7. Systematics of Renormalization

1| Goal: Classify UV-divergencies in QED

2| Def:

N_e = # ext. e-lines

N_γ = # ext phot.-lines

P_e = # e-propagators: $\prod^{P_e} \frac{1}{\not{k}_i - m}$

P_γ = # phot.-prop.: $\prod_i^{P_\gamma} \frac{1}{k_i^2}$

V = # vertices

L = # independent loops: $\prod_i^{L} \int \frac{d^4 k_i}{(2\pi)^4}$

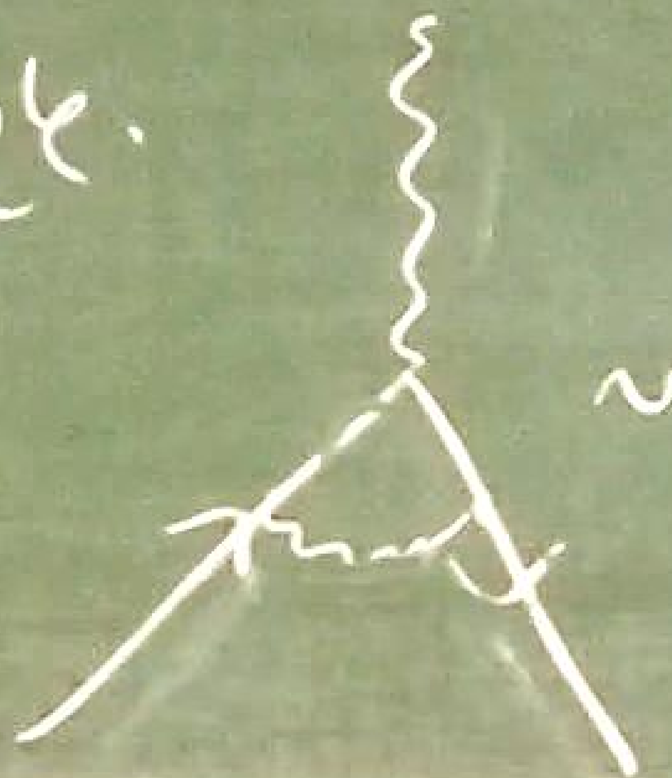
3| Superficial degree of divergence:

$$\boxed{D_{QED} = (3L + L) - (P_e + 2P_\gamma)}$$

Intuition:

$$D_{QED} \begin{cases} > 0 & : \text{Divergence with } \Lambda^D \\ = 0 & : \text{Divergence with } \log\Lambda \\ < 0 & : \text{No divergence} \end{cases}$$

Example:

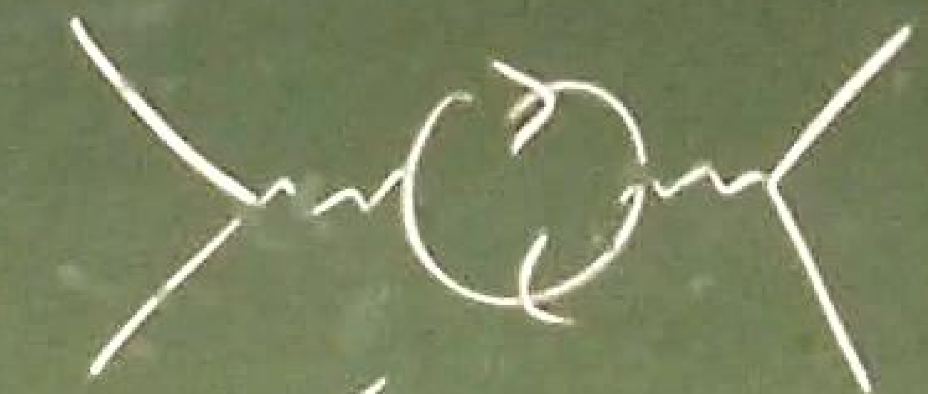


$\sim \log\Lambda$, $D_{QED} = 4 - (2 + 2\cdot 1) = 0$

However: Not always correct.

