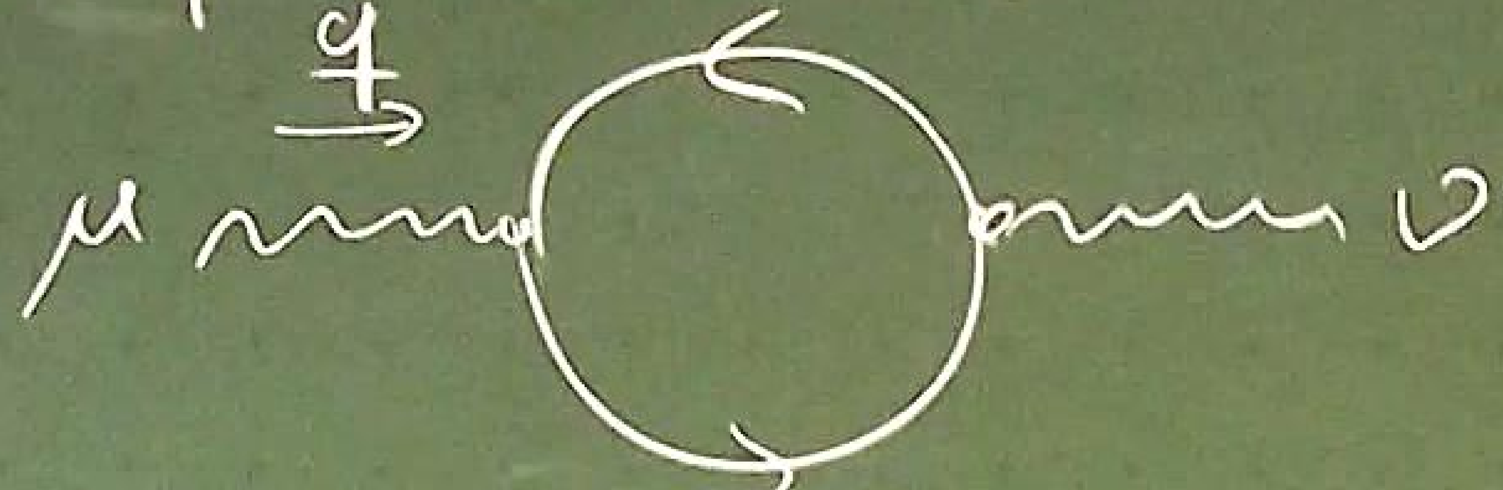
Recap.

6.5. Electric Charge Renormalization

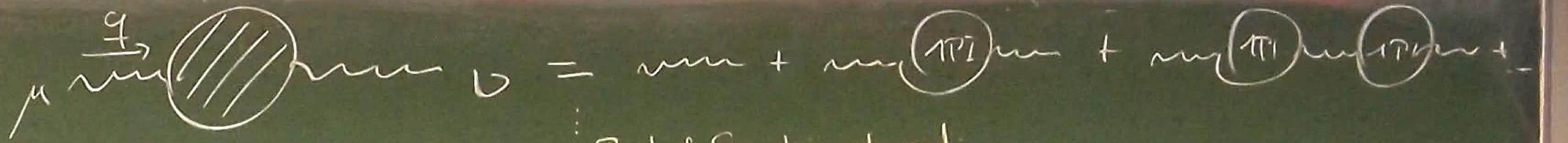
Vacuum polarization diagram:



$$= i\Pi_2^{\mu\nu}(q)$$

$$= \boxed{i\Pi^{\mu\nu}(q)} = i\left[\Pi_2^{\mu\nu}(q) + O(\alpha^2)\right]$$

$$\equiv (q^2 g^{\mu\nu} - q^\mu q^\nu)\cdot \Pi(q^2)$$

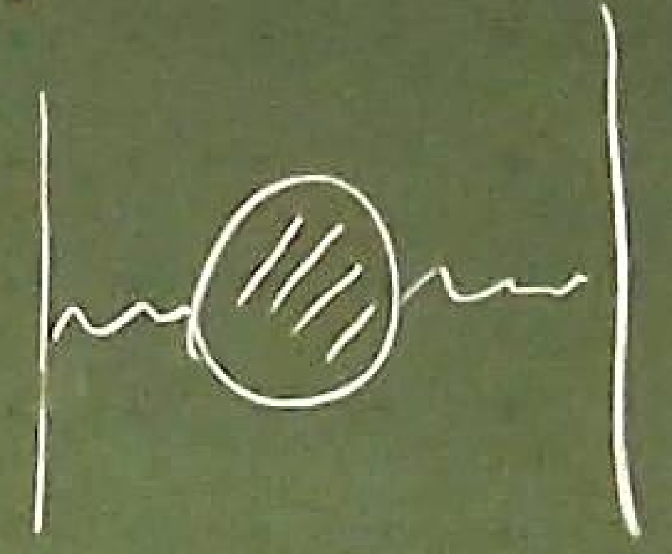


Part of S-matrix element:

$$= \frac{-ig_{\mu\nu}}{q^2\left[1-\Pi(q^2)\right]}$$

- $Z_3 = \dfrac{1}{1-\Pi(0)}$

(1) Physical charge: $\boxed{e = \sqrt{Z_3}\, e_0}$ (e_0: Bare charge)



$$= -\frac{ig_{\mu\nu}}{q^2}\,\underset{\overset{\parallel}{Z_3 e_0^2}}{e^2}$$

(2) q-dependent fine-structure constant.

