Recap:

9.2 The Yang-Mills Lagrangian

Goal: Gauge theories for non-abelian Lie groups $G = SU(2), SU(3), \dots$

1] Lie group G with defining representation:
V: unitary $n \times n$ matrix
↑ dimension of the representation

2] V acts on n-plets of Dirac spinor fields.

$$\tilde{\psi}(x) = V(x)\,\psi(x)$$

$V: \mathbb{R}^{3,1} \to G$ local gauge transformation

$\psi(x) = (\psi_1, \dots, \psi_n)^T$, Dirac bispinor

3] Lie algebra $\mathfrak{g}$ describes Lie group G:

$t^a \in \mathfrak{g}$: $n \times n$ Hermitian matrix, generators ($a = 1, \dots, N$) ← Dimension of Lie group

$$\boxed{[t^a, t^b] = i f^{abc} t^c}$$

Lie bracket (here: commutator)

$f^{abc} \in \mathbb{C}$: structure constants

$$\rightarrow \underbrace{V(x) = \exp[i \underbrace{\alpha^a(x) t^a}_{\in \mathfrak{g}}]}_{\in G} = 1 + i\alpha^a(x) t^a + O(\alpha^2)$$

$\alpha^a: \mathbb{R}^{1,3} \to \mathbb{R}$ "Real Lie algebra"

Example: $G = SU(2)$
$= \{ 2\times 2$ unitary matrices with determinant $1 \}$

- $n = 2$, $N = 3$
- $t^i = \frac{\sigma^i}{2}$ ← Pauli matrices, x, y, z; $i = 1, 2, 3$
- $[t^i, t^j] = i \varepsilon^{ijk} t^k$, $\varepsilon^{ijk} = f^{ijk}$ for $SU(2)$

Representation with $n = 3$
⇓
Spin 1

4| "Comparator".

$$\tilde U(x,y) = V(y)\, U(y,x)\, V^\dagger(x)$$

n×n unitary matrix

$$U(y,y) = \mathbb{1}$$

$$\to U(x+\varepsilon n, x) = 1 + ig\varepsilon n^\mu A^a_\mu t^a + O(\varepsilon^2)$$

g: arbitrary constant

A^a_μ: N vector fields

5| Covariant derivative:

$$\boxed{D_\mu \overset{\circ}{=} \partial_\mu - ig A^a_\mu t^a}$$

$A_\mu \leftarrow$ n×n matrix

6| Transformation of A^a_μ:

i)

$$\boxed{\tilde A^a_\mu t^a \overset{\circ}{=} V(x)\left[A^a_\mu t^a + \frac{i}{g}\partial_\mu\right] V^\dagger(x)}$$

only

Valid for all $V \in G$

$D_\mu \psi$ transforms like ψ

$\Downarrow$

$\bar\Psi \not{D} \psi$

ii) ✗ Infinitesimal gauge trafo:

$$V \approx \mathbb{1} + \alpha \ldots + O(\alpha^2)$$

$$\boxed{\tilde A^a_\mu \overset{\circ}{\approx} A^a_\mu + \frac{1}{g}\partial_\mu \alpha^a + \boxed{f^{abc} A^b_\mu \alpha^c}}$$

Infinit. transformation is $O(\alpha)$

new for non-abelian groups

7|

$$\boxed{\tilde D_\mu \tilde \psi \overset{\circ}{=} V D_\mu \psi} \qquad (\partial_\mu V^\dagger)\, V = -V^\dagger(\partial_\mu V)$$

$\Leftarrow$

$$0 = \partial_\mu \mathbb{1} = \partial_\mu(V^\dagger V) = (\partial_\mu V^\dagger)V + V^\dagger(\partial_\mu V)$$

8) Kinetic term for A_μ^a

i) $\tilde{D}_\mu \underbrace{\tilde{D}_\nu \tilde{\Psi}}_{\tilde{\phi}} = V D_\mu D_\nu \Psi$

$\Rightarrow \underbrace{[\tilde{D}_\mu, \tilde{D}_\nu]}\tilde{\Psi} = V[D_\mu, D_\nu]\Psi$

$= V[D_\mu, D_\nu]V^\dagger \tilde{\Psi}$

ii) On the other hand:

