Recap:

8. Functional Methods

8.2 Path integrals for scalar fields

$$U(\phi_a,\phi_b;T) = \langle \phi_b | e^{-i\hat{H}T} | \phi_a \rangle$$

$$= \int_{\phi(\vec{x},0)\equiv\phi_a(\vec{x})}^{\phi(\vec{x},T)\equiv\phi_b(\vec{x})} \mathcal{D}\phi \, \exp\Big[\frac{i}{\hbar}\underbrace{\int_0^T d^4x\, \mathcal{L}(\phi,\partial\phi)}_{\text{Action } S[\phi]}\Big]$$

Functional/Path integral over all field trajectories $\phi(x)=\phi(\vec{x},t)$

Note on $\mathcal{D}\phi$:

Fourier transform: $\phi(x) = \sum_k \phi_k e^{ikx}$

Fourier coefficients $\in \mathbb{C}$

$\not\sum$ finite volume $L^3 T$

$k = \frac{2\pi}{L}\cdot n \quad n\in\mathbb{Z}$

→ Use $\{\phi_k\}$ to parametrize ϕ

→ Integrate over all $\{\phi_k\}$:

$$\int \mathcal{D}\phi := \lim_{L\to\infty} \prod_k \int d\phi_k^I \int d\phi_k^R$$

imaginary part, real part

$$\phi_k = \phi_k^R + i\,\phi_k^I \in \mathbb{C}$$

- If ϕ is real: $\phi_k^* = \phi_{-k}$ ⟹ restrict k to half space: $k^0>0$

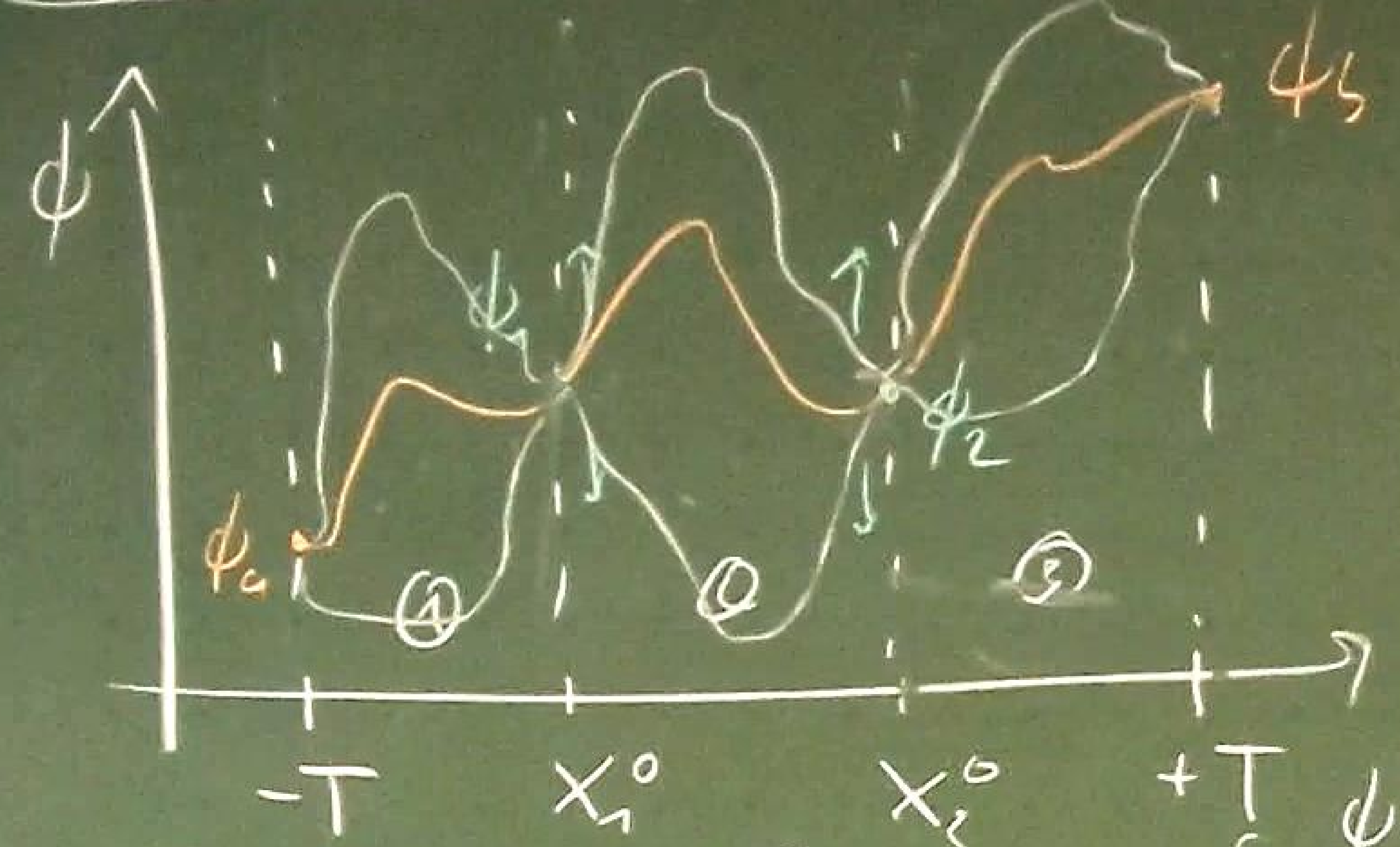
Correlation functions:

