Recap:

ϕ^4-theory:

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{n!}\phi^n$$

Superficial degree of divergence:

$$\boxed{D_{\phi^n} = d - \log_\mu[\lambda]\, V - \left(\frac{d-2}{2}\right) N_\phi}$$

Mass dimension of λ $\quad n=4,\ d=4$

$$\log_\mu[\lambda] = d - n\,\frac{d-2}{2} = 0$$

- $\log_\mu[\lambda] > 0 \Rightarrow$ Super-renormalizable (QED in $d<4$)
- $\log_\mu[\lambda] = 0 \rightarrow$ Renormalizable (QED in $d=4$, $\boxed{\phi^4 \text{ in } d=4}$) today
- $\log_\mu[\lambda] < 0 \Rightarrow$ Non-renormalizable (Einstein gravity)

7.2. Renormalized Perturbation Theory

Recipe:

(i) Compute UV-divergent amplitude with UV-regulator Λ:

$$\mathcal{M} = \mathcal{M}(m_0, e_0; \Lambda) + \mathcal{O}(\alpha_0^4)$$

(ii) Compute physical mass, charge, field strength: $Z = Z(m_0, e_0, \Lambda) + \mathcal{O}$

$m = m(m_0, e_0; \Lambda) + \mathcal{O}$, $e = e(m_0, e_0, \Lambda) + \mathcal{O}$

(iii) Renormalization:

Eliminate bare parameters in favour of physical parameters:

$$e_0 = e_0(m,e,\Lambda)\,,\quad m_0 = m_0(m,e,\Lambda)$$

(iv)

$$M(m,e) = \lim_{\Lambda\to\infty} M(m_0(m,e,\Lambda), e_0(m,e,\Lambda), \Lambda)$$

is finite and independent of Λ in all orders of α.

⇒ Bare perturbation theory

Note:  $\Rightarrow \dfrac{e_0^2}{q^2(1-\Pi(q))}$

$$\Pi = \Pi_2 + O(\alpha^2)$$

$$e = \sqrt{Z_3}\, e_0 \qquad Z_3 = \frac{1}{1-\Pi(0)}$$

$$\frac{e^2}{q^2(1-\underbrace{(\Pi_2(q)-\Pi_2(0))}_{\hat{\Pi}_2}}$$

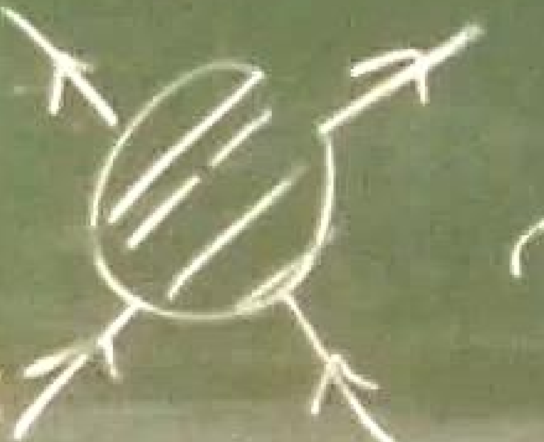
→ Equivalent formalism:

Renormalized perturbation theory

1| $\lambda\phi^4$-theory

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \frac{\lambda_0}{4!}\phi^4$$

2| $D_{\phi^4} = 4 - N_\phi$

- $N_\phi = 0$, $D_{\phi^4} = 4$.  Vacuum energy shift
- $N_\phi = 2$, $D_{\phi^4} = 2$.  $\sim \Lambda^2 + p^2 \log\Lambda$
- $N_\phi = 4$, $D_{\phi^4} = 0$.  $\sim \log\Lambda$

→ 3 diverging quantities

3| Recall:

$$\int d^4x\, e^{ipx} \langle \Omega | T \phi_r(x) \phi_r(0) | \Omega \rangle = \frac{i\cancel{Z}\,1}{p^2-m^2} + \dots \qquad p^2 \geq (2m)^2$$

Absorb Z in the fields:

$$\boxed{\phi_r := \frac{\phi}{\sqrt{Z}}}$$

4|

$$\mathcal{L}_{\phi^4} = \frac{1}{2} Z (\partial_\mu \phi_r)^2 - \frac{1}{2} m_0^2 Z \phi_r^2 - \frac{\lambda_0}{4!} Z^2 \phi_r^4$$

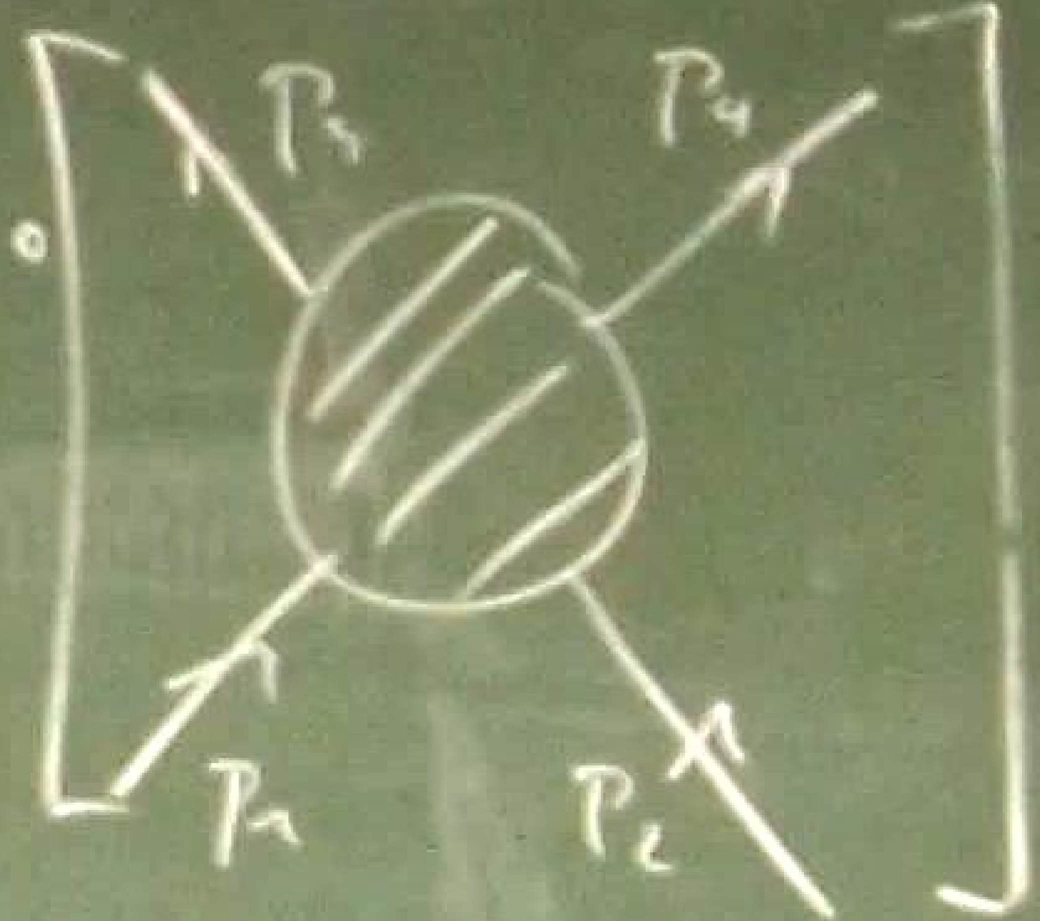
5|

$$\mathcal{L}_{\phi^4} = \frac{1}{2} (\partial_\mu \phi_r)^2 - \frac{1}{2} m^2 \phi_r^2 - \frac{\lambda}{4!} \phi_r^4 + \underbrace{\frac{1}{2} \underbrace{(Z-1)}_{\delta_Z} (\partial_\mu \phi_r)^2 - \frac{1}{2} \underbrace{(m_0^2 Z - m^2)}_{\delta_m} \phi_r^2 - \frac{1}{4!} \underbrace{(\lambda_0 Z^2 - \lambda)}_{\delta_\lambda} \phi_r^4}_{\text{Counter terms}}$$

$\rightarrow \delta_Z, \delta_m, \delta_\lambda$ absorb unobservable diverging shifts of bare and physical quantities.

6| Renormalization conditions:

$$\stackrel{!}{=} \frac{i}{p^2 - m^2} + \dots$$



$$\stackrel{!}{=} -i\lambda$$

fct 4

$p_i = (m, 0)$

7| Perturbation theory

→ Feynman rules for renormalized perturbation theory.

