3. The Dirac Field

3.1. The Dirac Equation

1] Observation I:

i) $\forall$ $x' = \Lambda x$, $\phi'(x') = \phi(x)$

ii) $\forall$ $(\partial^2 + m^2)\phi(x) = 0$

iii) $\phi'(x) = \phi(\Lambda^{-1}x)$ is a new solution

Proof: $(\partial^2 + m^2)\phi'(x) = \dots \overset{!}{=} 0$

2] Observation II:

$\forall$ Vector fields: $\vec{\phi}'(x) = R\,\vec{\phi}(R^{-1}x)$

$$\phi'_a(x) = M_{ab}(\Lambda)\,\phi_b(\Lambda^{-1}x) \qquad a,b = 1,\dots,n$$

where $\underbrace{M(\Lambda')M(\Lambda)}\,\phi(\Lambda^{-1}\Lambda'^{-1}x) \overset{!}{=} \underbrace{M(\Lambda'\Lambda)}\,\phi((\Lambda'\Lambda)^{-1}x)$

$\rightarrow$ M n-dimensional representation of the Lorentz group

$(M: SO^+(1,3) \longrightarrow \mathrm{End}(V))$

3] Goal: first-order relativistic field equation

$$(i\Box^\mu\partial_\mu + \text{const})\,\phi = 0 \qquad \vec{\gamma} = \begin{pmatrix} \gamma^0 \\ \vdots \end{pmatrix}$$

$$\boxed{M^{-1}(\Lambda)\,\gamma^\mu\,M(\Lambda) = \Lambda^\mu{}_\nu\,\gamma^\nu} \quad (*) \Leftarrow$$

4] i) $\forall$ $x' = \Lambda x$, $\phi'(x') = M(\Lambda)\phi(x)$

ii) $\forall$ ϕ: $(i\Box^\mu\partial_\mu + \text{const})\,\phi(x) = 0$ $\forall x$

iii) $\phi'(x) = M(\Lambda)\,\phi(\Lambda^{-1}x)$ new solution?

$(i\Box^\mu\partial_\mu + \text{const})\,\phi'(x)$

$= \left(i\Box^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu + \text{const}\right) M(\Lambda)\,\phi(x)\Big|_{x=\Lambda^{-1}x} \overset{!}{=} 0$

$\overset{M^{-1}(\Lambda)}{\Longrightarrow} \left[i\,\underbrace{M^{-1}(\Lambda)\,\Box^\mu M(\Lambda)(\Lambda^{-1})^\nu{}_\mu}_{\overset{!}{=}\,\Box^\nu}\,\partial_\nu + \text{const}\right]\phi(x)\Big|_{x=\Lambda^{-1}x} = 0$

$\rightarrow$ $\Box^\mu \equiv \gamma^\mu$ must be $n\times n$ matrices

5] $SO^+(1,3)$ is a Lie group.

- $\Lambda = \exp\left[-\frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}\right]$ $\Lambda \in SO^+(1,3)$

$\approx \mathbb{1} - \frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}$

- $M(\Lambda) = \exp\left[-\frac{i}{2}\omega_{\alpha\beta} S^{\alpha\beta}\right]$

$\approx \mathbb{1} - \frac{i}{2}\omega_{\alpha\beta} S^{\alpha\beta}$

$(\mathcal{J}^{\alpha\beta})_{\mu\nu} = i(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu)$

- Infinitesimal form of (*):

$$\boxed{\begin{aligned}[\gamma^\mu, S^{\alpha\beta}] &\overset{!}{=} (\mathcal{J}^{\alpha\beta})^\mu{}_\nu \gamma^\nu \\ &= i(g^{\alpha\mu}\gamma^\beta - g^{\beta\mu}\gamma^\alpha)\end{aligned}} \quad (1)$$

- $\mathcal{J}^{\alpha\beta} \to$ Lie algebra of Lorentz group:

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] \overset{!}{=} i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho})} \quad (2)$$

$J \in \{\mathcal{J}, S\}$

- Solution: Dirac: & γ^μ

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_{n\times n}}$$

Dirac algebra

- If: $\boxed{S^{\mu\nu} := \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$

$\Rightarrow$ (1) ✓ (2) ✓

7] Representations.

- At least 4-dimensional
- All 4-D reps are unitarily equivalent

Weyl representation.

