Recap: $\psi^\dagger\gamma^0$ Dirac adjoint

$$\mathcal{L}_{Dirac} = \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$$

$$\Rightarrow \quad (i\gamma^\mu\partial_\mu - m)\Psi = 0$$

$$(i\not{\partial} - m)\psi = 0, \quad \psi \in \mathbb{C}^2 \oplus \mathbb{C}^2 \cong \mathbb{C}^4$$

Solutions:

$$\psi^+_{\vec{p}s} = \underbrace{\begin{pmatrix}\sqrt{p\sigma}\,\xi^s \\ \sqrt{p\bar\sigma}\,\xi^s\end{pmatrix}}_{u^s(p)} e^{-ipx}$$

$$\boxed{\xi^1, \eta^1 = \begin{pmatrix}1\\0\end{pmatrix} \qquad \xi^2, \eta^2 = \begin{pmatrix}0\\1\end{pmatrix}}$$

$$\psi^-_{\vec{p}s} = \underbrace{\begin{pmatrix}\sqrt{p\sigma}\,\eta^s \\ -\sqrt{p\bar\sigma}\,\eta^s\end{pmatrix}}_{v^s(p)} e^{+ipx}$$

Relations:

- $\underset{v}{\bar{u}}^s = \underset{v}{u}^{s\dagger}\gamma^0$
- $\underset{v}{\bar{u}}^r \underset{v}{u}^s = -2m\delta^{rs}$, $\boxed{(\underset{v}{u}^{r\dagger})\,\underset{v}{u}^s = 2E_{\vec{p}}\delta^{rs}}$
- $\bar{u}^s v^r = 0$
- $\boxed{u^{r\dagger}(\vec{p})\,v^s(-\vec{p}) = 0}$
- $\not{p} := \gamma^\mu p_\mu$ Feynman slash notation
- $\sum_s u^s(p)\,\bar{u}^s(p) = \not{p} + m\mathbb{1}$
- $\sum_s v^s(p)\,\bar{v}^s(p) = \not{p} - m\mathbb{1}$

3.3. Dirac Field Bilinears

1) Weyl basis

- $\boxed{\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix}-\mathbb{1} & \\ & \mathbb{1}\end{pmatrix}}$

$\gamma^{5\dagger} = \gamma^5$, $(\gamma^5)^2 = \mathbb{1}$,

$\{\gamma^5, \gamma^\mu\} = 0$ $\quad \mu = 0,1,2,3$

$$S^{\mu\nu} = [\gamma^\mu, \gamma^\nu]$$

$$S = \begin{pmatrix}\Xi & 0 \\ 0 & \Xi\end{pmatrix}$$

2) Bilinears: $\bar\Psi \Gamma \Psi$

$$\Gamma = \begin{cases} \mathbb{1} & \cdot \text{ scalar} \\ \gamma^\mu & \cdot \text{ vector} \\ \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] & \cdot \text{ tensor} \\ \gamma^\mu \gamma^5 & \cdot \text{ pseudo vector} \\ \gamma^5 & \cdot \text{ pseudo scalar} \end{cases}$$

Example:

$$(j^\mu)' = \bar\Psi' \gamma^\mu \Psi' = \bar\Psi \underbrace{\Lambda_{\frac{1}{2}}^{-1} \gamma^\mu \Lambda_{\frac{1}{2}}}_{\Lambda^\mu{}_\nu \gamma^\nu} \Psi$$

$$= \Lambda^\mu{}_\nu \underbrace{\bar\Psi \gamma^\nu \Psi}_{j^\mu} = \Lambda^\mu{}_\nu j^\mu$$

3.4 Quantization of the Dirac Field

1) $\mathcal{L} = \bar\Psi (i\gamma^\mu \partial_\mu - m) \Psi$

2) Can. momentum: $\Pi_a = i \Psi_a^*$

3) Hamiltonian:

$$H = \int d\vec{x}^3 \, \Psi^+ \underbrace{\left[-i\vec\alpha \nabla + m\beta\right]}_{H_0} \Psi$$

