8. Functional Methods

So far:

Hamiltonian → Canonical quant.

→ Feynman rules

Alternative:

Lagrangian → Path integral

→ Feynman rules

8.1. Path integrals in QM

1) 1 nonrel. particle in 1D: $H = \frac{P^2}{2m} + V(x)$

2) Time evolution operator:
$$U(x_a, x_b, T) = \langle x_b | e^{-\frac{i}{\hbar}\hat{H}T} | x_a \rangle$$

3) PI formalism → Alternative expression for U.

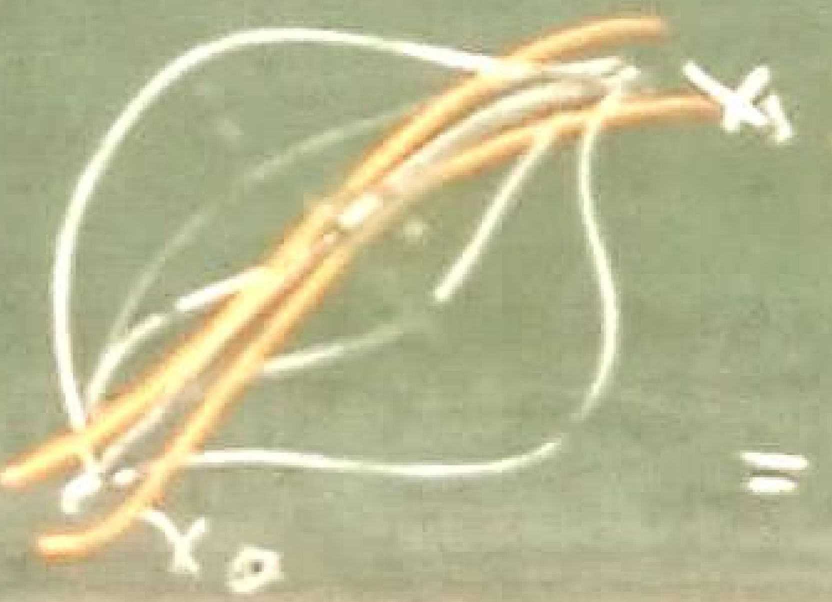
Functional

$$U(x_a, x_b, T) = \sum_{\text{All paths } x(t)} e^{i F[x(t)]} \, dx$$

Superpos. principle

$x_b = x(T)$

$$= \int_{x_a = x(0)} \mathcal{D}x(t) \, e^{i F[x(t)]}$$



4) F?