$$\boxed{\alpha_{eff}(q^2) = \frac{e_0^2/4\pi}{1-\Pi(q^2)} = \frac{\alpha}{1-\underbrace{\left[\Pi_2(q^2)-\Pi_2(0)\right]}_{\hat{\Pi}_2(q^2)}} + O(\alpha^2)}$$

6] Computation of $\Pi_2^{\mu\nu}$:

i] Feynman parameters, Substitution $l = k + xq$,

ii] Wick rotation $l^0 = i l_E^0$

iii] Dimensional regularization

$$d = 4 - \varepsilon$$

spacetime dimension

- $$\int \frac{d^d l_E}{(2\pi)^d} \frac{1 \, l_E^2}{(l_E^2 + \Delta)^2} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2 - \frac{d}{2} - 1)}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2 - \frac{d}{2} - 1} \boxed{\frac{d}{2}}$$

- $$\Gamma\left(2 - \frac{d}{2}\right) = \Gamma\left(\frac{\varepsilon}{2}\right) = \frac{2}{\varepsilon} - \gamma + O(\varepsilon)$$

$$\ln \Lambda \sim \frac{1}{\varepsilon}$$

iv] $$i\Pi_2^{\mu\nu}(q) \doteq (q^2 g^{\mu\nu} - q^\mu q^\nu)\, i\Pi_2(q^2)$$

$$\boxed{\Pi_2(q^2) = \frac{-8e^2}{(4\pi)^{d/2}} \int_0^1 dx \, \frac{x(1-x)\,\Gamma(2 - \frac{d}{2})}{\underbrace{[m^2 - x(1-x)q^2]}_{=\Delta}{}^{2 - \frac{d}{2}}}}$$

$$-\left(-\frac{2}{d} g^{\mu\nu} l_E^2 + g^{\mu\nu} l_E^2\right) = -\left(1 - \frac{2}{d}\right) g^{\mu\nu} l_E^2$$

$$\downarrow \int d^d l_E$$

$$= \underbrace{\left(-\frac{d}{2} + 1\right)}_{z} \Gamma\Big(\underbrace{2 - \frac{d}{2} - 1}_{z}\Big)$$

$$\Gamma(z+1) = \Gamma\left(2 - \frac{d}{2}\right)$$

$$z\,\Gamma(z) = \underbrace{z(z-1)!}_{z!}$$

V] $$\Pi_2(q^2) \overset{0}{=} -\frac{2\alpha}{\pi} \int_0^1 dx \, x(1-x) \left[\frac{2}{\varepsilon} - \log(\Delta) - \gamma + \log(4\pi)\right] + O(\varepsilon)$$

7] $O(\alpha)$ charge renormalization.

$$\frac{e^2 - e_0^2}{e_0^2} = Z_3 - 1 = \delta Z_3 = \frac{\Pi(0)}{1 - \Pi(0)}$$

$$= \Pi_2(0) + O(\alpha^2) \qquad \frac{1}{1-x} = 1 + x + \dots$$

$$\sim -\frac{2\alpha}{3\pi\varepsilon} \xrightarrow{\varepsilon \to 0} -\infty$$

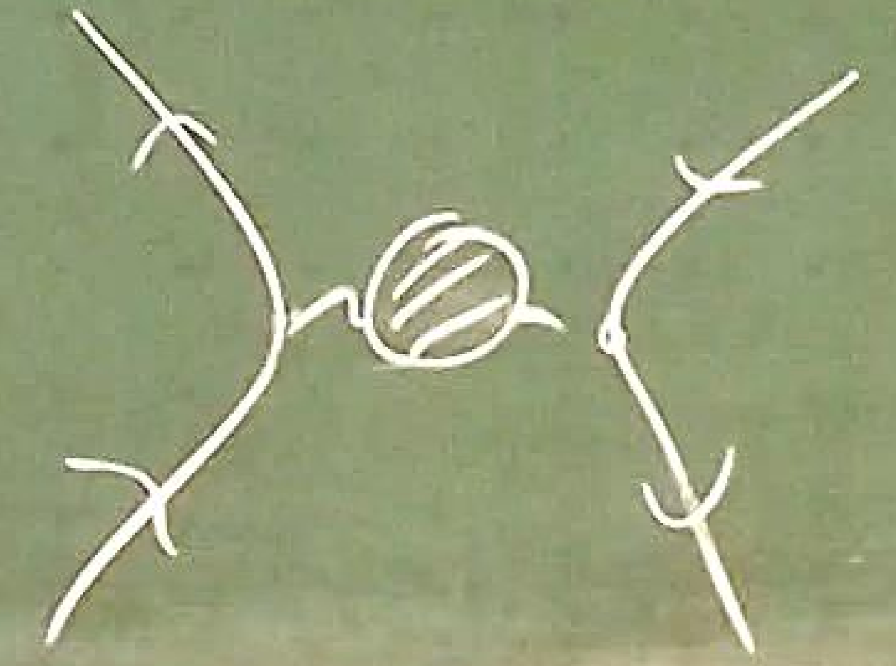
8] $O(\alpha)$ q^2-dependence of $\alpha_{eff}(q^2)$:

