2. The Klein-Gordon Field

2.1. Canonical Quantization

1. Theory:

$\phi = \phi^*$

i) Real field $\phi(x)$

ii) Lagrangian: $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$

iii) EOM: $(\partial^2 + m^2)\phi = 0$

iv) Hamiltonian: $\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} m^2 \phi^2$

2. Canonical quantization:

$$\boxed{\begin{aligned} [\phi(\vec{x}), \phi(\vec{y})] &= 0 \\ [\pi(\vec{x}), \pi(\vec{y})] &= 0 \\ [\phi(\vec{x}), \pi(\vec{y})] &= i\delta^{(3)}(\vec{x}-\vec{y}) \end{aligned}} \qquad (3)$$

- $\phi^\dagger = \phi$, $\pi^\dagger = \pi$

3. Goals:

- Representation of field operators
- Spectrum of Hamiltonian
- Time evolution

4. Motivation:

i) FT KG equation in space.

$$\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\vec{x}} \tilde{\phi}(\vec{p},t)$$

KG $\hookrightarrow \left(\partial_t^2 + \underbrace{|\vec{p}|^2 + m^2}_{\omega_p^2}\right)\tilde{\phi}(\vec{p},t) = 0$

$\rightarrow$ Decoupled Harmonic oscillators.

and constraint $\tilde{\phi}^*(\vec{p},t) = \tilde{\phi}(-\vec{p},t)$

ii) $\mathcal{L}$ $H_{SHO} = \frac{1}{2}\tilde{\pi}^2 + \frac{1}{2}\omega_p^2 \tilde{\phi}^2$

$\tilde{\phi} = \frac{1}{\sqrt{2\omega}}(a + a^\dagger)$

$\tilde{\pi} = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$

with $[a, a^\dagger] = 1$

$\Rightarrow H_{SHO} = \omega\left(a^\dagger a + \frac{1}{2}\right)$

5. Field operators:

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} (a_{\vec{p}} + a^\dagger_{-\vec{p}}) e^{i\vec{p}\vec{x}}$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(a_{\vec{p}} e^{i\vec{p}\vec{x}} + a^\dagger_{\vec{p}} e^{-i\vec{p}\vec{x}}\right) \quad (1)$$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} -i\sqrt{\frac{\omega_{\vec{p}}}{2}} (a_{\vec{p}} - a^\dagger_{-\vec{p}}) e^{i\vec{p}\vec{x}}$$

$$(\phi^\dagger = \phi, \quad \pi^\dagger = \pi)$$

with momentum modes

$$\boxed{[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3 \delta(\vec{p} - \vec{q})} \quad (2)$$

Check: (1) + (2) => (3) $\quad \omega_p = \sqrt{|\vec{p}|^2 + m^2} = E_p$

6. Hamiltonian:

$$H \overset{\circ}{=} \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \omega_p \left(a^\dagger_{\vec{p}} a_{\vec{p}} + \cancel{\frac{1}{2}[a_{\vec{p}}, a^\dagger_{\vec{p}}]}\right)$$

$\delta(0) = \infty$

($\to$ Normal ordering)

7. Eigenstates + Spectrum.

i) $\overset{\circ}{\longrightarrow} [H, a^\dagger_{\vec{p}}] = \omega_{\vec{p}} a^\dagger_{\vec{p}}$

ii) Vacuum $|0\rangle \longrightarrow$ Eigenstates

$a_{\vec{p}}|0\rangle = 0 \ \forall \vec{p}$ $\qquad a^\dagger_{\vec{p}} a^\dagger_{\vec{q}} \dots |0\rangle$

iii) Energy: $E_p = \omega_p = \sqrt{|\vec{p}|^2 + m^2}$

iv) Kinetic momentum

$$P^i = \int d^3x \, \pi(\vec{x})(-\partial_i \phi(\vec{x}))$$

$$\overset{\circ}{=} \int \frac{d^3p}{(2\pi)^3} p_i \, a^\dagger_{\vec{p}} a_{\vec{p}}$$

v) Statistics: $\quad a^\dagger_{\vec{p}} a^\dagger_{\vec{q}} |0\rangle = a^\dagger_{\vec{q}} a^\dagger_{\vec{p}} |0\rangle$

$\to$ Excitations $a^\dagger_{\vec{p}}$ commute and carry additive energy + momentum.