1) Goal:

$$\langle \Omega | T\, \phi_H(x_1)\phi_H(x_2) | \Omega \rangle$$

$$\overset{?}{\longleftrightarrow} \int_{\phi(-T)=\phi_a}^{\phi(+T)=\phi_b} \mathcal{D}\phi\, \phi(x_1)\phi(x_2)\, e^{i\int_{-T}^{T} d^4x\,\mathcal{L}} = (*)$$

2| Split functional integral.



$\phi(\vec{x}, x_2^0) = \phi_2(\vec{x})$

$$\int \mathcal{D}\phi = \int \mathcal{D}\phi_1(\vec{x}) \int \mathcal{D}\phi_2(\vec{x}) \int_{\phi(\vec{x}, x_1) = \phi_1(\vec{x})} \mathcal{D}\phi$$

3|

$$(*) = \int \mathcal{D}\phi_1 \int \mathcal{D}\phi_2 \; \phi_1(\vec{x})\phi_2(\vec{x}) \underbrace{\langle \phi_b | e^{-iH(T - x_2^0)} | \phi_2 \rangle}_{3} \underbrace{\langle \phi_2 | e^{-iH(x_2^0 - x_1^0)} | \phi_1 \rangle}_{2} \underbrace{\langle \phi_1 | e^{-iH(x_1^0 + T)} | \phi_a \rangle}_{1} = (**)$$

4|

- $\hat{\phi}_S(\vec{x}_1) | \phi_1 \rangle = \phi_1(\vec{x}_1) | \phi_1 \rangle$
- $\int \mathcal{D}\phi_1(\vec{x}) \, |\phi_1\rangle\langle\phi_1| = \mathbb{1}$ $\quad \{ \int dx \, |x\rangle\langle x| = \mathbb{1}$

$$(**) \overset{x_2^0 > x_1^0}{=} \langle \phi_b | \underbrace{e^{-iH(T - x_2^0)}}_{\to |\Omega\rangle\langle\Omega|} \underbrace{\phi_S(\vec{x}_2) e^{-iH(x_2^0 - x_1^0)}}_{\phi_H(x_2)} \underbrace{\phi_S(\vec{x}_1) e^{-iH(x_1^0 + T)}}_{\phi_H(x_1)} | \phi_a \rangle$$

$$e^{-iH(x_1^0 + T)} \to |\Omega\rangle\langle\Omega|$$

$T \to \infty(1 - i\varepsilon)$

$$\longrightarrow C \langle \Omega | T\{\phi_H(x_1) \phi_H(x_2)\} | \Omega \rangle$$