1. Edges:  $= \dfrac{i}{p^2 - m^2 + i\varepsilon}$

2. Vertices:  $= -i\lambda$

 $= -i\delta_\lambda$

 $= i(p^2\delta_Z - \delta_m)$

3. External lines:  $= 1$

4. Momentum conservation at vertices

5. Integrate undetermined momenta

6. Divide by sym. factor.

8| Procedure:

(i) Sum all relevant Feynman diagrams

(ii) UV diverging integrals $\Rightarrow$ Regulator Λ, ε

(iii) Result depends on
- λ, m
- $\delta_\lambda, \delta_m, \delta_Z$
- Regulators Λ, ε

(iv) Choose ('renormalize') parameters $\{\delta\}$ to satisfy renorm. conditions.

$$\delta = \delta(m, \lambda, \Lambda)$$

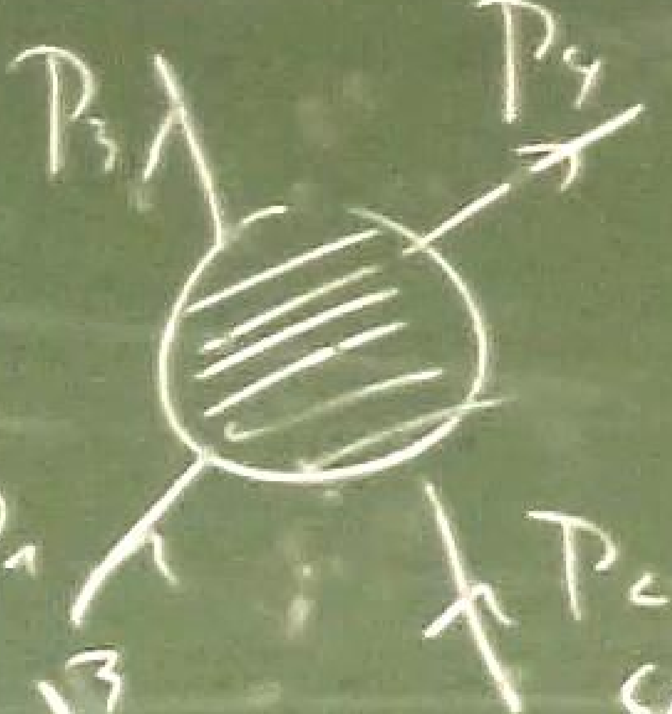
(v) With these δ the amplitude is finite and independent of the regulator

9] Bare and renormalized perturbation theory are equivalent !

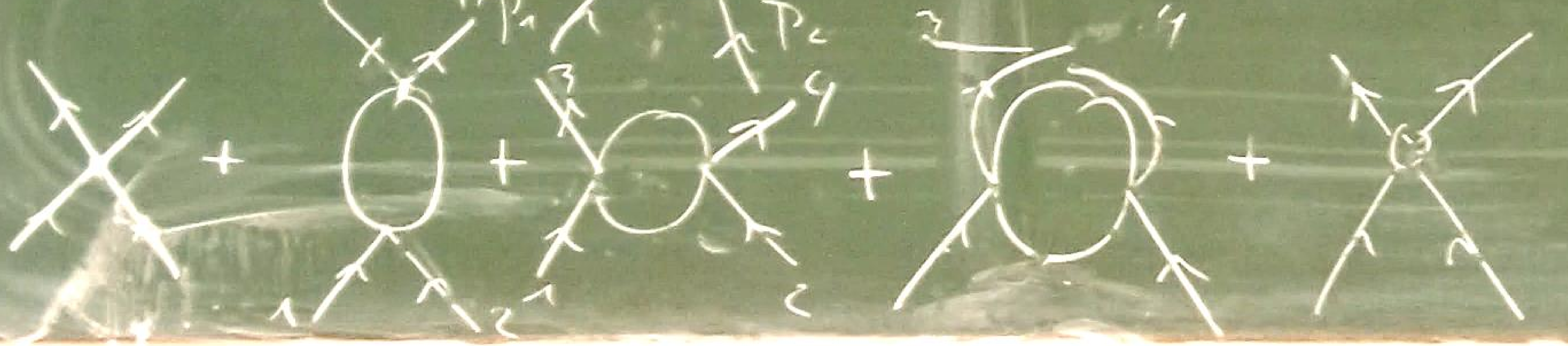
10] Example:

i) $\mathcal{M}$ Amplitude.