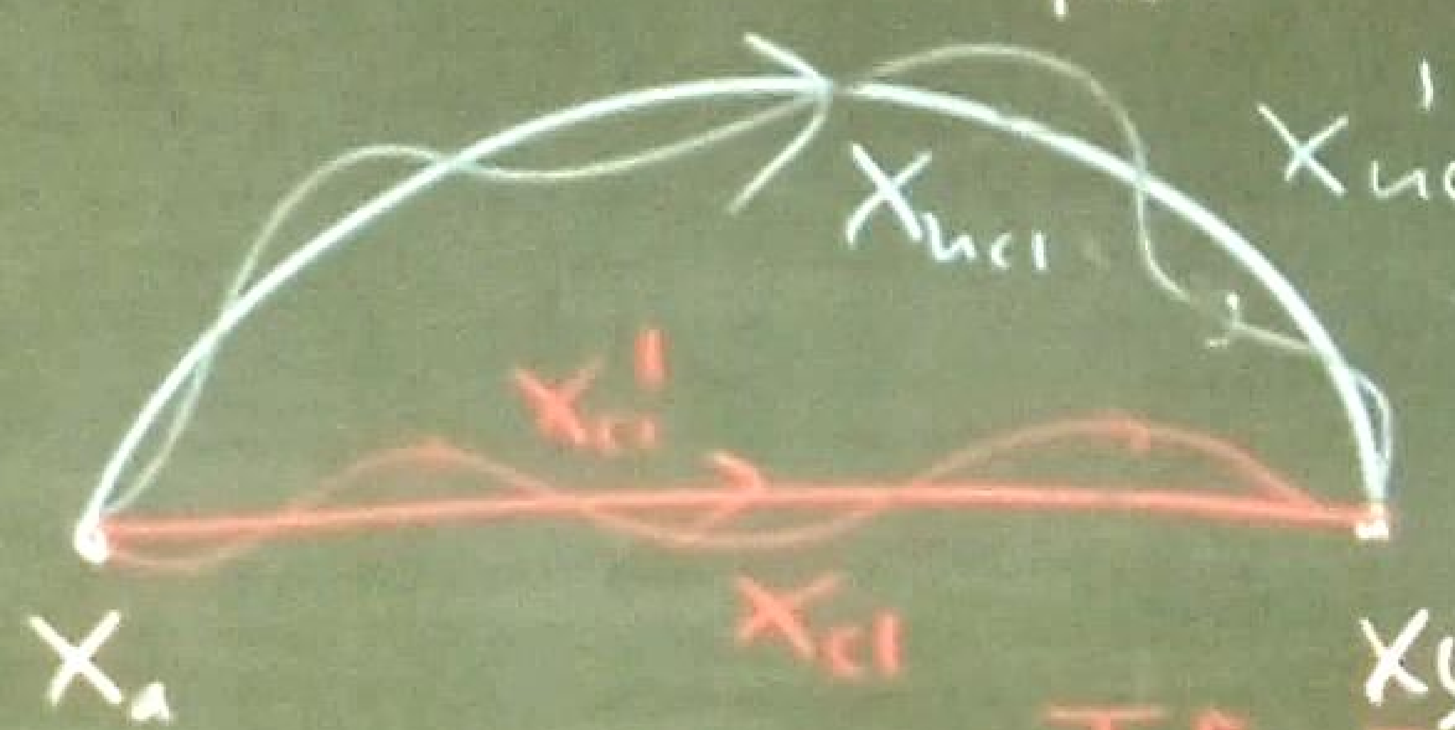
i) Descr. System

ii) Functional of path

iii) Classical paths $x_{cl}(t)$ should dominate $\hbar \to 0$

$$U(x_a, x_b, T) \overset{\hbar \to 0}{\approx} \sum e^{i F[x(t)]}$$

paths close to x_{cl}



$$e^{i F[x_{ncl}]} + e^{i F[x_{ncl}']} \approx 0$$

$$e^{i F[x_{cl}]} + e^{i F[x_{cl}']} \neq 0$$

Therefore

$$\left.\frac{\delta F}{\delta x}\right|_{x=x_{cl}} \stackrel{!}{=} 0$$

$$\Rightarrow \quad F = \frac{S}{\hbar} - \frac{1}{\hbar}\int dt\, L(x(t))$$

<u>Note</u> $\hbar \to 0$

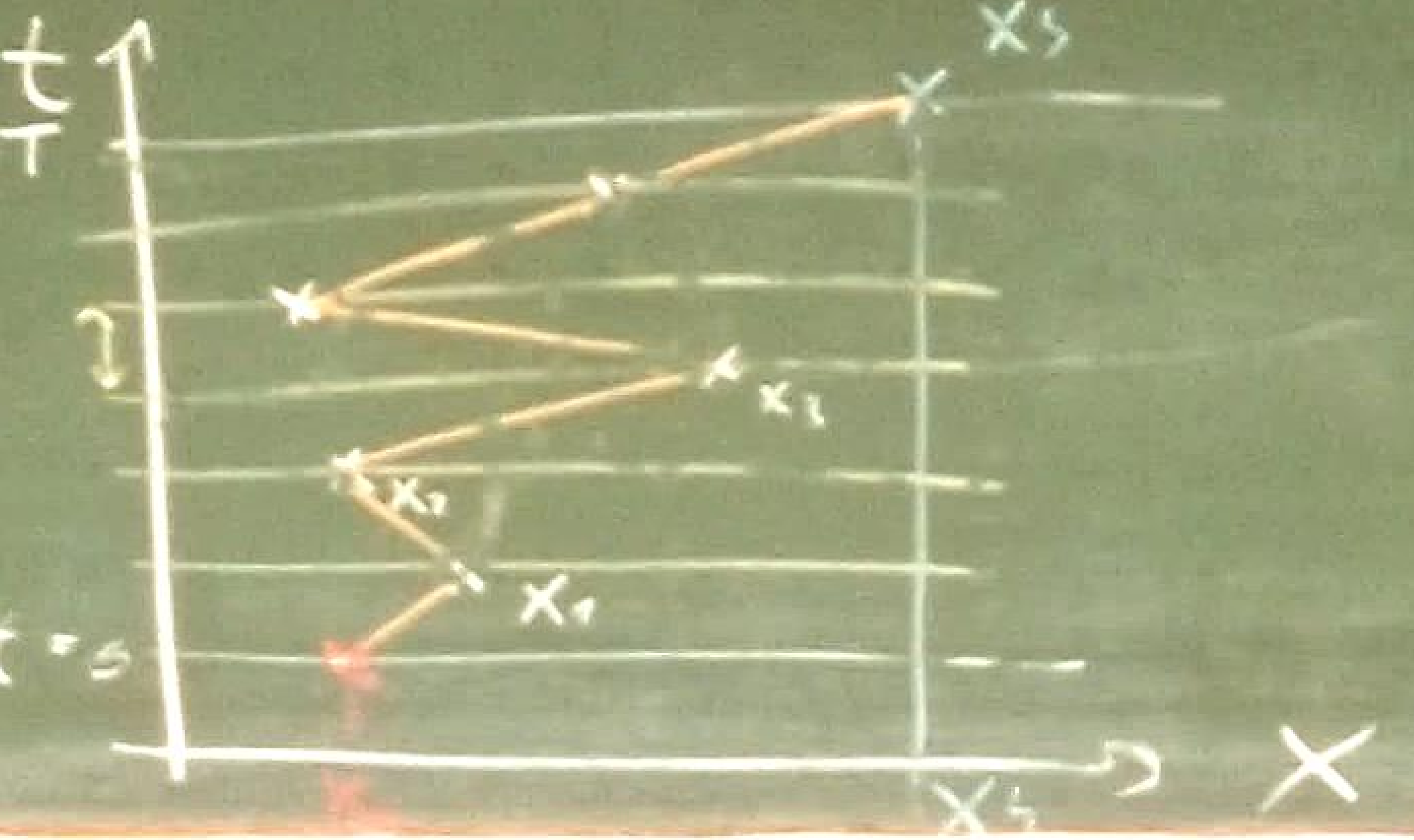
$$U(x_a, x_b, T) \sim e^{\frac{i}{\hbar}\underbrace{S[x_{cl}(t)]}_{S(x_a, x_b, T)}}$$

<u>5) Propagation amplitude:</u>