*  $\sim \log\Lambda$ $D_{QED} = 4 - (2+2\cdot 0) = 2$

(Cancellations can reduce divergence)

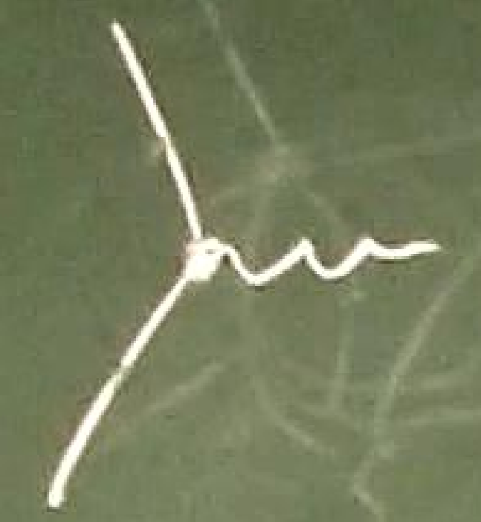
*  $\sim \log\Lambda$ $D_{QED} = 4 - (2+2\cdot 2) = -2$

*  ~ 1 $D_{QED} = 4 \cdot 0 - (0 + 2\cdot 0) = 0$

4] • $L = \underbrace{P_e + P_\gamma}_{E} - V + 1$

• $V = 2P_\gamma + N_\gamma$

$= \frac{1}{2}(2P_e + N_e)$



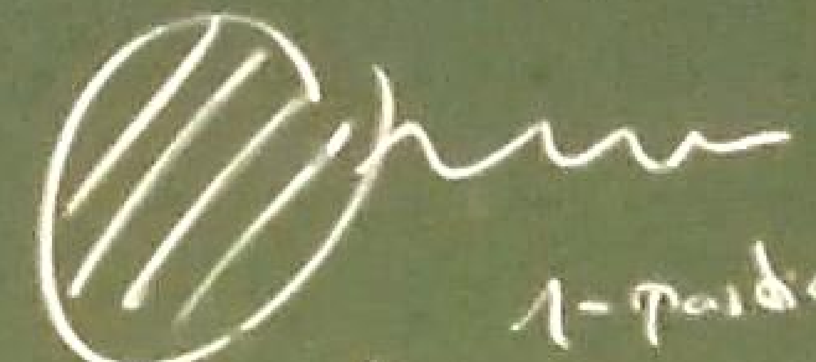
$$\Downarrow$$

$$\boxed{D_{QED} = 4 - N_\gamma - \frac{3}{2} N_e}$$

→ Independent of number of vertices

5] Furry's theorem:

> Feynman diagrams with an odd # of external photons as only external lines vanish.

Example:  $= 0$

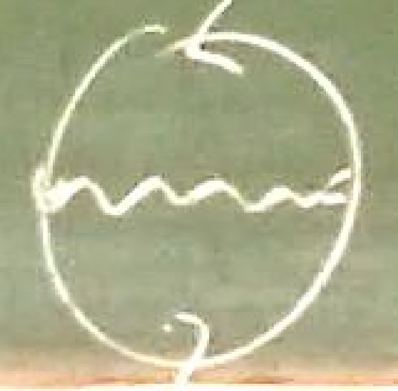
1-particle irreducible

6] Enumerate all amplitudes with $D_{QED} \geq 0$:

1] $N_e = 0$:

