-\vec p}$ ← flip spins
- $[U(T), H] = 0$
- $U^{-1}(T) = U^{+}(T)$ (Wigner's theorem)

2| Problem

$$\psi(t,\vec x) = e^{iHt}\,\psi(\vec x)\,e^{-iHt}$$

$U \cdot U^{-1}$ $\Rightarrow$ $\underline{U\,\psi(t,\vec x)\,U^{-1}} = e^{iHt}\,\underline{U\,\psi(\vec x)\,U^{-1}}\,e^{-iHt}$

$|0\rangle$ $\Rightarrow$ $T_{\frac12}\,\psi(-t,\vec x)\,|0\rangle = e^{iHt}\,T_{\frac12}\,\psi(\vec x)\,|0\rangle$

$\Rightarrow T_{\frac12}\, e^{-iHt}\,\psi(\vec x)\,|0\rangle = $ — " —

$$\rightarrow \underbrace{e^{-2iHt}}_{t\ ?} \; T_{\frac12}\,\psi(\vec x)\,|0\rangle = \underbrace{T_{\frac12}\,\psi(\vec x)\,|0\rangle}_{\text{no } t}$$

3) Solution:

$U(T)$ must be antiunitary / antilinear.

$$U(T)\, c = c^* \, U(T)$$

($c \in \mathbb{C}$)

A	U
A·K	U·K

4) Transformation of Spin

Spin flipped under TR

$$\overline{a^1_p} \to a^2_p$$

$$\overline{a^2_p} \to -a^1_p$$

5) Definition

- $U(T)$ antiunitary
- $U(T)\, a^s_p U^{-1}(T) = \overline{a^s_{-p}}$ (b, b)

$$\Rightarrow U(T)\,\Psi(t,x)\,U^{-1}(T) \overset{\circ}{=} \underbrace{(\gamma^1\gamma^3)}_{T_{1/2}}\Psi(-t,\vec{x})$$

$$\begin{pmatrix} i\sigma^y & \\ & i\sigma^y \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

2) Problem

$$\Psi(t,\vec{x}) = e^{iHt}\,\psi(\vec{x})\,e^{-iHt}$$

$$U\cdot U^{-1}:\quad \Rightarrow U\,\Psi(t,\vec{x})\,U^{-1} = e^{-iHt}\,U\,\psi(\vec{x})\,U^{-1}\,e^{+iHt}$$

$$|0\rangle:\quad \Rightarrow T_{\frac{1}{2}}\,\psi(-t,\vec{x})\,|0\rangle = e^{-iHt}\,T_{\frac{1}{2}}\,\psi(\vec{x})\,|0\rangle$$

$$\Rightarrow T_{\frac{1}{2}}\,e^{-iHt}\,\psi(\vec{x})\,|0\rangle = \text{— '' —}$$

$$\Rightarrow \underbrace{e^{-2iHt}}_{t\ ?}\,T_{\frac{1}{2}}\,\psi(\vec{x})\,|0\rangle = \underbrace{T_{\frac{1}{2}}\,\psi(\vec{x})\,|0\rangle}_{\text{no } t}$$

($0\,Ht$; no t)

6) Examples:

$$U(T) j^\mu(t,x) U^{-1}(T) = \begin{cases} + j^\mu(-t,\vec{x}) & \mu = 0 \\ - j^\mu(-t,\vec{x}) & \mu = 1,2,3 \end{cases}$$

$\bar{\Psi}\gamma^\mu\psi$

Charge Conjugation

1) Discrete, non-spacetime symmetry.

$$U(C) a_{\vec{p}}^{s} U^{-1}(C) = b_{\vec{p}}^{s}$$

b a

2) —

3) $U(C)\psi U^{-1}(C)$

$\overset{\circ}{=} -i\gamma^2 (\psi^\dagger)^T$

$= -i(\bar{\Psi}\gamma^0\gamma^2)^T$

$$U(C)\bar{\Psi}U^{-1}(C) \overset{\circ}{=} -i(\gamma^0\gamma^2\psi)^T$$

5) Examples:

$$U(C)\bar{\Psi}\psi U^{-1}(C) \overset{\circ}{=} \bar{\Psi}\psi$$

$$U(C)\bar{\Psi}\gamma^\mu\psi U^{-1}(C) \overset{\circ}{=} -\bar{\Psi}\gamma^\mu\psi$$

j^μ

- $SO^+(1,3)$ invariance
- Causality
- Locality
- Stable vacuum $\Longrightarrow$ CPT is symmetry.

Note 33

- relativistic QFT: invariant $SO^+(1,3)$
- $(i\gamma^\mu\partial_\mu - m)\psi = 0$ is $\{C,P,T\}$ invariant
- The (quantized) Dirac theory is $\{C,P,T\}$ invariant:

$$[H, U(X)] = 0 \qquad X = C,P,T$$

- Weak interactions violate C and P but preserve CP and T (Wu exp)
- Rare processes (neutral Kaon decay) violate CP and T
- CPT Theorem: