Causality

* Amplitude for particle to propagate from y to x:

$$\boxed{D(x-y) = \langle 0|\phi(y)\,\phi(x)|0\rangle \doteq \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\,e^{-ip(x-y)}}$$

* $D(\Lambda(x-y)) = D(x-y)$

$\Lambda \in SO^+(1,3)$

1] * Time-like distance:

$x^0 - y^0 = t \qquad \vec{x} - \vec{y} = 0$

$$D(x-y) = \frac{4\pi}{(2\pi)^3}\int_0^\infty dp\,\frac{p^2}{2\sqrt{p^2+m^2}}\,e^{-i\sqrt{p^2+m^2}\,t}$$

$$= \frac{1}{4\pi^2}\int_m^\infty dE\,\sqrt{E^2-m^2}\,e^{-iEt}$$

$\neq 0 \quad \Rightarrow \quad$ Propagation possible

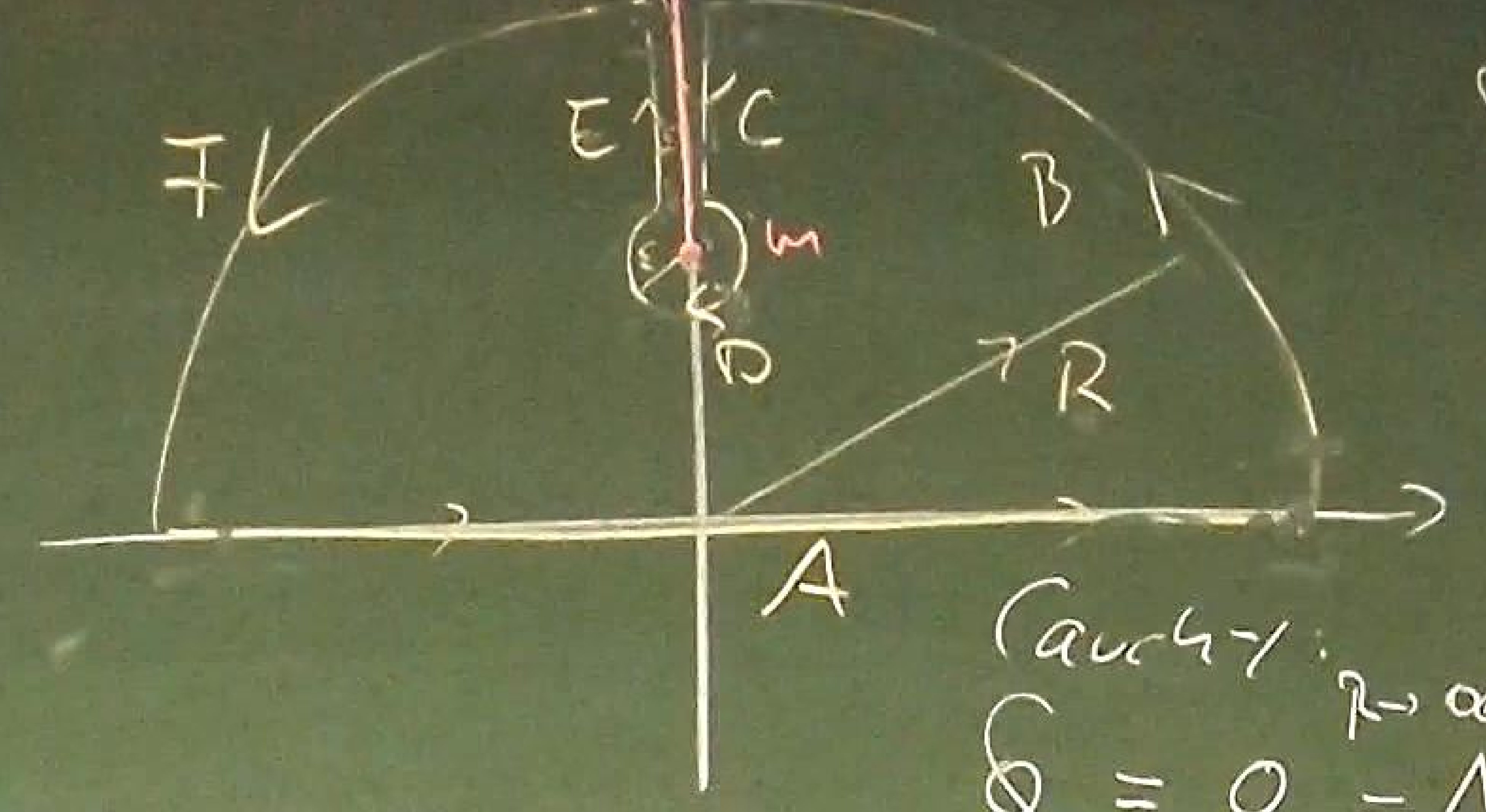
2] * Space like distance:

$x^0 - y^0 = 0 \quad$ and $\quad \vec{x} - \vec{y} = \vec{r}$

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\,e^{i\vec{p}\vec{r}} \qquad \vec{p}\vec{r} = p\,r\,\overbrace{\cos\theta}^{u}$$

$$= \frac{2\pi}{(2\pi)^3}\int_0^\infty dp\,\frac{p^2}{2E_p}\,\frac{e^{ipr}-e^{-ipr}}{ipr}$$

$$= \frac{-i}{2(2\pi)^2 r}\int_{-\infty}^{\infty} dp\,\frac{p\,e^{ipr}}{\sqrt{p^2+m^2}}$$



$p \in \mathbb{C}$

Cauchy:

$$\oint \ldots = 0 \overset{R\to\infty}{=} A + C + \underset{C}{\underset{\|}{E}}$$

$$D(x-y) = -C - E = -2C$$

$$= \frac{-i}{(2\pi)^2 r}\int^{i\infty} dp\,\frac{p\,e^{ipr}}{\sqrt{p^2+m^2}}$$

$p = -i\rho$

$$= \frac{1}{4\pi^2 r}\int_{m}^{\infty} d\rho\,\frac{\rho\,e^{-\rho r}}{\sqrt{\rho^2-m^2}}$$

$$\xrightarrow{r\to\infty} e^{-mr}$$

→ Vanishes exponentially in r

3.) & Measurements: A, B

can affect each other iff $[A, B] \neq 0$

$$[\phi(x), \phi(y)] \qquad \Rightarrow A = \phi(x),\ B = \phi(y)$$

$$\stackrel{?}{=} \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}\left[e^{-ip(x-y)} - e^{ip(x-y)}\right] \cdot \mathbb{1}}_{\in \text{ number}}$$