$\mathcal{M}(p_1 p_2 \to p_3 p_4) =$



$\approx$



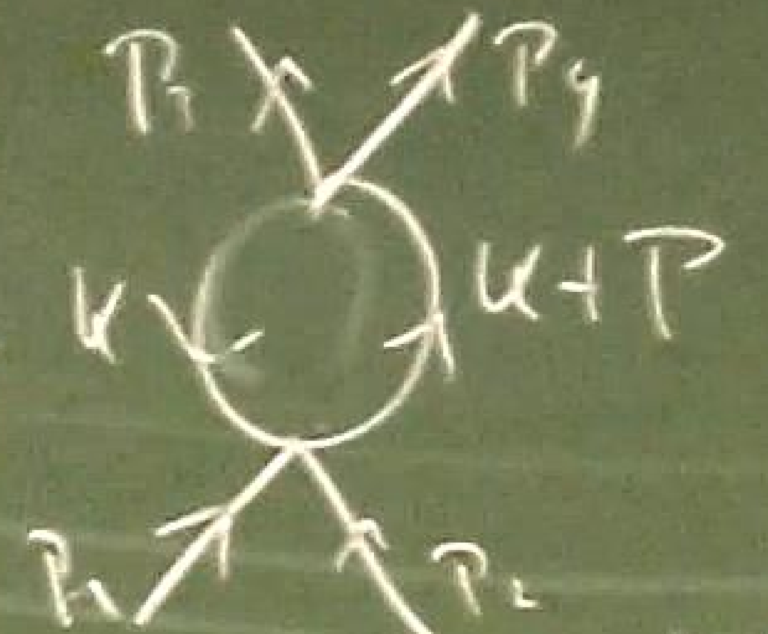
$\overset{\circ}{=} -i\lambda + (-i\lambda)^2 \left[i V(s) + i V(t) + i V(u) \right]$

$-i\delta_\lambda$

Mandelstam variables

$s = (p_1+p_2)^2 \quad t = (p_3-p_1)^2 \quad u = (p_4-p_1)^2$

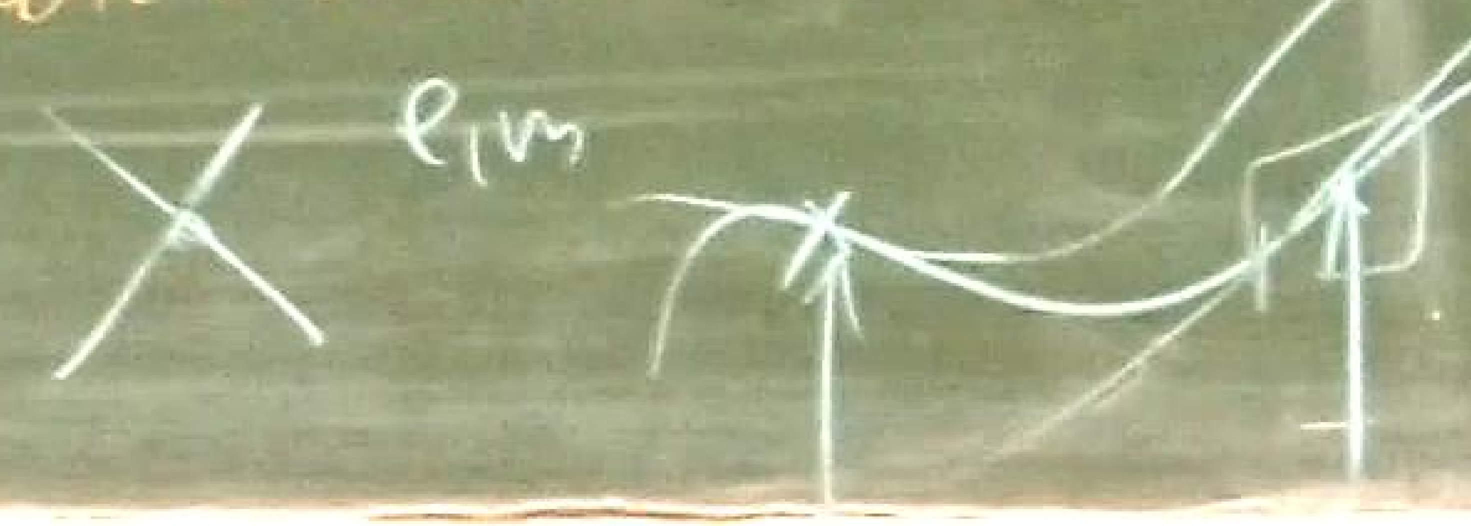
ii) $(-i\lambda)^2 \, i V(s) =$



$p^2 = (p_1+p_2)^2 = s$

$= \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} \frac{i}{(k+p)^2-m^2}$