$$\boxed{\begin{aligned} U(x_a, x_b, T) &= \int_{x_a}^{x_b} \mathcal{D}x(t)\, e^{\frac{i}{\hbar}S[x(t)]} \\ &\stackrel{?}{=} \langle x_b | e^{-\frac{i}{\hbar}\hat{H}T} | x_a \rangle \end{aligned}}$$

<u>6) Def. PI via timeslices:</u>

$\varepsilon = \frac{T}{N}$



$$\int \mathcal{D}x(t) = \lim_{N\to\infty} \frac{1}{C_\varepsilon}\int \frac{dx_1}{C_\varepsilon} \cdots \int \frac{dx_{N-1}}{C_\varepsilon}$$

$$= \lim_{N\to\infty} \frac{1}{C_\varepsilon} \prod_{k=1}^{N-1} \int \frac{dx_k}{C_\varepsilon}$$

$\varepsilon = \frac{T}{N}$, C_ε constant

<u>Example 8.1</u>: Particle in potential $V(x)$

1) Lagrangian: $L = \frac{m}{2}\dot{x}^2 - V(x)$

2) <u>Action:</u>

$$S = \int_0^T dt\, L \simeq \sum_{k=0}^{N-1} \left[\frac{m}{2}\left(\frac{x_{k+1} - x_k}{\varepsilon}\right)^2 - \varepsilon V\left(\frac{x_{k+1} + x_k}{2}\right)\right]$$