$$\hat{\Pi}_2(q^2) = \Pi_2(q^2) - \Pi_2(0)$$
$$= -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\log\left[\frac{m^2}{m^2 - x(1-x)q^2}\right]$$

9) Analysis + Interpretation of $\hat{\Pi}_2(q)$:

i] ✗ Effective potential in non-rel limit.

$(|q^2| \ll m^2,\quad q^2 = (p-p')^2 \approx -|\vec{p} - \vec{p}\,'|^2 = -\vec{q}^{\,2}$



$$\frac{e^2}{q^2[1-\hat{\Pi}(q^2)]}$$

ii] $$V(\vec{x}) = \underbrace{\int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\vec{x}}}\ \frac{-e^2}{|\vec{q}|^2\underbrace{(1-\hat{\Pi}_2(-\vec{q}^{\,2}))}}$$

$$\frac{1}{1-x} = 1 + x + \dots$$

$$1 + \hat{\Pi}_2(-\vec{q}^{\,2}) + O(\alpha^2)$$

$-q^2 \ll m^2$

$$\frac{-e^2}{\vec{q}^{\,2}}\left[1 + \frac{\alpha}{15\pi m^2}\vec{q}^{\,2}\right] + O(\alpha^3)$$

$$\hat{=} -\frac{\alpha}{|\vec{x}|}\ \boxed{-\frac{4\boxed{\alpha^2}}{15m^2}\delta^{(3)}(\vec{x})} + O(\alpha^3)$$

→ EM force becomes stronger for $|\vec{x}| \to 0$

iii] $$\Delta E = \int d^3x\, |\psi(x)|^2\left(-\frac{4\alpha^2}{15m^2}\delta^{(3)}(x)\right)$$

$$= -\frac{4\alpha^2}{15m^2}|\psi(0)|^2 < 0$$ $l=0$

· Darwin term:

$$H_{Darwin} = \frac{\pi\boxed{\alpha}}{2m^2}\delta^{(3)}(\vec{x})$$

$${}^2S_{\frac{1}{2}} = {}^2P_{\frac{1}{2}}$$

⇒ Part of Lamb shift.