Feynman param, Substitution, Wick rotation, Dimensional reg

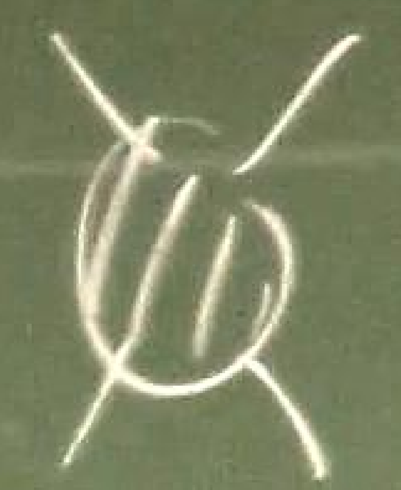


$p_i = \begin{pmatrix} m \\ \vec{0} \end{pmatrix}$

$\overset{\varepsilon \to 0}{\sim} (-i\lambda)^2 \frac{i}{32\pi^2} \int_0^1 dx \left\{ \boxed{\frac{2}{\varepsilon}} - \gamma + \log(4\pi) - \log\left[m^2 - x(1-x)p^2\right] \right\}$

iii] Enforce renormalization condition:

$$\boxed{i\mathcal{M}\Big|_{\substack{s=4m^2\\ t=u=0}} \overset{!}{=} -i\lambda}$$



$$\delta_\lambda = -\lambda^2\left[V(4m^2)+2V(0)\right]$$

$$\overset{\varepsilon\to 0}{\sim} \frac{\lambda^2}{32\pi^2}\int_0^1 dx \left\{\boxed{\frac{6}{\varepsilon}} - 3\gamma + 3\log 4\pi - \log\left[m^2-(x(1-x))4m^2\right] - 2\log\left[m^2\right]\right\}$$

iv] Amplitude:

$$i\mathcal{M} = -i\lambda - i\lambda^2\, \mathcal{F}(\{p_i\}, m)$$

finite function of momenta

Regulator ε drops out!

$$-iM^2(p^2) \simeq$$

$$+$$

$$d = 4-\varepsilon$$

$$= -\frac{i\lambda}{2}\frac{1}{(4\pi)^{d/2}}\frac{\Gamma\left(1-\frac{d}{2}\right)}{(m^2)^{1-d/2}} + i\left(p^2\delta_Z - \boxed{\delta_m}\right)$$

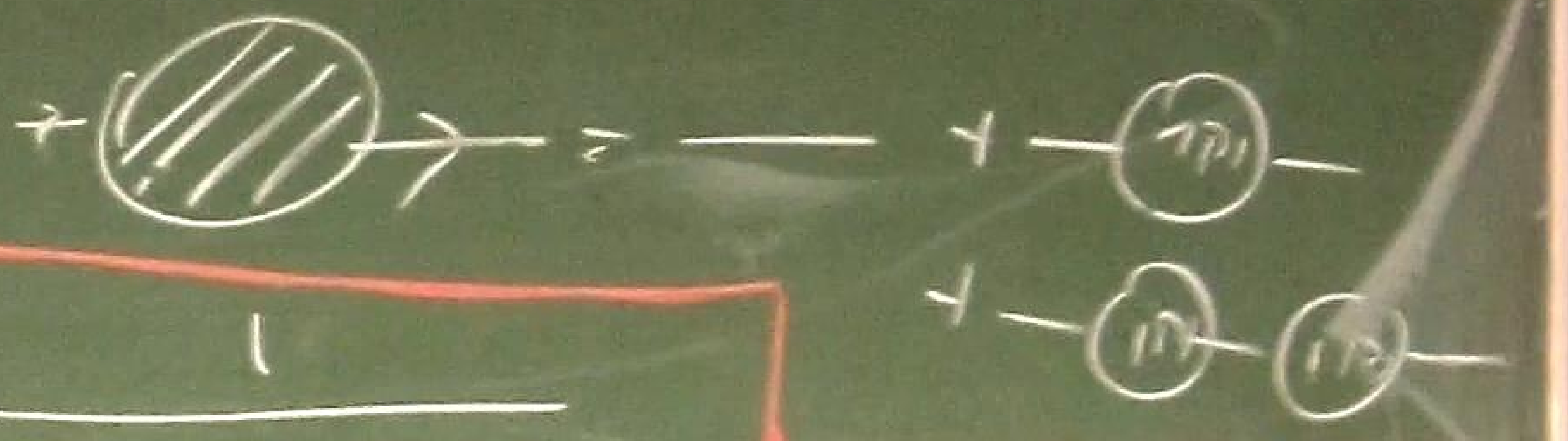
$$\Rightarrow \boxed{\delta_Z = 0}$$

v]

a)

$$= -iM^2(p^2)$$

b)



$$= \frac{1}{p^2-m^2-M^2(p^2)} \overset{!}{=} \frac{i\cdot 1}{p^2-m} + \dots$$

- $M^2(m^2) \overset{!}{=} 0$
- $\frac{dM^2}{dp^2}\Big|_{p^2=m^2} \overset{!}{=} 0$