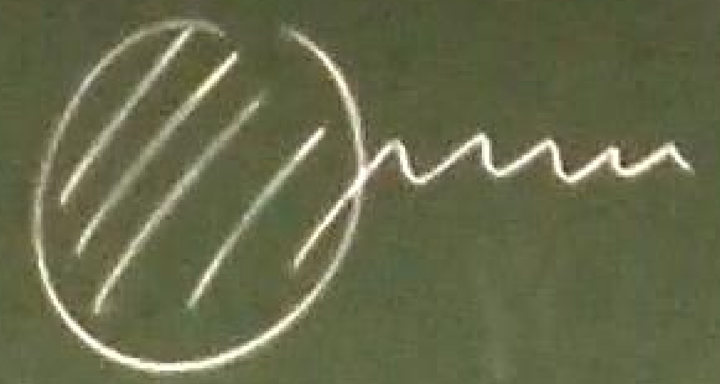
a] $N_\gamma = 0$:  ~ badly divergent

→ $D = 4$

 ↳ Unobservable

Vacuum energy shift → ignore

b] $N_\gamma = 1$ $\quad D = 3$

 $\overset{\text{Furry}}{=} 0$

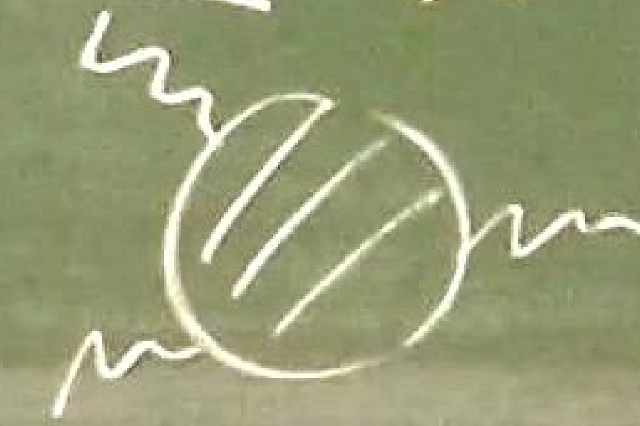
c] $N_\gamma = 2$ $\quad D = 2$

$= (g^{\mu\nu} q^2 - q^\mu q^\nu)\,\Pi(q^2)$

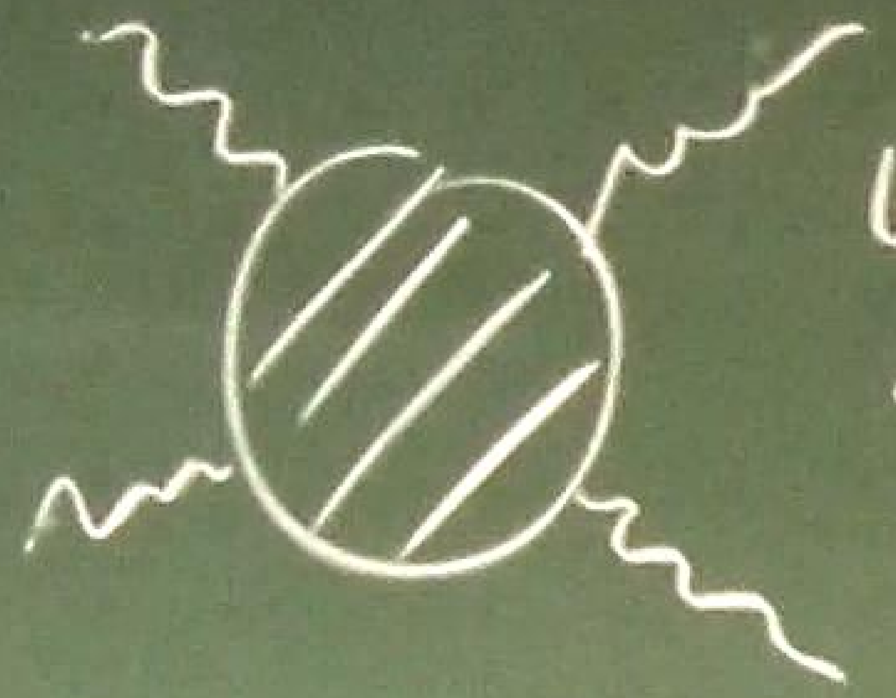
$\sim$ —"— $\frac{\text{const}}{\varepsilon}$