$-ig F^a_{\mu\nu} t^a := [D_\mu, D_\nu]$

with

$$\boxed{F^a_{\mu\nu} \hat{=} \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu}$$

$F^a_{\mu\nu}$: N field-strength tensors

$F_{\mu\nu} = F^a_{\mu\nu} t^a$: $n \times n$ matrix

iii) $$\boxed{\tilde{F}_{\mu\nu} = \tilde{F}^a_{\mu\nu} t^a = V F_{\mu\nu} V^\dagger}$$

$\rightarrow F_{\mu\nu}$ is no longer gauge invariant

iv)

$$\boxed{\begin{aligned} \mathcal{L}_{YM} &= -\frac{1}{2}\mathrm{Tr}[F^2] \\ &= -\frac{1}{2}\mathrm{Tr}[F^a_{\mu\nu} t^a F^{\mu\nu b} t^b] \\ &= -\frac{1}{2}(F^a_{\mu\nu} F^{\mu\nu b}) \underbrace{\mathrm{Tr}[t^a t^b]}_{\frac{1}{2}\delta^{ab}} \\ &= -\frac{1}{4}(F^a_{\mu\nu} F^{\mu\nu a}) \\ &= -\frac{1}{4}(F^a_{\mu\nu})^2 \end{aligned}}$$