$$= D(x-y) - D(y-x)$$

• Let $(x-y)^2 < 0$

$\to \exists\, \Lambda^* \in SO^+(1,3)$

$$\Lambda^*(x-y) = -(x-y)$$

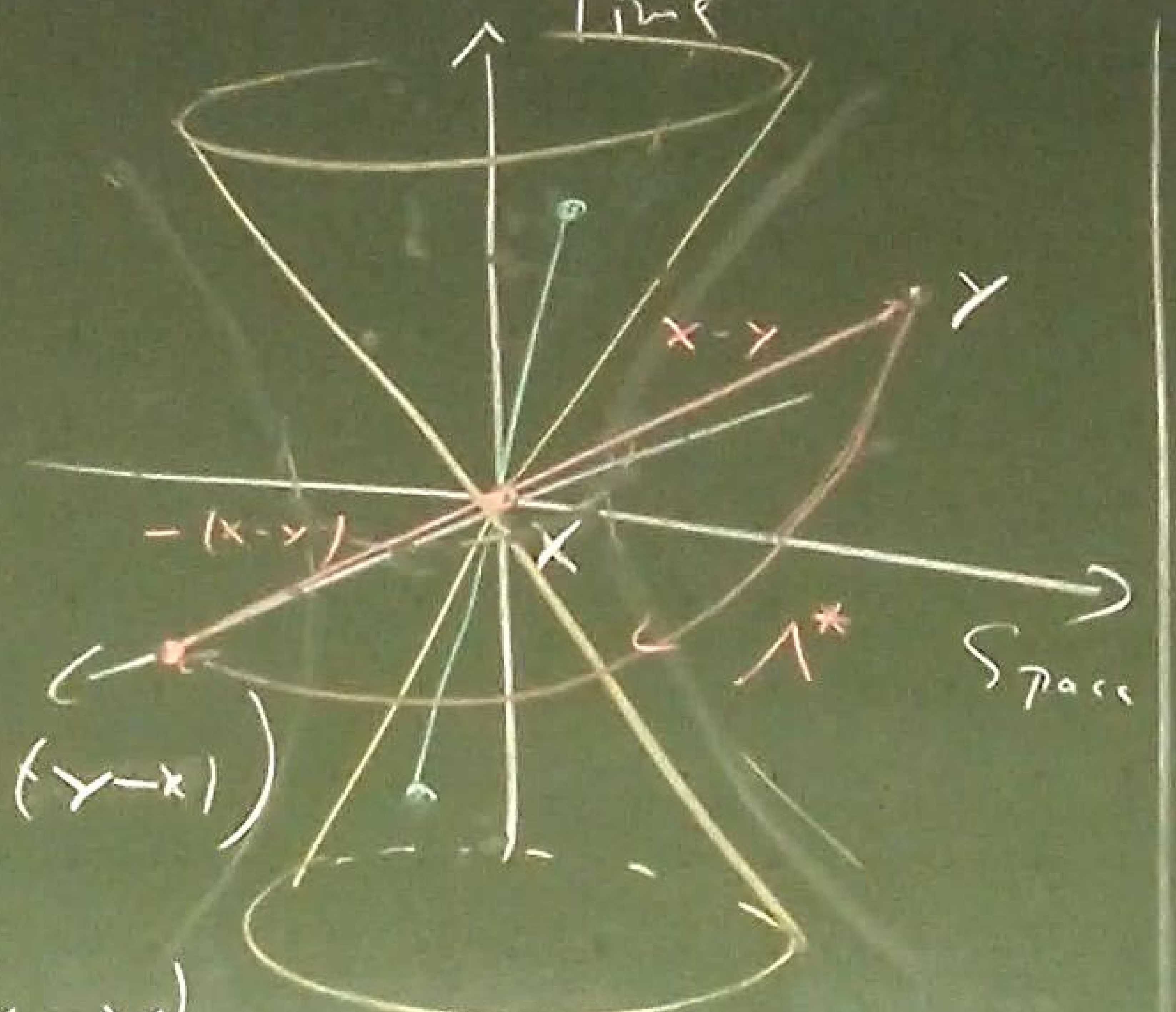
$\Rightarrow$

$$[\phi(x), \phi(y)]$$

$$= D(x-y) - D(\Lambda^*(y-x))$$

$$= D(x-y) - D(x-y)$$

$$= \underline{\underline{0}}$$

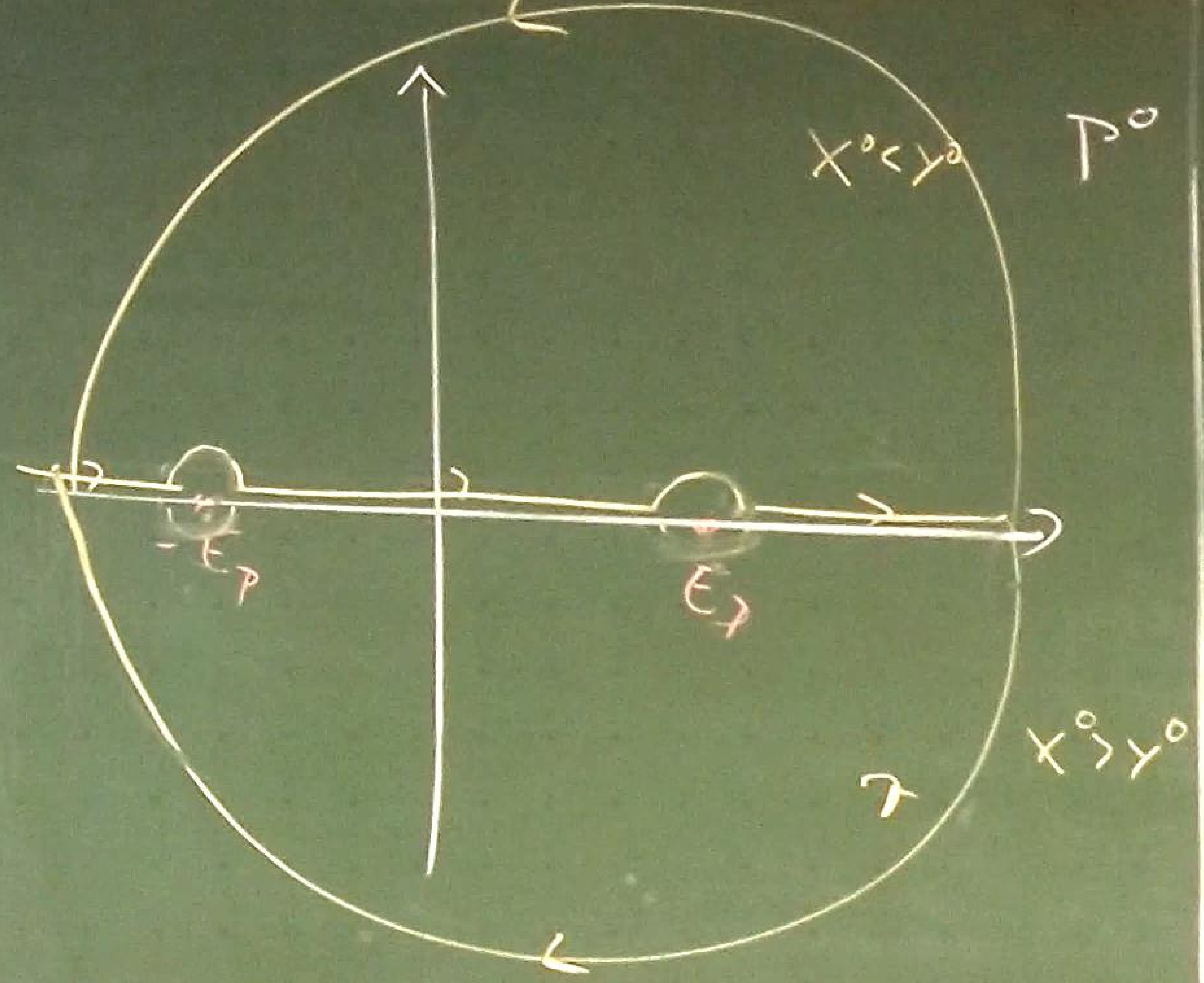


The Propagator

• $$\langle 0|[\phi(x), \phi(y)]|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}\left[e^{-ip(x-y)} - e^{ip(x-y)}\right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \left[\frac{e^{-ip(x-y)}}{2E_p}\Bigg|_{p^0 = E_p} + \frac{e^{-ip(x-y)}}{-2E_p}\Bigg|_{p^0 = -E_p} \right]$$