$\sim$ —"— $\underbrace{\log\Lambda \cdot \text{const.}}_{a_0(\Lambda)}$

d] $N_\gamma = 3$ $\quad D = 1$

 $\overset{\text{Furry}}{=} 0$

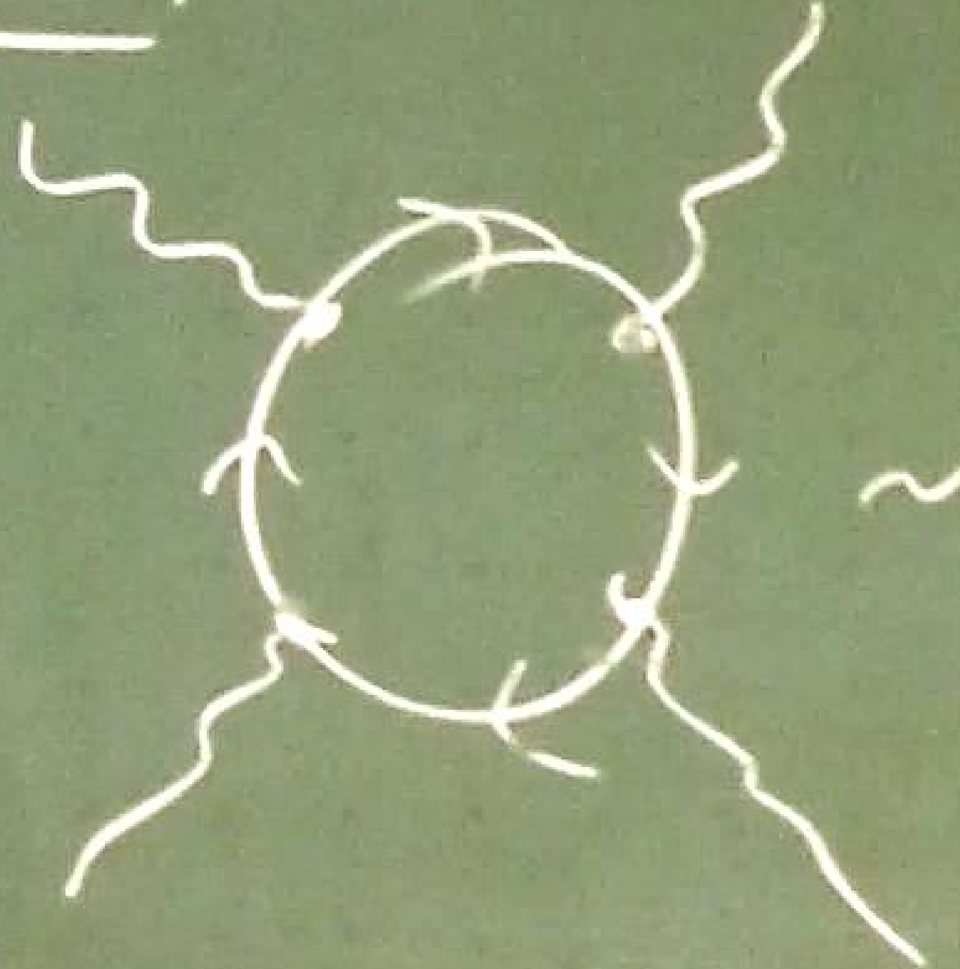
e] $N_\gamma = 4$ $\quad D = 0$



Ward

~ 1

Note:



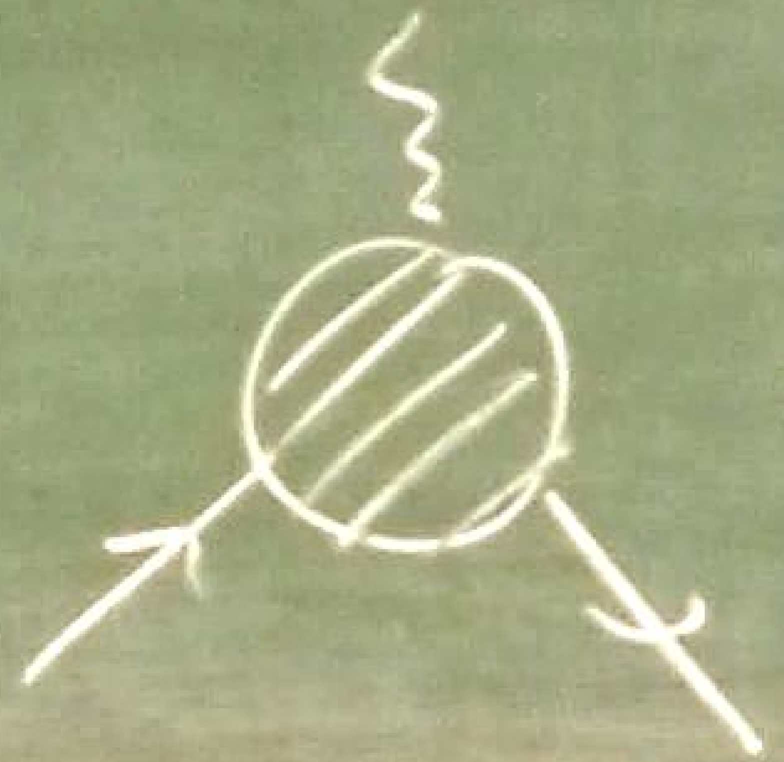
$\sim \alpha^2$

ii] $N_e = 2$ $\quad D = 4 - N_\gamma - \frac{3}{2} N_e$

a] $N_\gamma = 0$ $\quad D = 1$

$$= \underbrace{\text{const}\cdot \log\Lambda}_{a_1(\Lambda)} + \not{p}\, \underbrace{\log\Lambda \cdot \text{const}}_{a_2(\Lambda)}$$

b] $N_\gamma = 1$ $\quad D = 0$



$$\sim \underbrace{-ie\gamma^\mu \overbrace{\log(\Lambda)}^{F_1}}_{a_3(\Lambda)}$$