5)

$$\boxed{\langle\Omega|T\,\phi_H(x_1)\phi_H(x_2)|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\int \mathcal{D}\phi\,\phi(x_1)\phi(x_2)\,e^{i\int_{-T}^{T} d^4x\,\mathcal{L}}}{\int \mathcal{D}\phi\,e^{i\int_{-T}^{T} d^4x\,\mathcal{L}}}}$$

8.3 Application: Quantization of the Electromagnetic Field

Goal: PI $\rightarrow \frac{-i g_{\mu\nu}}{k^2+i\varepsilon}$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

1) $S[A] = \int d^4x \left[-\frac{1}{4}(F_{\mu\nu})^2\right]$

Partial integration

$$\overset{\circ}{=} \frac{1}{2}\int d^4x\, A_\mu(x)\left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right)A_\nu(x)$$

Fourier transform

$$\overset{\circ}{=} \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\,\tilde{A}_\mu(k)\underbrace{\left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right)\tilde{A}_\nu(-k)}_{\square}$$

2) $\forall\ \tilde{A}_\mu(k) = k_\mu\,\alpha(k)$

$\square = 0 \rightarrow S[A] = 0$

$\rightarrow \int \mathcal{D}A\,\underbrace{e^{i0}}_{1} = \infty$

3) Problem: Gauge invariance

$A_\mu \rightarrow A_\mu + \frac{1}{e}\partial_\mu\alpha$

$A_\mu = 0 \rightarrow A_\mu \propto \partial_\mu\alpha$

$\rightarrow \tilde{A}_\mu \sim k_\mu\alpha$

4] Solution: Count each physical configuration only once: Faddeev & Popov procedure

i] Gauge fixing: $G(A) = 0$ $\quad \{ G(A) = \partial_\mu A^\mu$

ii] $A_\mu^\alpha = A_\mu + \frac{1}{e}\partial_\mu \alpha$

Note:

$$1 = \prod_1^N \int dg_i\, \delta^{(n)}(\vec{g}) \underset{\uparrow}{=} \left[\prod_i \int da_i\right] \delta^{(n)}(g(\vec{a})) \det\left(\frac{\partial \vec{g}}{\partial \vec{a}}\right)$$

$\vec{g} = \vec{g}(\vec{a})$

$$1 = \int \mathcal{D}\alpha\, \delta(\underbrace{G(A^\alpha)}_{\sim g(\alpha)}) \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right)$$

iii] Assumption: $\frac{\delta G(A^\alpha)}{\delta \alpha}$ independent of α and A $\sim$ does not hold non-abelian $\to$ Ghost fields

iv] $$\int \mathcal{D}A\, e^{iS[A]} \underset{1=}{} \det\left(\frac{\delta G(A^\alpha)}{\delta \alpha}\right) \int \mathcal{D}\alpha \int \mathcal{D}A\, e^{iS[A]}\, \delta(G(A^\alpha))$$

- $\tilde{A} = A^\alpha = A + \frac{1}{e}\partial\alpha \longrightarrow \mathcal{D}A = \mathcal{D}\tilde{A}$
- $S[A] = S[\tilde{A}]$ Gauge invariance

$$= \det\left(\frac{\delta(G(A^\alpha))}{\delta\alpha}\right)\left(\underbrace{\int \mathcal{D}\alpha}_{\infty}\right) \underbrace{\int \mathcal{D}\tilde{A}\, e^{iS[\tilde{A}]}\, \delta(G(\tilde{A}))}_{\text{Physically distinct configurations}}$$

v) Choose $G(A^\alpha) = \partial^\mu A^\alpha_\mu - \omega(x)$ $\quad A_\mu + \frac{1}{e}\partial_\mu \alpha \quad \frac{\delta \alpha}{\delta \alpha} = 1$