$\Delta E_{VT} \approx -27\ MHz$

$\Delta E_{ES} \approx 1011\ MHz$

$\Delta E_{AM} \approx 68\ MHz$

Lamb shift $\approx 1058\ MHz$

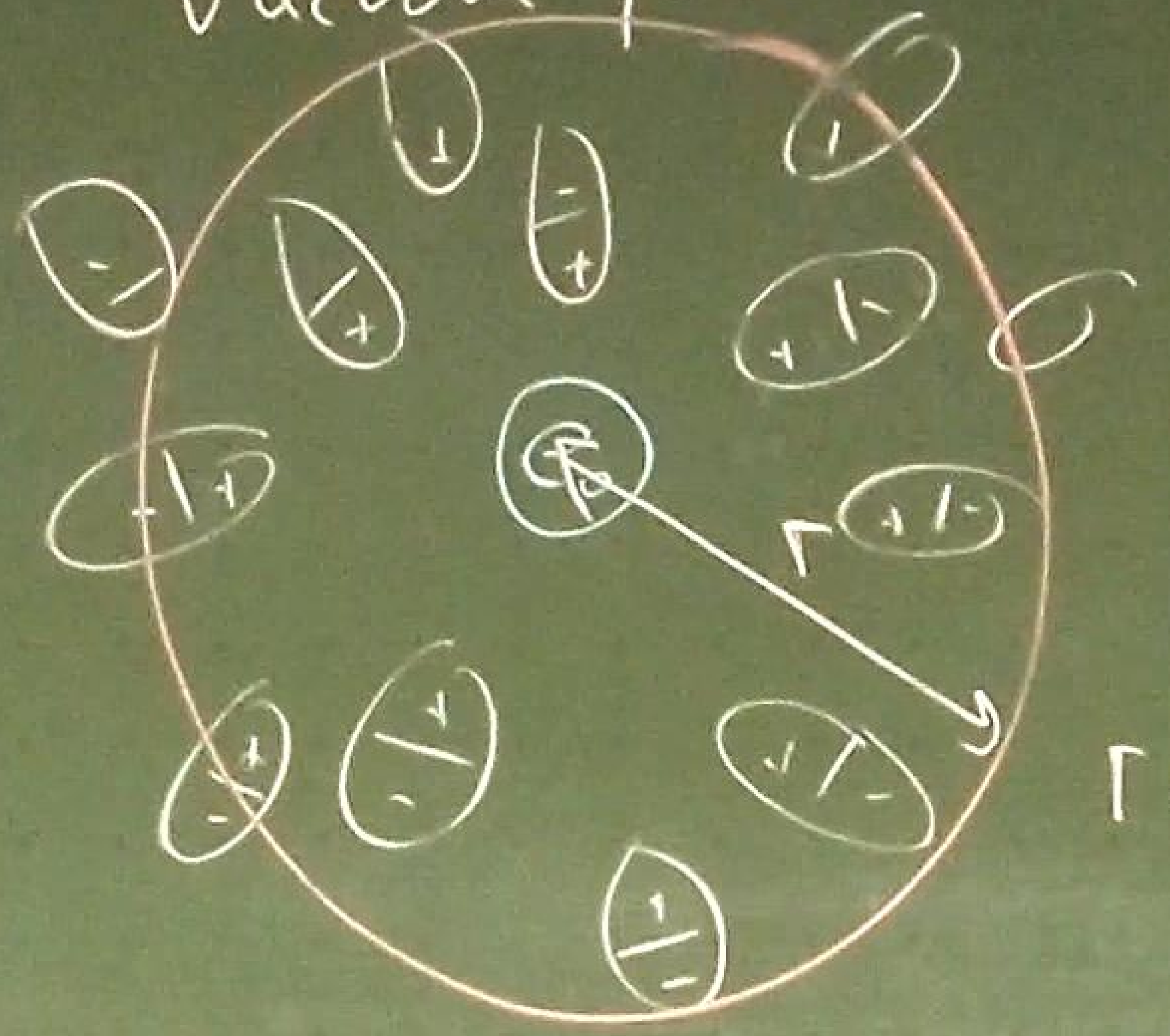
iv] More generally: Uehling potential:

$$V(r) = -\frac{\alpha}{r}\left(1 + \frac{\alpha}{4\sqrt{\pi}} \frac{e^{-2mr}}{(mr)^{3/2}} + \dots\right)$$

- $\lambda_C = \frac{h}{mc} = \frac{2\pi}{m}$
- $a_0 = \frac{\lambda_c}{2\pi\alpha} \approx 22 \cdot \lambda_c$

V] Interpretation:

Vacuum polarization



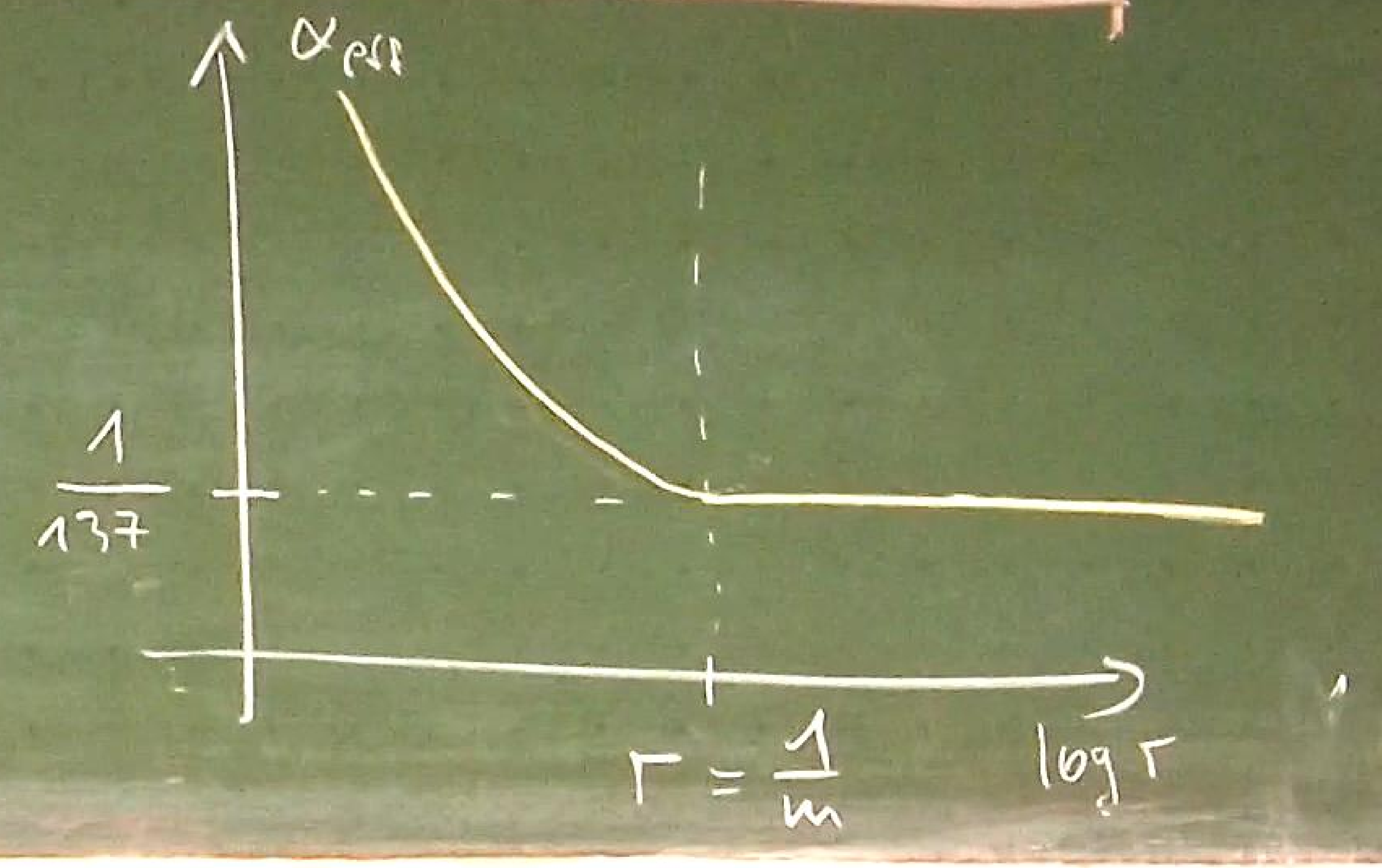
$r \sim \frac{1}{q}$

VI] Relativistic limit $-q^2 \gg m^2$

$$\hat{\Pi}_2(q^2) \doteq \frac{\alpha}{3\pi}\left[\log\left(\frac{-q^2}{m^2}\right) - \frac{5}{3} + O\left(\frac{m^2}{q^2}\right)\right]$$

$$\boxed{\alpha_{eff}(q) \underset{-q^2 \gg m^2}{\approx} \frac{\alpha}{1 - \frac{\alpha}{3\pi}\log\left(\frac{-q^2}{A m^2}\right)}}$$

$A = e^{5/3}$



Note:

$$1 - \frac{\alpha}{3\pi}\log\left(\frac{\Lambda_L^2}{\Lambda_{ML}}\right) \overset{!}{=} 0$$