→ Diagrams only diverge if they contain one of the three diverging subdiagrams

→ QED contains only four UV-divergent numbers: a_0, a_1, a_2, a_3

7] Idea: Absorb finite number of div. quantities into finite number of diverging unobservable Lagrangian parameters

8| Generalization: ∢ QED in d spacetime dimensions.

$$\boxed{\begin{aligned} D_{QED} &= dL - (P_e + 2P_\gamma) \\ &\hat= d + \left(\frac{d-4}{2}\right)V - \left(\frac{d-2}{2}\right)N_\gamma - \frac{(d-1)}{2}N_e \end{aligned}}$$

→ For $d<4$: diagrams of high-enough order converge

→ For $d=4$: D_{QED} is independent of the order of V

→ For $d>4$: all diagrams of high-enough order diverge

9| • Super-Renormalizable theory
(example: QED in $d<4$)
· Only finite number of divergent diagrams

• Renormalizable theory
(ex. QED $d=4$)
· Only finite number of divergent amplitudes

• Non-renormalizable theory
(ex. QED in $d>4$)
All amplitudes diverge at sufficiently high order in perturbation theory.

Alternative Approach.

1| ∢ ϕ^4-theory

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

2| N_ϕ = # external lines

P_ϕ = # propagators

V = # vertices

L = # indep. loops

3| Superficial degree of divergence

$$\boxed{D_{\phi^4} = dL - 2P_\phi \hat= d + \left[4\left(\frac{d-2}{2}\right) - d\right]V - \left(\frac{d-2}{2}\right)N_\phi}$$

4) Alternative approach: Dimensional analysis

i) $\hbar = c = 1$

$$\lambda_c = \frac{h}{mc} = \frac{2\pi}{m}$$

$$\to \quad [\lambda_c] = M^{-1}$$

← dimension of mass

ii) $[S] = 1$

iii) $S = \int d^d x \, \mathcal{L}$

$$[d^d x] = M^{-d}$$

$$\Rightarrow \quad [\mathcal{L}] = M^d$$

iv) $\mathcal{L} = (\partial_\mu \phi)^2 - m^2 \phi^2 - \lambda \phi^4$

$$[\partial_\mu] = M \quad \Rightarrow \quad [\phi] = M^{\frac{d-2}{2}}$$

$$[m] = M^1$$

$$\boxed{[\lambda] = M^{d - \frac{n(d-2)}{2}}}$$

v) Amplitude $\mathcal{M}$ with N_ϕ external lines

$$\to \quad \boxed{[\mathcal{M}] = M^{d - \frac{N_\phi(d-2)}{2}}}$$

vi) Diagram with V vertices

$$\mathcal{M} \sim \lambda^V \Lambda^D$$

$$\to \quad [\lambda]^V [\Lambda]^D = [\mathcal{M}] = M^{d - \frac{N_\phi(d-2)}{2}}$$

$$\hookrightarrow \quad [\Lambda] = M$$

$$\boxed{D = d - \log_M [\lambda] \, V - \left(\frac{d-2}{2}\right) N_\phi}$$

$$\log_M [\lambda] = d - \frac{n(d-2)}{2}$$

- Super renormalizable: $\log_\mu[\lambda] > 0$
- Renormalizable: $\log_\mu[\lambda] = 0$
- Non-renormalizable: $\log_\mu[\lambda] < 0$

$\hbar = c = 1$

QED $[e] = 1$
$d = 4$

$$G = \frac{\hbar c}{m_p^2} = \frac{1}{m_p^2} \Rightarrow [G] = M^{-2}$$

$$S = \frac{1}{16\pi G}\int d^4(x)\sqrt{-g}\,R$$