Note:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ipx} + a_p^\dagger e^{ipx}\right)\Big|_{p^0 = E_p}$$

$$\phi(\vec{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-i\vec{p}\vec{x}} |\vec{p}\rangle \equiv |\vec{x}\rangle$$

$\frac{1}{m} = \lambda$

$$D_F(x-y) = \langle 0| T\{\phi(x)\phi(y)\}|0\rangle$$

$$\overset{x^0 > y^0}{=} \langle 0|\phi(x)\phi(y)|0\rangle$$

$$= \langle 0| e^{iHx^0} \phi(\vec{x}) e^{-iHx_0} e^{iHy^0} \phi(\vec{y}) e^{-iHy^0}|0\rangle$$

$\langle 0|$ $\quad U = e^{-iH(x^0-y^0)}$ $\quad |0\rangle$

$$= \langle \vec{x}| U(y^0, x^0) |\vec{y}\rangle$$

$$= U(\vec{y}, \vec{x}, T = x^0 - y^0)$$

9. Non-Abelian Gauge Theories

Motivation:

* Interactions? $\quad \bar{\psi}\gamma^\mu\psi A_\mu \quad$ others?
  $A^4, (\partial A)A^2 \quad$?
* Massless particle + Vector field A_μ

$$ISO(2) = SO(2) \times \text{Trans}$$

Helicity ± 1

$$A^\mu = \int d\tilde{p} \sum_{\sigma=\pm} a_{\vec{p}\sigma} + a^\dagger_{\vec{p},\sigma}$$

$\longrightarrow$

$$U(\Lambda) A^\mu(x) U^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu A^\nu(\Lambda x) + \partial_\mu \Omega(\Lambda, x)$$

cf.

$$U(\Lambda)\psi(x)U^{-1}(\Lambda) = \Lambda^{-1}_{\frac{1}{2}} \psi(\Lambda x)$$

* 't Hooft: Gauge symmetry $\Downarrow$ Renormalizability

9.1. The Geometry of Gauge Invariance

1| ∢ Local $U(1)$ symmetry G of Dirac field:

$$\tilde{\psi}(x) = e^{i\alpha(x)}\psi(x)$$

arbitrary $\alpha(x): \mathbb{R}^{1,3} \to \mathbb{R}$

2| Goal: Construct invariant Lagrangians

3| No problem without derivatives:

global $U(1)$ invariance $\longrightarrow$ local $U(1)$ inv.

Example: $\bar{\psi}\psi$

4| ∢ Directional derivative along $n \in \mathbb{R}^{1,3}$:

$$n^\mu \partial_\mu \psi := \lim_{\varepsilon \to 0} \frac{\psi(x+\varepsilon n) - \psi(x)}{\varepsilon}$$

↳ $n^\mu \partial_\mu \psi$ no simple transformation law under G

5| ∢ "Comparator" $U: \mathbb{R}^{1,3} \times \mathbb{R}^{1,3} \longrightarrow \mathbb{C}$

$$\tilde{U}(y,x) = e^{i\alpha(y)}U(y,x)\,e^{-i\alpha(x)}$$

$$U(y,x) = e^{i\phi(y,x)}$$

$$U(y,y) = 1$$

↳ $\psi(y)$, $U(y,x)\cdot\psi(x)$

Differential geometry

* fiber bundles
* principal bundles
* connection / parallel transport

6| Covariant derivative:

$$\boxed{n^\mu D_\mu \psi := \lim_{\varepsilon \to 0} \frac{\psi(x+\varepsilon n) - U(x+\varepsilon n, x)\psi(x)}{\varepsilon} \quad (1)}$$

7| $U(x+\varepsilon n, x) = 1 - ie\varepsilon n^\mu A_\mu(x) + O(\varepsilon^2) \quad (2)$

arbitrary constant

new vector field — gauge connection

8| (2) in (1)

$$\boxed{D_\mu\psi(x) \overset{\circ}{=} \partial_\mu\psi(x) + ieA_\mu\psi(x)}$$

9| $\boxed{\tilde{A}_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)}$

10| $\boxed{\tilde{D}_\mu \tilde{\Psi}(x) = e^{i\alpha(x)} D_\mu \Psi(x)}$

→ $D_\mu \Psi$ transforms like Ψ

→ All globally $U(1)$ invariant terms are allowed if $\partial \to D$

$$\bar{\Psi} \not\partial \Psi \longrightarrow \bar{\Psi} \not D \Psi$$

11| Conclusions:

Local symmetry → Gauge field A_μ needed for covariant derivatives

12| Kinetic energy for A_μ?