$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

• Henceforth: $\Lambda_{\frac{1}{2}} = M(\Lambda)$ $(n=4)$

8| $$\boxed{(i\gamma^\mu \partial_\mu - m)\,\psi_{(x)} = 0}$$

Dirac equation

ψ: Bispinor field

$\psi(x) \in \mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$

9| $$\underbrace{(-i\gamma^\mu \partial_\mu - m)(i\gamma^\nu \partial_\nu - m)}_{\hat{=}\,(\partial^2 + m^2)}\psi = 0$$

10| Dirac adjoint:

i) First try: $\psi'^{\dagger}\psi' = \psi^\dagger \underbrace{\Lambda^\dagger_{\frac{1}{2}}\Lambda_{\frac{1}{2}}}_{\neq \mathbb{1}}\psi \neq \psi^\dagger\psi$

ii) $$\boxed{\overline{\psi} = \psi^\dagger \gamma^0}$$ Dirac adjoint

$\rightarrow$ $\overline{\psi}'\psi' = \overline{\psi}\,\underbrace{\Lambda^{-1}_{\frac{1}{2}}\Lambda_{\frac{1}{2}}}_{\mathbb{1}}\,\psi = \overline{\psi}\psi$

11| Lagrangian:

$$\boxed{\mathcal{L}_{\text{Dirac}} = \overline{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi}$$

Note 3.1

• $\sigma^\mu = (\mathbb{1}, \vec{\sigma})^T$

$\bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})^T$

$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$

$\rightarrow \begin{pmatrix} -m & i\sigma\partial \\ i\bar{\sigma}\partial & -m \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$

$\mathbb{C}^2$ $\mathbb{C}^2$

ψ_L, ψ_R left / right handed Weyl spinors

$m=0$ $\quad i\bar\sigma\partial\psi_L=0$, $\quad i\sigma\partial\psi_R=0$ } Weyl equations

3.7 Free Particle Solutions of the Dirac Equation

1.) $(\partial^2+m^2)\psi=0$

$p^2 \overset{!}{=} m^2, \quad p^0>0$

$E_p=p^0=\sqrt{\vec p^{\,2}+m^2}$

$$\psi^\pm(x)=\underset{\in\mathbb{C}^4}{\psi^\pm(p)}\,e^{\mp ipx}$$

2.) $(\pm\gamma^\mu p_\mu-m)\psi^\pm(p)$

$$=\begin{pmatrix}-m & \pm p\sigma\\ \pm p\bar\sigma & -m\end{pmatrix}\begin{pmatrix}\psi_L^\pm\\ \psi_R^\pm\end{pmatrix}\overset{!}{=}0$$

$\boxed{s=1,2 \quad \xi^1=\begin{pmatrix}1\\0\end{pmatrix} \quad \xi^2=\begin{pmatrix}0\\1\end{pmatrix}}$

3.) Note

- $(p\sigma)(p\bar\sigma)=p^2=m^2$
- Eigenvalues $\sqrt{\vec p^{\,2}+m^2}$: $p^0\pm|\vec p|\Rightarrow$ $p^0>0$, $m>0$

4.) $\psi_L^\pm=\sqrt{p\sigma}\,\xi^\pm$, $\xi^\pm\in\mathbb{C}^2$, $\|\xi^\pm\|=1$

$-m\sqrt{p\sigma}\,\xi^\pm\pm p\sigma\,\psi_R^\pm=0$

$\sqrt{p\sigma}\sqrt{p\bar\sigma}=m$

$\Leftrightarrow \psi_R^\pm=\pm\frac{m}{\sqrt{p\sigma}}\xi^\pm=\pm\sqrt{p\bar\sigma}\,\xi^\pm$

5.) $\boxed{\text{Solutions:}}$

$$\psi^+(x)=\underbrace{\begin{pmatrix}\sqrt{p\sigma}\,\xi^s\\ \sqrt{p\bar\sigma}\,\xi^s\end{pmatrix}}_{u_s(p)}e^{-ipx} \quad \text{pos. freq. sol.}$$

$$\psi^-(x)=\underbrace{\begin{pmatrix}\sqrt{p\sigma}\,\eta^s\\ -\sqrt{p\bar\sigma}\,\eta^s\end{pmatrix}}_{v_s(p)}e^{+ipx} \quad \text{neg. freq. sol.}$$