Yang-Mills Lagrangian

Note 9.1

$$F^2 \sim (\partial A)^2 + f(\partial A)AA + f^2 AAAA$$

→ Interacting QFT for $f \neq 0$

↑ non-abelian

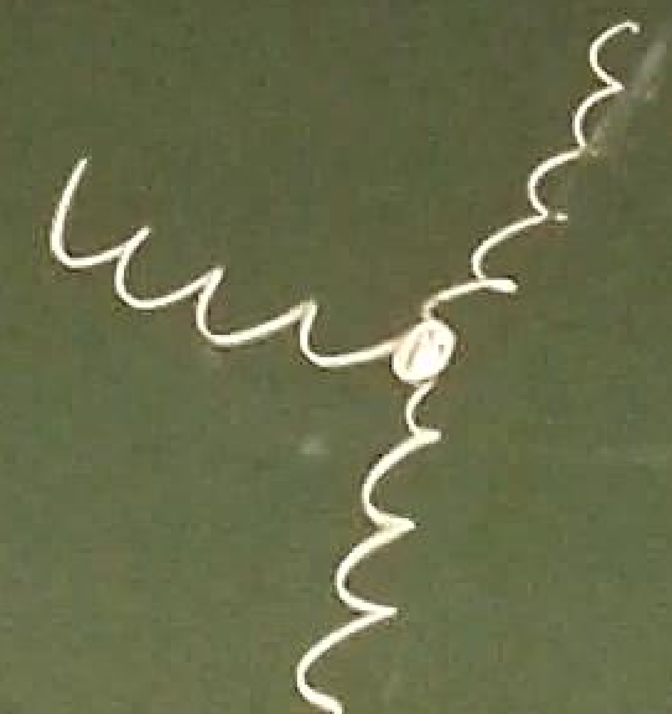
→ Gauge bosons can interact

Example: $G = SU(3)$

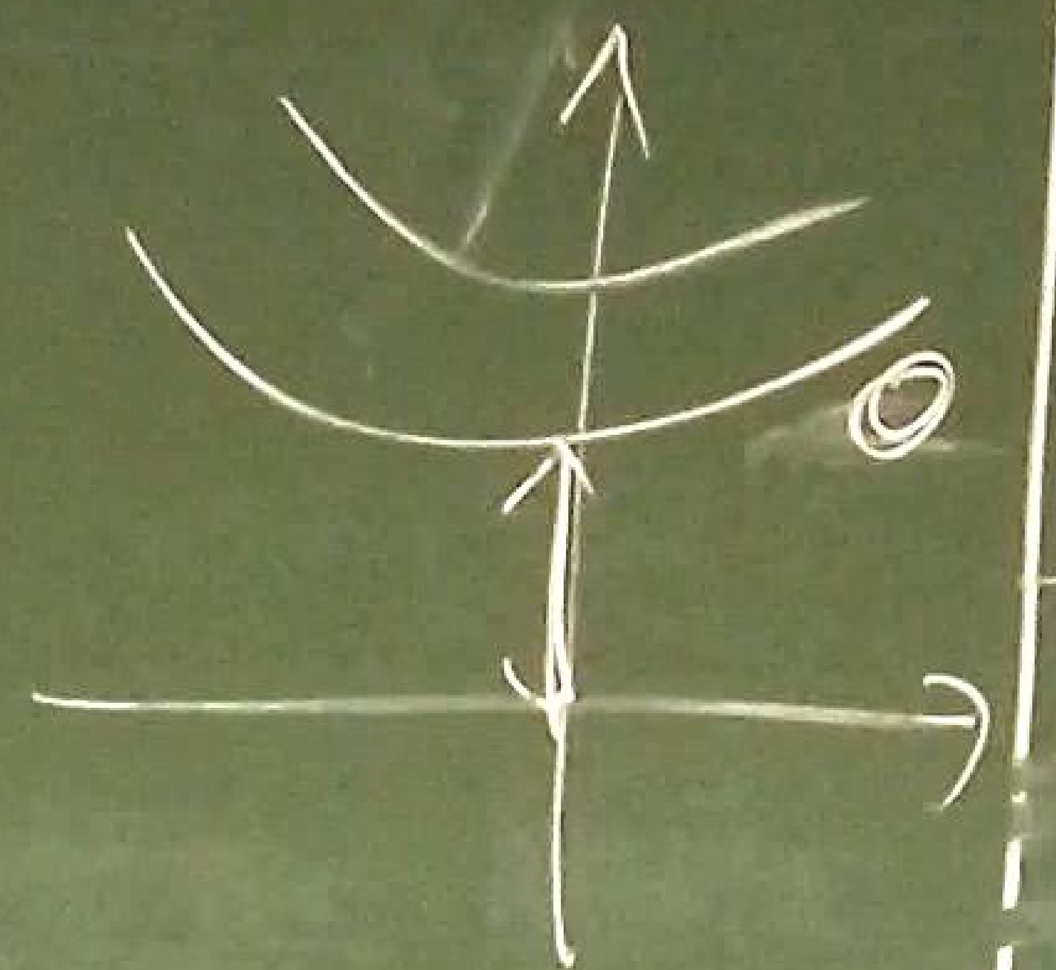
(Quantum Chromodynamics)

Gauge bosons = Gluons ($8 \times A_\mu^a$)

→ Pure gluon vertices:

$(\partial A)AA \hat{=}$ 

$A^4 \hat{=}$ 



→ Bound states of gluons

→ Glueballs

9.1

$$\boxed{\mathcal{L}_{YMD} = \bar{\Psi}(i\not{D} - m)\Psi - \frac{1}{4}\left(F^a_{\mu\nu}\right)^2}$$

Two parameters:

- m: mass of Fermions
- g: coupling constant

- gauge invariant
- Lorentz invariant
- renormalizable
- P- and T- symmetric

Note 9.2

- The mass term A^2 is not allowed because it is not gauge invariant

$\rightarrow$ Gauge bosons of YM are massless

Problem: Weak interaction is short ranged

$\Rightarrow$ gauge bosons $W^{\pm}, Z$ are massiv

Solution: Higgs mechanism

10 Excursion

10.1 The Higgs Mechanism

1) $\mathcal{L}$ Maxwell theory

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + |D_\mu \phi|^2 - V(\phi)$$

complex scalar field

- $V(\phi) = \mu^2 |\phi|^2 + \lambda |\phi|^4$
- $D_\mu = \partial_\mu + ieA_\mu$