$\to$ Bosonic particles

8. Normalization:

i) $\forall\ \Lambda = R' L_3(p) R \in SO^+(1,3)$ $\quad \frac{v}{c}$

$$p' = (E_{p'}, \vec{p}\,') = \Lambda\, p_{(E_p, \vec{p})}, \qquad E_p = \sqrt{|\vec{p}|^2 + m^2}$$

ii) Jacobian in space

$$\det\left(\frac{\partial \vec{p}\,'}{\partial \vec{p}}\right) \doteq \frac{dp'_3}{dp_3} \doteq \frac{E_{p'}}{E_p}$$

$$\to\ \underset{\text{not LI}}{\delta^{(3)}(\vec{p}-\vec{q})} = \frac{E_{p'}}{E_p}\, \delta^{(3)}(\vec{p}\,'-\vec{q}\,')$$

$\to\ E_p\, \delta(\vec{p}-\vec{q})$ is LI

iii) Single particle eigenstates.

$$|\vec{p}\rangle = \boxed{\sqrt{2E_{\vec{p}}}}\, a^\dagger_{\vec{p}} |0\rangle$$

$$\Rightarrow\ \langle \vec{p} | \vec{q} \rangle = \underbrace{(2\pi)^3\, 2 E_{\vec{p}}\, \delta(\vec{p}-\vec{q})}_{\text{LI}}$$

9. Lorentz transformations $\quad \Lambda \in SO^+(1,3)$

$$U(\Lambda)|\vec{p}\rangle := |\Lambda \vec{p}\rangle$$

$$\Leftrightarrow\ U(\Lambda)\, a^\dagger_{\vec{p}}\, U^\dagger(\Lambda) = \sqrt{\frac{E_{\Lambda \vec{p}}}{E_{\vec{p}}}}\; a^\dagger_{\Lambda \vec{p}} \qquad \Lambda^i{}_\mu p^\mu,\ i = 1,2,3$$

10. Interpretation of $\phi(\vec{x})$:

$$\phi(\vec{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-i\vec{p}\vec{x}} |\vec{p}\rangle$$

$|\vec{p}| \ll m \Rightarrow E_p \approx$ const.

$\to\ \phi(\vec{x})$ creates particle at $\vec{x}$

$$\left(\langle 0 | \phi(x) | \vec{p}\rangle = e^{i\vec{p}\vec{x}}\right)$$

Note 2.1:

- Projector on SP sector:

$$\mathbb{1}_1 = \int \frac{d^3p}{(2\pi)^3} |\vec{p}\rangle \frac{1}{2E_{\vec{p}}} \langle \vec{p} |$$

$\forall\ f(p)$ LI ($f(p) = f(p')$)

- $\displaystyle \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} f(p) \Rightarrow$ LI

2.2. The Klein-Gordon Field in Space-Time

1. Heisenberg operators:

$$\phi_{\pi}(\vec{x},t) = \phi_{\pi}(x) = e^{iHt}\,\phi_{\pi}(\vec{x})\,e^{-iHt}$$

2. Heisenberg equation:

$$i\partial_t \underset{\phi,\pi}{O} = [O, H]$$

$(\vec{y},t)$

- $i\partial_t\phi(x) = \left[\phi(x), \int d^3y \left\{\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right\}\right]$

$$\overset{\circ}{=} i\pi(x) \qquad \boxed{\pi(x) = \partial_t\phi(x)} \quad (*)$$

- $i\partial_t\pi(x) \overset{\circ}{=} -i(-\nabla^2 + m^2)\phi(x) \quad (**)$

$(*) + (**) \Rightarrow$

$$\boxed{(\partial_t^2 - \nabla^2 + m^2)\,\phi(x) = 0}$$

Klein Gordon equation

3. Time evolution of modes

$$e^{iHt}\,a_{\vec{p}}^{+}\,e^{-iHt} = a_{\vec{p}}^{+}\,e^{\pm iE_{\vec{p}}t}$$

4. Field operators.

$$\boxed{\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}\,e^{-ip\cdot x} + a_{\vec{p}}^{+}\,e^{ip\cdot x}\right)\Big|_{p^0 = E_{\vec{p}}}}$$

$$\pi(x) = \partial_t\phi(x)$$

$(x-y)^2 \geq 0$

$\langle 0|\,\phi(x)\,\phi(y)\,|0\rangle$

Note 2.2

1. $\phi(\vec{x},t) = e^{iHt}\,\phi(\vec{x},0)\,e^{-iHt}$ (Translation in time)

- $\phi(\vec{x}) = e^{-i\vec{P}\cdot\vec{x}}\,\phi_s(\overset{\vec{x}}{\overset{\shortparallel}{0}})\,e^{i\vec{P}\cdot\vec{x}}$

- $\phi(x) = e^{iP\cdot x}\,\phi(\overset{x}{\overset{\shortparallel}{0}})\,e^{-iP\cdot x}$

$P^{\mu}x_{\mu}$

Note 2.3

- $e^{\mp ip\cdot x} \leftrightarrow$ pos. (neg.) freq. solutions of KG equation

- $\leftrightarrow$ annihilation op. $a_{\vec{p}}$ (creation $a_{\vec{p}}^{+}$)