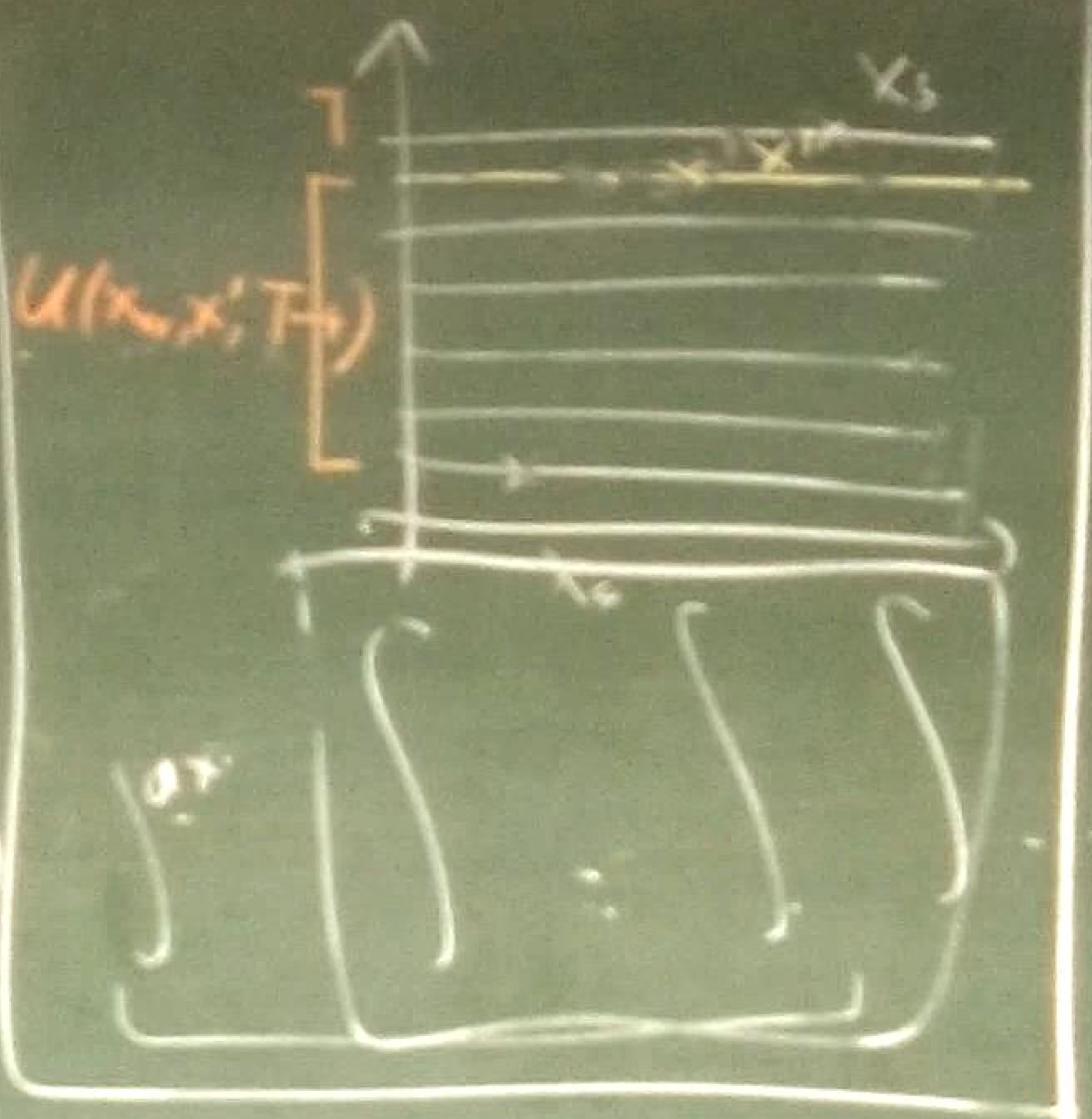
$$U(x_a, x_b, T) = \int_{-\infty}^{\infty} \frac{dx'}{C_\varepsilon} \exp\left[\frac{i}{\hbar}\frac{m(x_b-x')^2}{\varepsilon} - \frac{i}{\hbar}\varepsilon V\left(\frac{x_b+x'}{2}\right)\right]$$