$\vec{p} \to -\vec{p}$

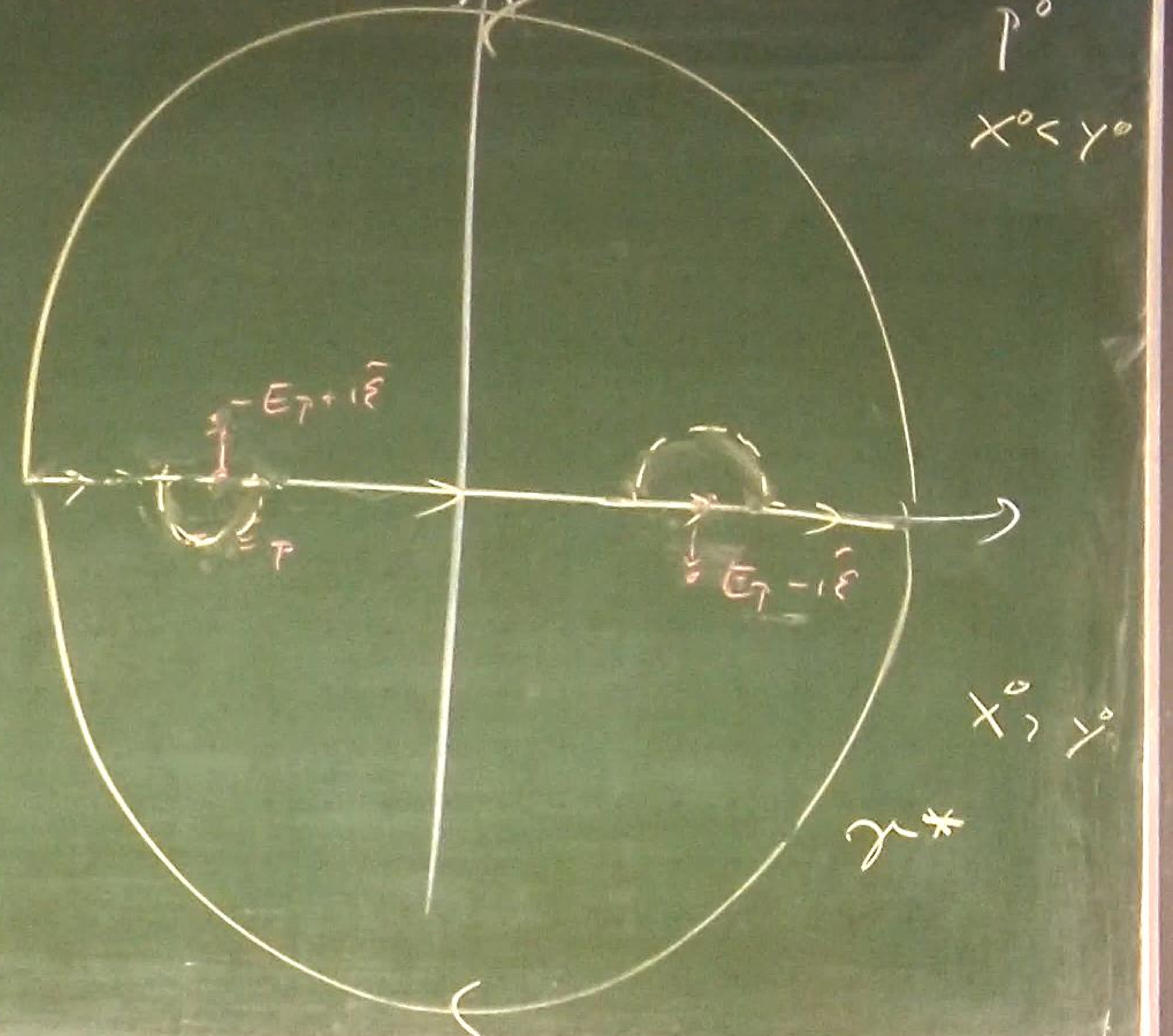


$$\boxed{\begin{aligned}(*) &= D_R(x-y)\\ &= \theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle\end{aligned}}$$

2] Interpretation:

$$(\partial^2+m^2)\,D_R(x-y) \overset{\circ}{=} -i\,\delta^{(4)}(x-y)$$

$\Rightarrow$ Retarded Green's function of KG operator



$$D_F(x-y) \equiv (*)_{\gamma \to \gamma^*}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$

Feynman propagator (of the KG field)

$A(H)$ H

$\phi(x)$ $T(P Q)$

$$\Rightarrow D_F(x-y) = \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & x^0 < y^0 \end{cases}$$

$$= \theta(x^0 - y^0) \langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0 - x^0)\langle 0|\phi(y)\phi(x)|0\rangle$$

$$\equiv \langle 0| T(\phi(x)\phi(y))|0\rangle$$

Time-ordering operator T

...

The Propagator

1). $\langle 0|[\phi(x), \phi(y)]|0\rangle = (D(x-y) - D(y-x))$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}\left[e^{-ip(x-y)} - e^{ip(x-y)}\right]$$

$$= \int \frac{d^3p}{(2\pi)^3}\left[\frac{e^{-ip(x-y)}}{2E_p}\Big|_{p^0=E_p} + \frac{e^{-ip(x-y)}}{-2E_p}\Big|_{p^0=-E_p}\right]_{\vec{p}\to-\vec{p}}$$

Residue Theorem $\uparrow$

$$\overset{x^0>y^0}{=} \int \frac{d^3p}{(2\pi)^3}\int_\gamma \frac{dp^0}{(2\pi i)} \frac{-1}{\underbrace{p^2 - m^2}_{(p^0 - E_{\vec{p}})(p^0 + E_{\vec{p}})}} e^{-ip(x-y)}$$

$$E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$$

$$= \int_\gamma \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip(x-y)} \quad (*)$$