$$\det\left(\frac{\delta G}{\delta \alpha}\right) = \det\left(\frac{1}{e}\partial^2\right)$$

$$(*) = \det\left(\frac{1}{e}\partial^2\right)\left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A \, e^{iS[A]} \delta(\partial^\mu A_\mu - \omega(x))$$

$$\boxed{N(\xi)\underbrace{\int \mathcal{D}\omega(x)\int \mathcal{D}A \, e^{iS[A]}}_{1} = N(\xi)\int \mathcal{D}\omega}$$

vi)

$$(*) \overbrace{N(\xi)\int \mathcal{D}\omega \, e^{-i\int d^4x \frac{\omega^2}{2\xi}}}^{1}$$

$$\times \det\left(\frac{1}{e}\partial^2\right)\left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A \, e^{iS[A]} \delta(\partial^\mu A_\mu - \omega(x))$$

$$= N(\xi) \det\left(\frac{1}{e}\partial^2\right)\left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A \, e^{iS[A]} \exp\left[-i\int d^4x \frac{(\partial^\mu A_\mu)^2}{2\xi}\right]$$

$\tilde{A} = A^\alpha \quad S[\tilde{A}] = S[A], \; O(\tilde{A}) = O(A)$

vii) $\langle \Omega | T \, O(\hat{A}) | \Omega \rangle$

gauge invariant

$$= \lim_{T \to \infty(1-i\epsilon)} \frac{\int \mathcal{D}A \, O(A) \exp\left\{i\int_{-T}^{T} d^4x \left[\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right]\right\}}{\int \mathcal{D}A \exp\{\cdots\}}$$

5) $\tilde{S}[A] = \int d^4x \left[-\frac{1}{4}(F_{\mu\nu})^2 - \boxed{\frac{1}{2\xi}(\partial^\mu A_\mu)^2}\right]$ $\quad$ Argument of 1) no longer works!

Part. int. / FT

$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(k)\left[-k^2 g^{\mu\nu} + (1 - \xi^{-1})k^\mu k^\nu\right]\tilde{A}_\nu(-k)$$

6) Propagator:

$D_F^{\mu\nu}(x-y) = \langle\Omega| T A^\mu(x) A^\nu(y) |\Omega\rangle$

$\rightarrow \langle\Omega| \tilde{A}^\mu(k) \tilde{A}^\nu(q) |\Omega\rangle = 0$ whenever $k \neq -q$

$\tilde{D}_F^{\mu\nu}(q) = \langle\Omega| \tilde{A}(q) \tilde{A}^\nu(-q) |\Omega\rangle$

Add $+i\varepsilon$ for regularization

$$= \frac{\int \mathcal{D}A\, \tilde{A}^\mu(q) \tilde{A}^\nu(-q) \exp\left\{ \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}_\mu(k) \overbrace{\left[-k^2 g^{\mu\nu} + (1-\xi^{-1}) k^\mu k^\nu \right]}^{M^{\mu\nu}(k)} \tilde{A}_\nu(-k) \right\}}{\int \mathcal{D}A \exp\{ - \cdots - \}}$$

$= i\left[M^{-1}(q) \right]^{\mu\nu}$

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$\xrightarrow{\text{Finally}}$ $\boxed{\tilde{D}_F^{\mu\nu}(q) \overset{6}{=} \frac{-i}{q^2+i\varepsilon} \left[g^{\mu\nu} - (1-\xi) \frac{q^\mu q^\nu}{q^2} \right]}$

$\cdot\ \nabla \cdot \vec{A} = 0$

$\cdot\ \partial_\mu A^\mu = 0$

7) Gauges:

- Set $\xi = 1$: Feynman gauge

$$\boxed{\tilde{D}_F^{\mu\nu}(q) = -\frac{i g^{\mu\nu}}{q^2+i\varepsilon}}$$

- Set $\xi = 0$: Landau gauge

$$\tilde{D}_F(q) = \frac{-i}{q^2+i\varepsilon} \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right)$$