$$\times\, U(x_a, x', T-\varepsilon) \qquad x_b + \frac{x'-x_b}{2} \sim \varepsilon$$

$$= \int \frac{dx'}{C_\varepsilon}\, e^{\frac{i}{\hbar} m \frac{(x_b-x')^2}{\varepsilon}} \Big[1 - \frac{i}{\hbar}\varepsilon \underbrace{V\left(\cancel{\tfrac{x_b+x'}{2}}\right)}_{V(x_b)+O(\varepsilon)} + O(\varepsilon^2)\Big]$$

$$\times\left[1 + (x'-x_b)\,\partial_{x_b} + \frac{(x'-x_b)^2}{2}\,\partial^2_{x_b} + \dots\right] U(x_a, x_b, T-\varepsilon)$$

$$\int dx\, e^{-x^2}\begin{Bmatrix}1\\ x\\ x^2\end{Bmatrix} \qquad \doteq \underbrace{\frac{1}{C_\varepsilon}\sqrt{\frac{2\pi\hbar\varepsilon}{-im}}}_{1} \left[1 - \frac{i}{\hbar}\varepsilon V(x_b) + \frac{i\hbar}{2m}\varepsilon\,\partial^2_{x_b} + O(\varepsilon^2)\right] \underbrace{U(x_a,x_b,T-\varepsilon)}_{U(x_a,x_b,T) - \varepsilon\partial_T U(x_a,x_b,T)+\dots}$$



4| & $\varepsilon \to 0$

$$\boxed{C_\varepsilon = \sqrt{\frac{2\pi\hbar\varepsilon}{-im}}}$$

5| & lin. Order in ε:

$$\boxed{i\hbar\,\partial_T U = \left[-\frac{\hbar^2}{2m} + V(x_b)\right]U}$$

Schrödinger equation

6| Check initial condition:

$$U(x_a, x_b, \varepsilon) \xrightarrow{\varepsilon\to 0} \delta(x_a - x_b)$$

$$\underset{\wedge=1}{1} \qquad = \langle x_b | x_a \rangle$$

$$\hat{1} = e^{-\frac{i}{\hbar} H\, 0}$$

7] $U_{PI} = U_{CQ}$

Generalization: (PSet 12)

1] $\vec{x}$ coordinates q_i, momenta p_i,
Hamiltonian $H(\vec{q},\vec{p})$

2] Canonical quant.: $[q_i, p_j] = i\hbar\,\delta_{ij}$

$$U(\vec{q}_b,\vec{q}_a,T) = \langle \vec{q}_b | e^{-i\hat{H}T} | \vec{q}_a \rangle$$

3] Time slicing:

$$e^{-i\hat{H}T} = \underbrace{e^{-i\hat{H}\varepsilon} \cdots\cdots e^{-i\hat{H}\varepsilon}}_{N}, \quad \varepsilon = \frac{T}{N}$$

$$\mathbb{1} = \int d\vec{q}_k \, |\vec{q}_k\rangle\langle\vec{q}_k|$$

$$e^{-iH\varepsilon}\int d\vec{q}_k\, |\vec{q}_k\rangle \boxed{\langle \vec{q}_k | e^{-iH\varepsilon} \int d\vec{q}_{k+1} |\vec{q}_{k+1}\rangle} \langle\vec{q}_{k+1}| \ldots$$

$$\langle \vec{q}_{k+1} | e^{-i\hat{H}\varepsilon} | \vec{q}_k \rangle = \langle q_{k+1} | \left[\mathbb{1} - i\hat{H}\varepsilon + O(\varepsilon^2) \right] | \vec{q}_k \rangle$$

$$\int d\vec{p}_k\, |\vec{p}_k\rangle\langle\vec{p}_k|$$

5] $\hat{H} = \hat{H}_1(\vec{q}) + \hat{H}_2(\vec{p})$

$$\langle \vec{q}_{k+1} | \hat{H} | \vec{q}_k \rangle \doteq \int \frac{d\vec{p}_k}{2\pi} H\left(\frac{\vec{q}_{k+1}+\vec{q}_k}{2}, \vec{p}_k\right) \times e^{i\vec{p}_k(\vec{q}_{k+1}-\vec{q}_k)}$$