$\vec\alpha^i = \gamma^0 \gamma^i$

$\beta = \gamma^0$

$$\Big[i\gamma^0\partial_0 + \underbrace{i\vec\gamma\nabla - m}_{\gamma^0 H_D}\Big]\psi = 0$$

$\pm E$ $\qquad e^{\pm ipx}$ $\qquad E_p t$

$$\Rightarrow H_D\, \underbrace{u^s(p)e^{i\vec p\vec x}} = E_p \text{ —"—}$$

$$H_D\, \underbrace{v^s(p)e^{-i\vec p\vec x}} = -E_p \text{ —"—}$$

5] Mode expansion:

$$\psi(\vec x) = \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}\Big[a^s_{\vec p}\, u^s(\vec p)e^{i\vec p\vec x} + b^{s\dagger}_{\vec p}\, v(\vec p)e^{-i\vec p\vec x}\Big]$$

6]

$$H_D\psi(\vec x) = \sum_s\int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_p}{2}}\Big[a^s_{\vec p}u_{\vec p}\, e^{ipx} - b^s_{\vec p}v(\vec p)e^{-i\vec p\vec x}\Big]$$

$$H = \int d^3x\, \psi^\dagger H_D\psi \overset{\circ}{=} \sum_s\int\frac{d^3p}{(2\pi)^3}E_{\vec p}\Big[a^{s\dagger}_{\vec p}a^s_{\vec p} - b^{s\dagger}_{\vec p}b^s_{\vec p}\Big]$$

First try: Commutator

7] • $[\psi_a(\vec x), \Pi_b(\vec y)] = i\delta_{ab}\,\delta^{(3)}(\vec x - \vec y)$

• $[\psi_a(\vec x), \psi^\dagger_b(\vec y)] = \delta_{ab}\,\delta^{(3)}(\vec x - \vec y)$

8] Mode algebra $\overset{\circ}{\longrightarrow}$

$$[a^r_{\vec p}, a^{s\dagger}_{\vec q}] = (2\pi)^3\delta^{rs}\delta^{(3)}(\vec p - \vec q)$$

$\quad b \qquad b$

$\Rightarrow$ Irred. Representation

Bosonic Fock space

9] Problem: $(b^{s+}_{\vec p})^N|0\rangle$

Energy: $-N E_{\vec p} \xrightarrow{N\to\infty} -\infty$

$(H = a^\dagger a \ominus b^\dagger b)$

$\hookrightarrow$ No stable vacuum state

10] Fix (?): $b \leftrightarrow b^+$

i) $\psi(\vec x) = [a^s + b^+]$

ii) $H = a^+ a - b^+_{\vec p} b^s_{\vec q}$ + const

iii) $[b^r_{\vec p}, b^{s+}_{\vec q}] = \ominus (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec p - \vec q)$

v) $[H, b^{s+}_{\vec p}] = \underbrace{E_{\vec p}}_{>0} b^+_{\vec p}$

b^+ creates a particle with positive energy

$\rightarrow H \geq 0$

vi) But:

$\| b^{s+}_{\vec p} |0\rangle \|^2 = \langle 0| b b^+ - b^+ b |0\rangle$

$= \langle 0| [b, b^+] |0\rangle$

$= - \underbrace{\langle 0|0\rangle}_{\delta^{(3)}(0)} < 0$

11] • instability of vacuum

• loss of unitarity

$\rightarrow$ No consistent quantization !

Second try: Anticommutator

7] $\boxed{\{\psi_a(\vec x), \psi^+_b(\vec y)\} = \delta_{ab} \delta^{(3)}(\vec x - \vec y)}$

8] $\boxed{\{a^s_{\vec p}, a^{r+}_{\vec q}\} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec p - \vec q)}$

b b

$\rightarrow$ Fermionic Fock space

9) Problem: $b^{s\dagger}_{\vec{p}}|0\rangle$: Energy $-E_{\vec{p}}$

$\rightarrow$ Still no stable vacuum state

10) Fix: $b \leftrightarrow b^\dagger$

i) $H = \sum_s \int \frac{d^3p}{(2\pi)^3} E_p \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - b^s_{\vec{p}} b^{s\dagger}_{\vec{p}} \right)$