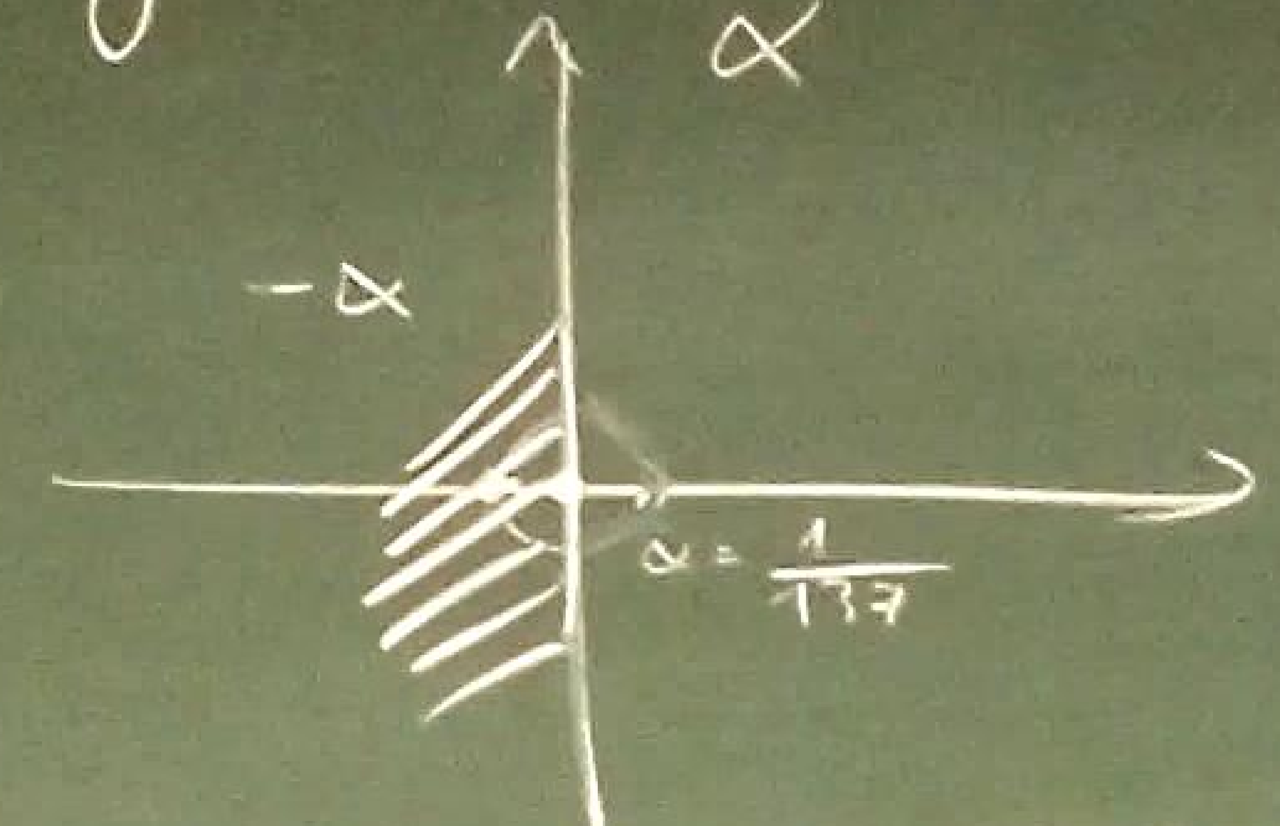
$A \sim O(1)$

$$\boxed{\Lambda_L \sim m e^{\frac{3\pi}{2\alpha}} \sim 10^{286}\,\mathrm{eV}}$$

$$E_{LHC} \approx 10^{13}\,\mathrm{eV} \ll E_{Pl} \approx 10^{28}\,\mathrm{eV} \leq \Lambda_L \approx 10^{286}\,\mathrm{eV}$$