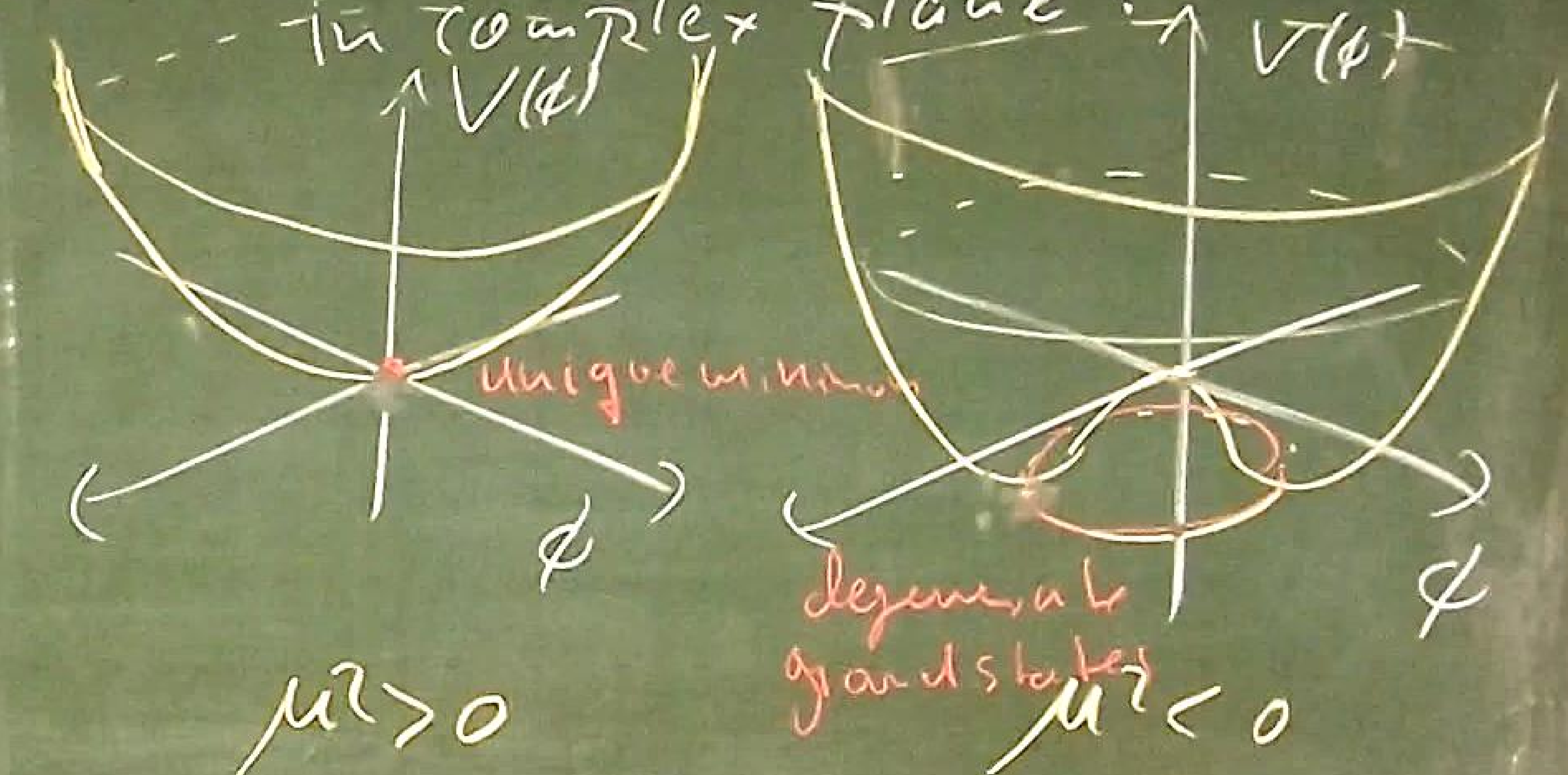
2) $U(1)$ gauge transformations:

$$\tilde{\phi}(x) = e^{i\alpha(x)}\phi(x)$$

$$\tilde{A}_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$$

3) $\mathcal{L}$ $V(\phi)$ for $\lambda > 0$

In complex plane:



• $\mu^2 < 0$: Mexican hat potential:

Degenerate minima with non-zero vacuum expectation value (VEV)

$$\phi_0 = \langle \phi \rangle \quad \text{and} \quad v = |\phi_0| = \sqrt{\frac{-\mu^2}{2\lambda}} \neq 0$$