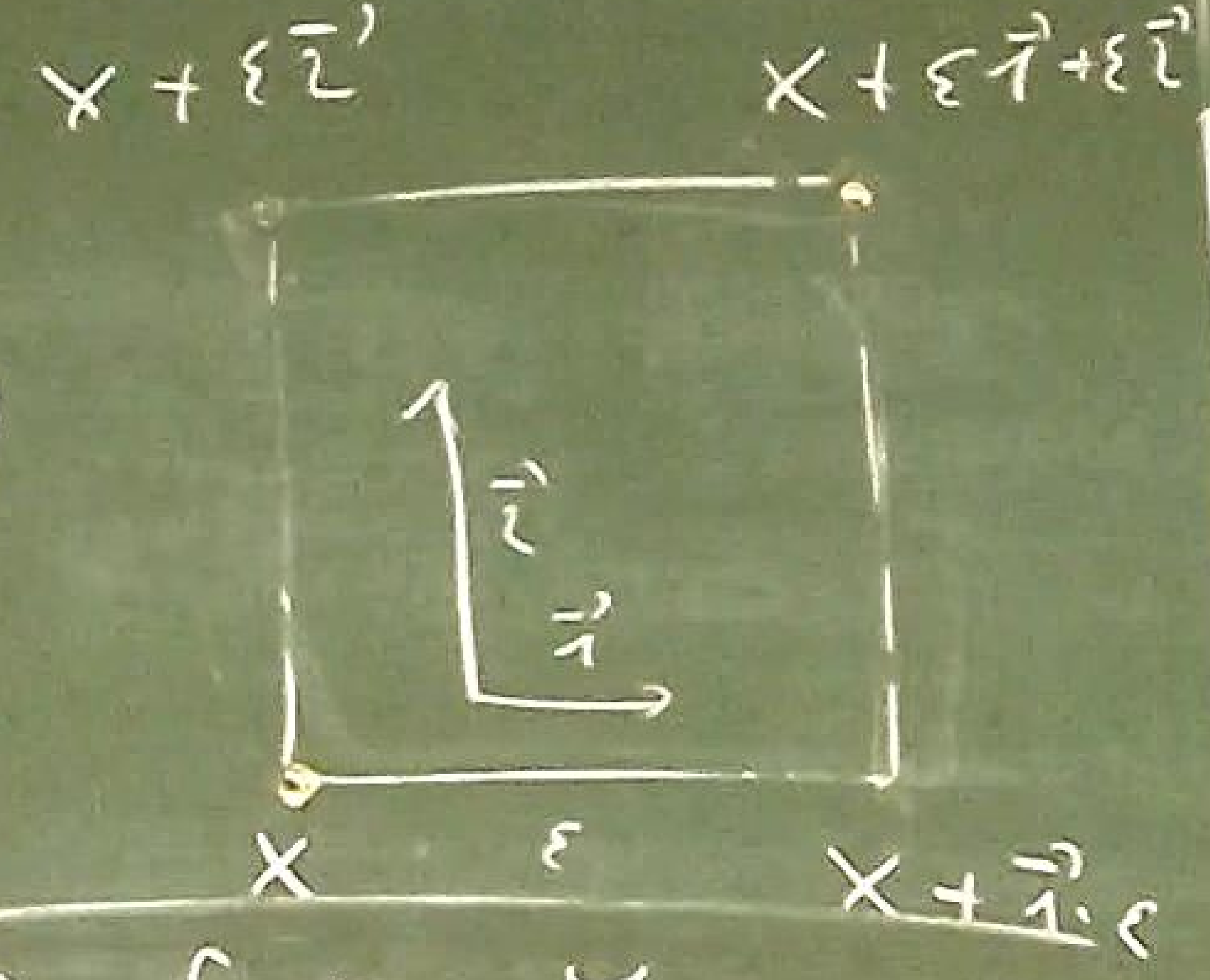
i] ∢ Locally invariant loop.

e^{ieA_μ}

$U(1) = \mathbb{R}$

$$\begin{aligned} U(x) := {} & U(x, x+\epsilon\vec{2}) \\ & \times U(x+\epsilon\vec{2}, x+\epsilon\vec{1}+\epsilon\vec{2}) \\ & \times U(x+\epsilon\vec{1}+\epsilon\vec{2}, x+\epsilon\vec{1}) \\ & \times U(x+\epsilon 1, x) \end{aligned}$$

→ $\tilde{U} = U$



$S = \int d^4x \, \mathcal{L}$

$$U(x) \doteq 1 - i\epsilon^2 e(\partial_1 A_2 - \partial_2 A_1) + O(\epsilon^3)$$

→ $\boxed{F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu}$ Field strength tensor

is gauge invariant by construction

13| Most general Lagrangian in $D = 1+3$:

* Gauge invariant → $\Psi, D_\mu\Psi, F_{\mu\nu}, \partial_\alpha F_{\mu\nu}$
* Relativistic → Lorentz scalar
* Renormalizable

→ Terms of mass dimension at most 4

→

$$\boxed{\mathcal{L} = \bar{\Psi} i \not{D} \Psi - m \bar{\Psi}\Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - c\,\varepsilon^{\alpha\beta\mu\nu} F_{\mu\nu} F_{\alpha\beta} + \cancel{G(\bar{\Psi}\Psi)^2} + \dots}$$

pseudo tensor
invariant under $SO^+(1,3)$
but not P, T ($O(1,3)$)

not renormalizable

→ Most general P/T symmetric Lagrangian: QED

9.2. The Yang-Mills Lagrangian

Goal: Replace local symmetry group U(1) by non-abelian Lie group G

Examples: SO(3), SU(2), SU(3)

1| ∀ Lie group G represented n×n unitary matrices V

Example: $G = SU(2)$

$$V = e^{i\omega_j \frac{\sigma^j}{2}}$$

2| Fields $\Psi = (\Psi_1, \dots, \Psi_n)$ n-plets of Dirac fields Ψ_i

$$\tilde{\Psi}(x) = V(x)\Psi(x)$$

$$\Psi : \mathbb{R}^{1,3} \to \underset{\text{Lorentz group}}{\mathbb{C}^4} \otimes \underset{\text{gauge group}}{\mathbb{C}^n} = \mathbb{C}^{4n}$$

3| G Lie group → Lie algebra $\mathfrak{g}$ with N Hermitian generators t^a (n×n matrices, $a = 1 \dots N$)

$$[t^a, t^b] = i \underset{\text{structure symbol}}{f^{abc}} t^c$$

Example: $SU(2)$: $N = 3$, $[L^i, L^j] = i \varepsilon^{ijk} L_k$