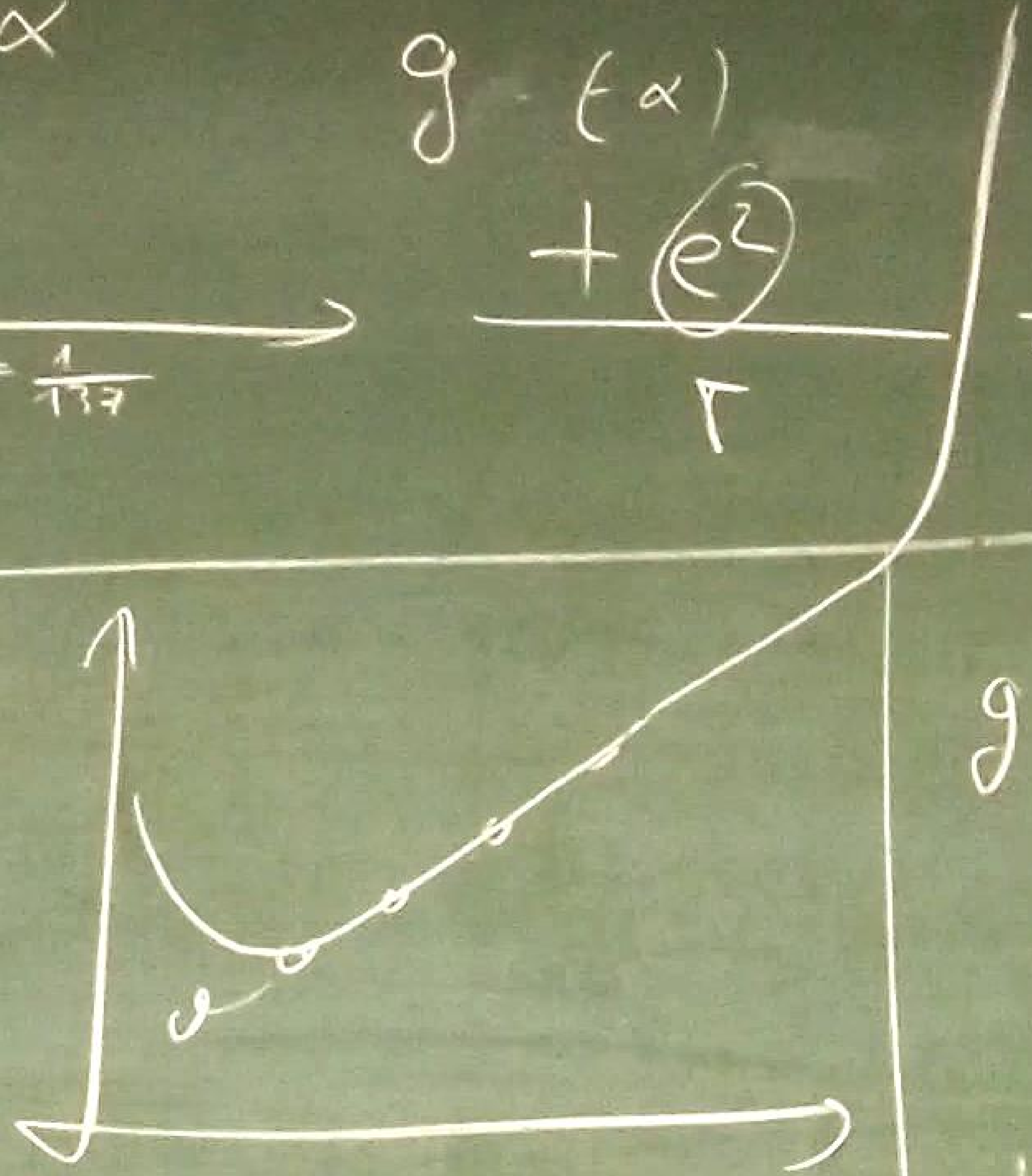
-> Landau pole

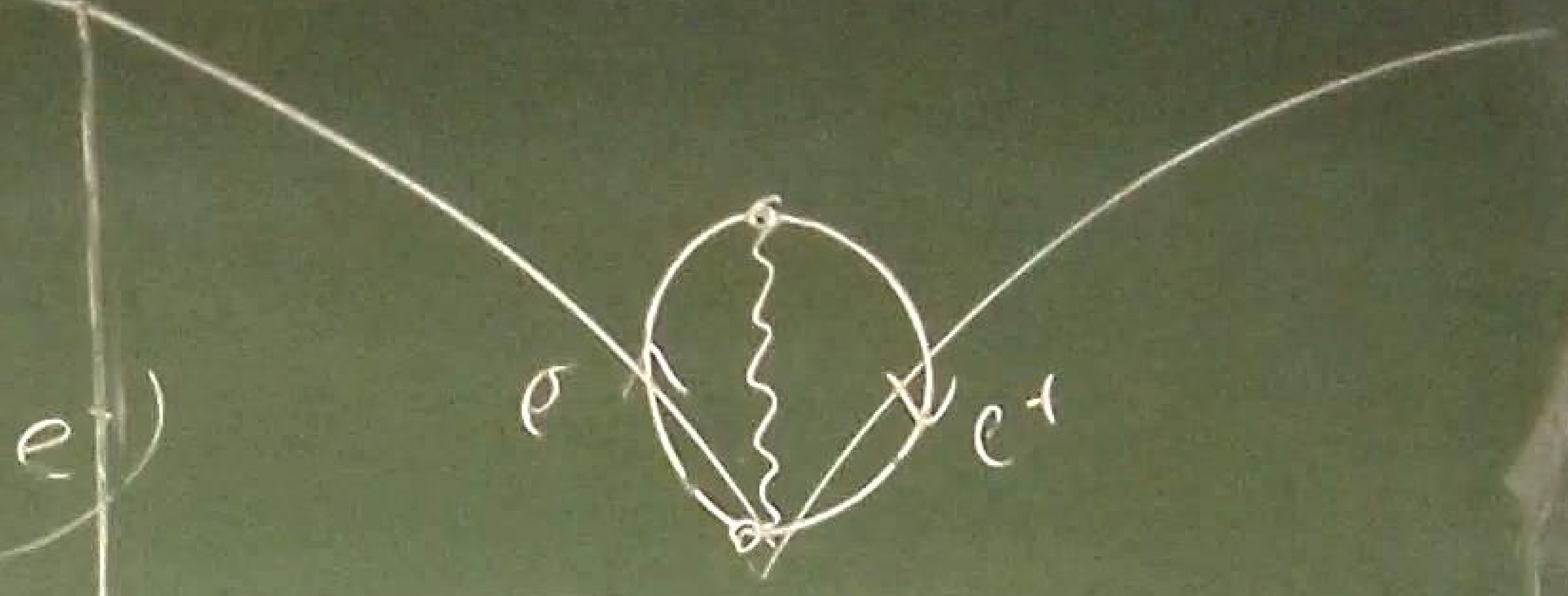
$$g(\alpha) = a_0 + a_1\alpha + a_2\alpha^2 + \dots$$