$= \sum_s \int \frac{d^3p}{(2\pi)^3} E_p \left[a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right]$

ii) The mode algebra is invariant under $b \leftrightarrow b^\dagger$

$\rightarrow$ Unitarity is preserved and Hamiltonian is lower-bounded

11) Heisenberg picture:

- $e^{iHt} a^s_{\vec{p}} e^{-iHt} \doteq a^s_{\vec{p}} e^{-iE_{\vec{p}}t}$

  b $\qquad$ b

  $\psi(x) = e^{iHt} \psi(\vec{x}) e^{-iHt}$

- $\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a^s_{\vec{p}} u^s(p) e^{-ipx} + b^{s(\dagger)}_{\vec{p}} v^s(p) e^{ipx} \right]$

  $\bar{\psi}(x)$

$\rightarrow Q = \int d^3x\, \psi^\dagger \psi \doteq \sum_s \int \frac{d^3p}{(2\pi)^3} \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right)$

up to ∞

Continuous Symmetries:

- Time translation $\leftrightarrow$ Hamiltonian
- Spatial translations $\leftrightarrow$ Momentum operator

$\vec{P} \doteq \int d^3x\, \psi^\dagger (-i\nabla) \psi \doteq \sum_s \int \frac{d^3p}{(2\pi)^3}\, \vec{p} \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right)$

- Rotations $\leftrightarrow$ Angular momentum operator
- Global phase rotation $e^{i\alpha}\psi$

$\hookrightarrow$ Conserved current $j^\mu = \bar{\psi} \gamma^\mu \psi$

$\hookrightarrow$ Conserved charge. $\rightarrow$

Excitations = Particles

$a^{s\dagger}_{\vec p}|0\rangle$: Fermion
- energy $E_{\vec p}$
- momentum $\vec p$
- spin $J = \frac{1}{2}$ (polarizations)
- charge $Q = +1$

$b^{s\dagger}_{\vec p}|0\rangle$: Antifermion
- energy $E_{\vec p}$
- momentum $\vec p$
- spin $J = \frac{1}{2}$ (polarizations s)
- Charge $Q = -1$

Lorentz transformations

$$\boxed{U(\Lambda)\Psi(x)U^{-1}(\Lambda) = \Lambda^{-1}_{\frac{1}{2}}\Psi(\Lambda x)}$$

1) $\Lambda \in SO^+(1,3)$

$|\vec p, s\rangle := \sqrt{2E_{\vec p}}\, a^{s\dagger}_{\vec p}|0\rangle$

$|\vec p, s\rangle \longmapsto \underbrace{U(\Lambda)}_{\text{representation of } SO^+(1,3) \text{ on Fock space}}|\vec p, s\rangle$

2) $\measuredangle$ Special case.

→ Spin pol do not mix

$$\boxed{U(\Lambda)a^s_{\vec p}U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\vec p}}{E_{\vec p}}}\, a^s_{\Lambda\vec p}}$$

3)

$\langle \vec p s, \vec q, r\rangle = 2E_{\vec p}(2\pi)^3\delta^{(3)}(\vec p' - \vec q')\delta^{rs}$ 1)

$= \langle \vec p', s|\vec q', r\rangle = \langle p, s|\underbrace{U^\dagger(\Lambda)U(\Lambda)}|q, r\rangle$

→ $U(\Lambda)$ are unitary

$SO^+(1,3) \longrightarrow \mathcal{A}(\mathcal{H})$ Fock space

4)
- $\boxed{\Lambda}$ acts on 4-vectors in $\mathbb{R}^{1,3}$
  $D = 4$ → not unitary
- $\boxed{\Lambda_{\frac{1}{2}}}$ acts on bispinors $\mathbb{C}^2 \oplus \mathbb{C}^2$
  $D = 4$ → not unitary
- $\boxed{U(\Lambda)}$ acts on states in fermionic Fock space
  $D = \infty$ → unitary