Weyl transform / quantization

6) Hamiltonian phase-space path integral:

$$e^{i p_k \left(\frac{q_k - q_{k-1}}{\epsilon}\right)\epsilon} \qquad \dot q \qquad p = \frac{\partial \mathcal{L}}{\partial \dot q}$$

$$\boxed{U(\vec q_a, \vec q_b, T) \hat{=} \int_{\vec q_a}^{\vec q_b} \mathcal{D}\vec q(t) \int \mathcal{D}p(t)\, \exp\left[\frac{i}{\hbar}\underbrace{\int_0^T dt \left(\overbrace{\vec p\cdot\dot{\vec q} - H(\vec q,\vec p)}^{\text{"}\mathcal{L}(q,\dot q)\text{"}}\right)}_{S[\vec q,\vec p]}\right]}$$

$$\lim_{N\to\infty} \prod_k \int \frac{d\vec q_k\, d\vec p_k}{2\pi\hbar}$$

$$\frac{\partial S}{\partial \vec p} = \dot q - \frac{\partial H}{\partial \vec p} = 0$$

$$\dot q = \frac{\partial H}{\partial \vec p}$$

8.2. Path Integrals for scalar fields

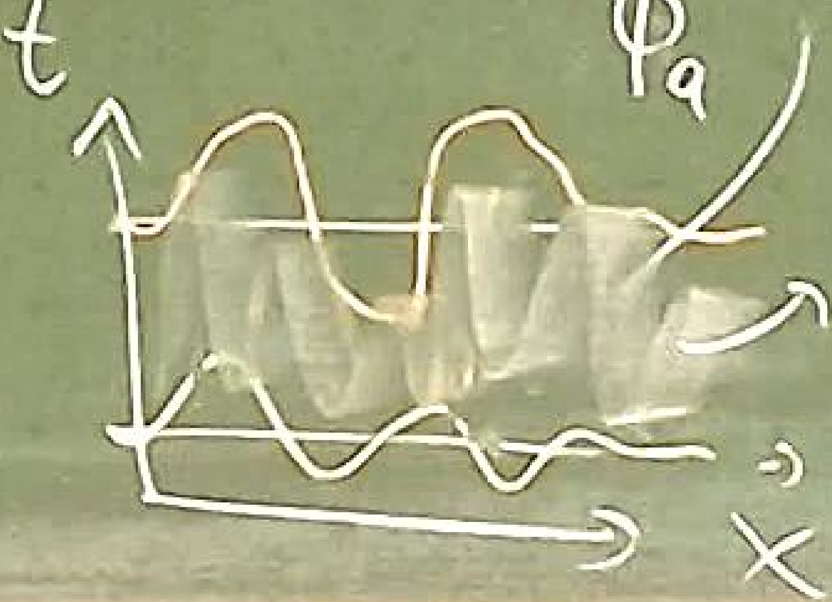
Identification: $q_i \leftrightarrow \phi(x)$

Example 8.2: Real scalar field:

$$\langle \phi_b | e^{-iHT} | \phi_a \rangle$$

$$= \int_{\phi_a}^{\phi_b} \mathcal{D}\phi\, \mathcal{D}\pi\, \exp\left[\frac{i}{\hbar}\int_0^T d^4x \left[\pi\dot\phi - \frac{1}{2}\pi^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right]\right]$$

ϕ^2



$\phi(x) \leftrightarrow q_i(t)$

$\phi_a(\vec x) \leftrightarrow q_a$

$$= \int_{\phi_a}^{\phi_b} \mathcal{D}\phi\, \exp\left[\frac{i}{\hbar}\int_0^T d^4x\, \mathcal{L}(\phi, \partial\phi)\right]$$

- Lagrangian: $\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$
- Boundaries: $\phi(\vec x, 0) \equiv \phi_a(\vec x)$
  $\phi(\vec x, T) \equiv \phi_b(\vec x)$

$\partial_\mu\phi\, \partial^\mu\phi$

$F_{\mu\nu} F^{\mu\nu}$ $\quad \phi^2$