$g = f(\alpha)$

$$+\frac{e^2}{r} = V(e^-, e^+)$$



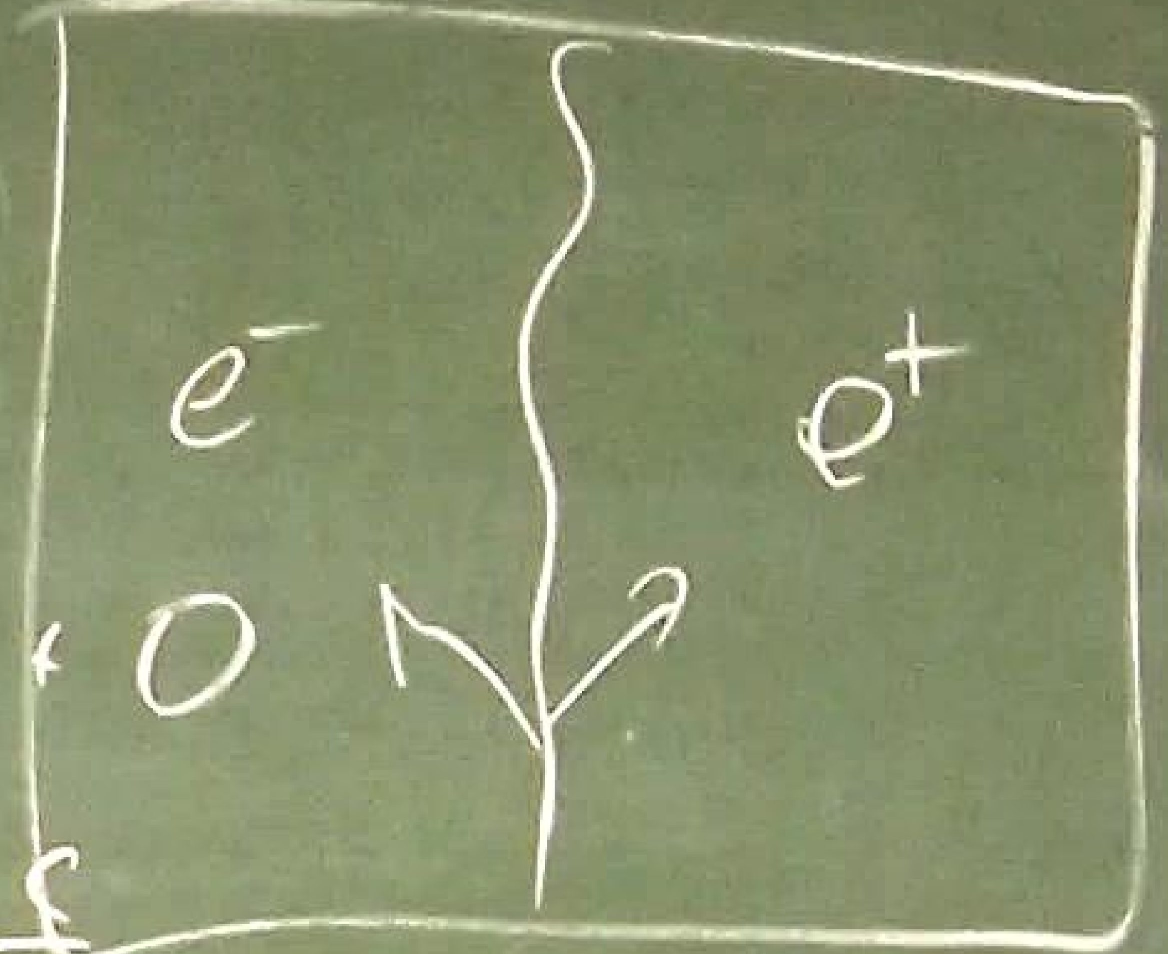


$$g = e^{-\frac{c}{\alpha}}$$

$$= 0 + 0 + 0 + 0$$

$$\alpha^N \sim e^{-\frac{c}{\alpha}}$$

$$\boxed{N \sim \frac{1}{\alpha} = 137}$$

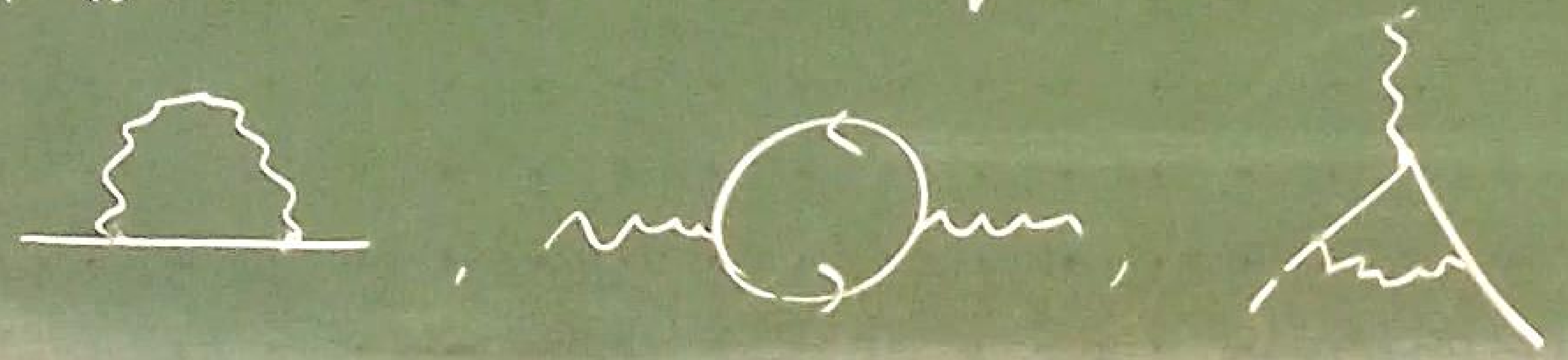


7. Systematics of Renormalization

* IR-divergences
  - Due to massless particle
  - Cancel with soft bremsstrahlung

  → Not a fundamental problem

* UV-divergences
  - Due to unbounded high momenta:



  - Diverging differences between physical and bare quantities

→ Fundamental problem

→ Study UV-divergences systematically