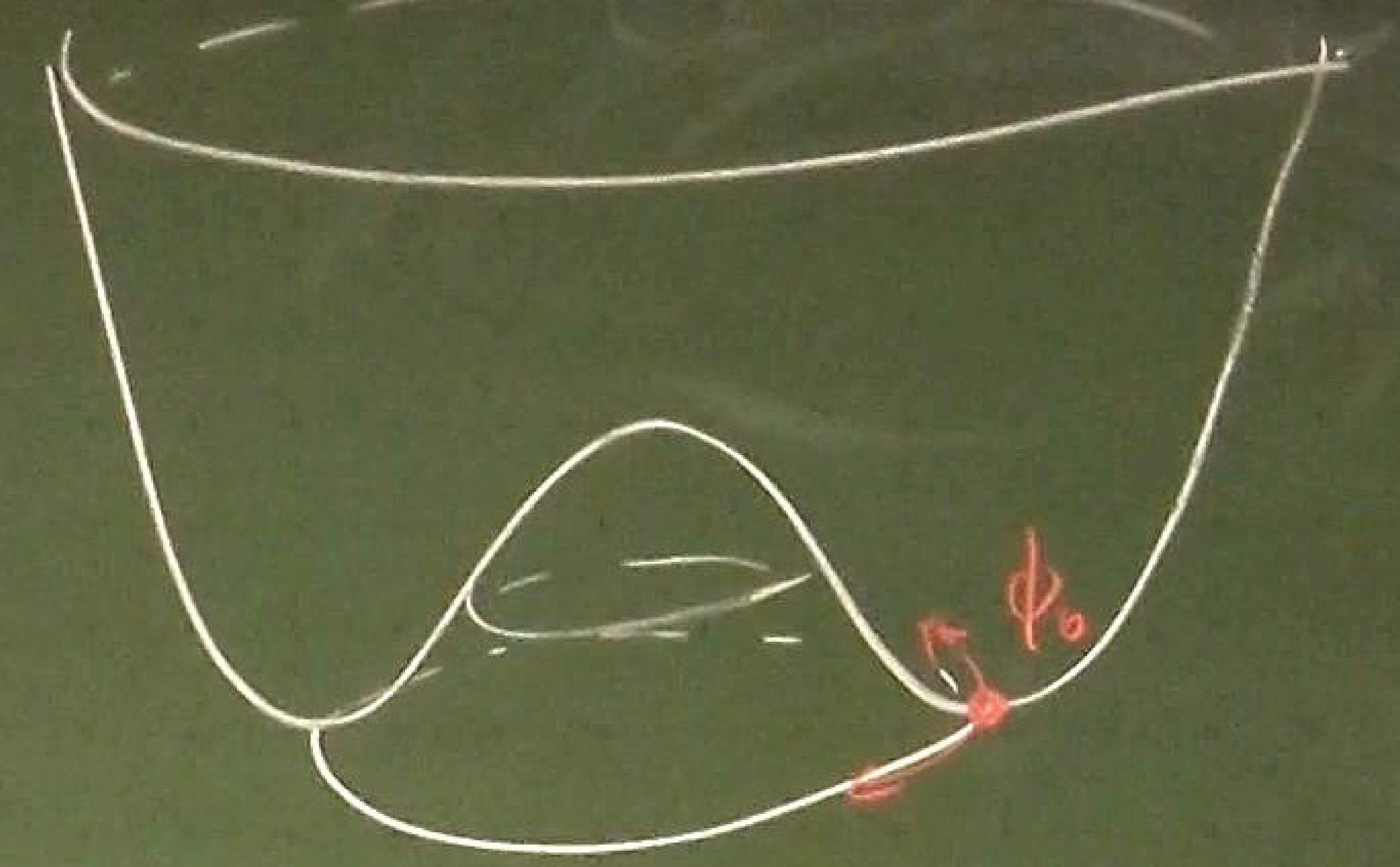
→ Ground states are not symmetric under global phase rotations

→ Spontaneous Symmetry Breaking (SSB)

4) Aside: The Goldstone theorem

> If a global, continuous symmetry is spontaneously broken, there is one massless scalar (spin-0) particle for each broken symmetry generator; these are called (Nambu-) Goldstone bosons

Proof by picture:



Example:

• Breaking translation invariance in crystals → Transversal and longitudinal phonons