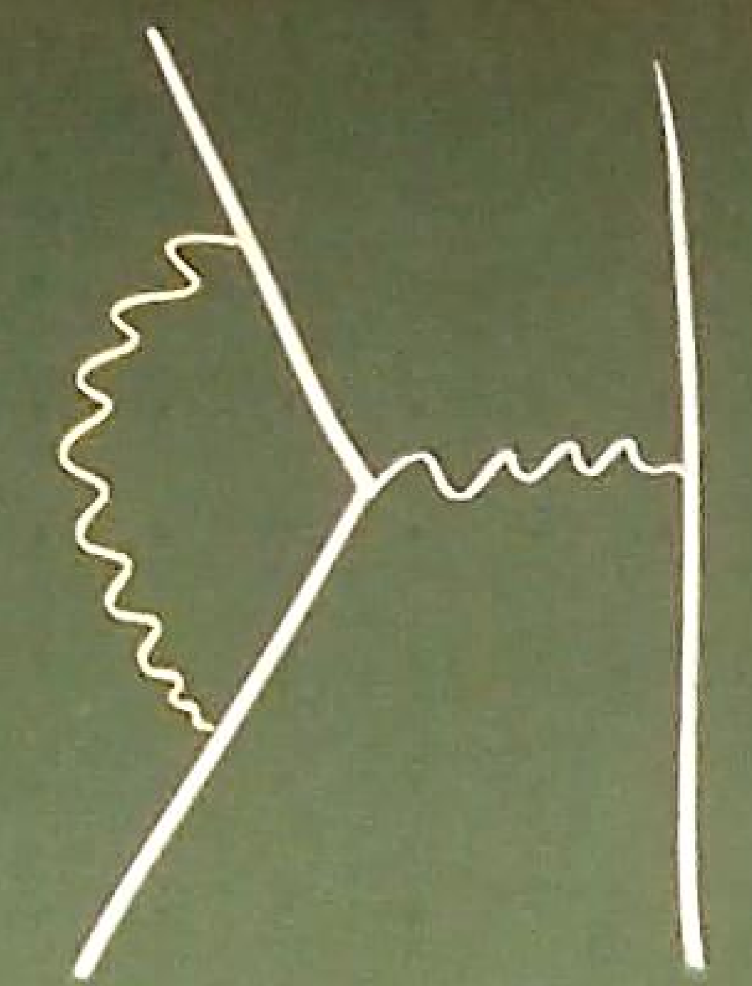
Recap:

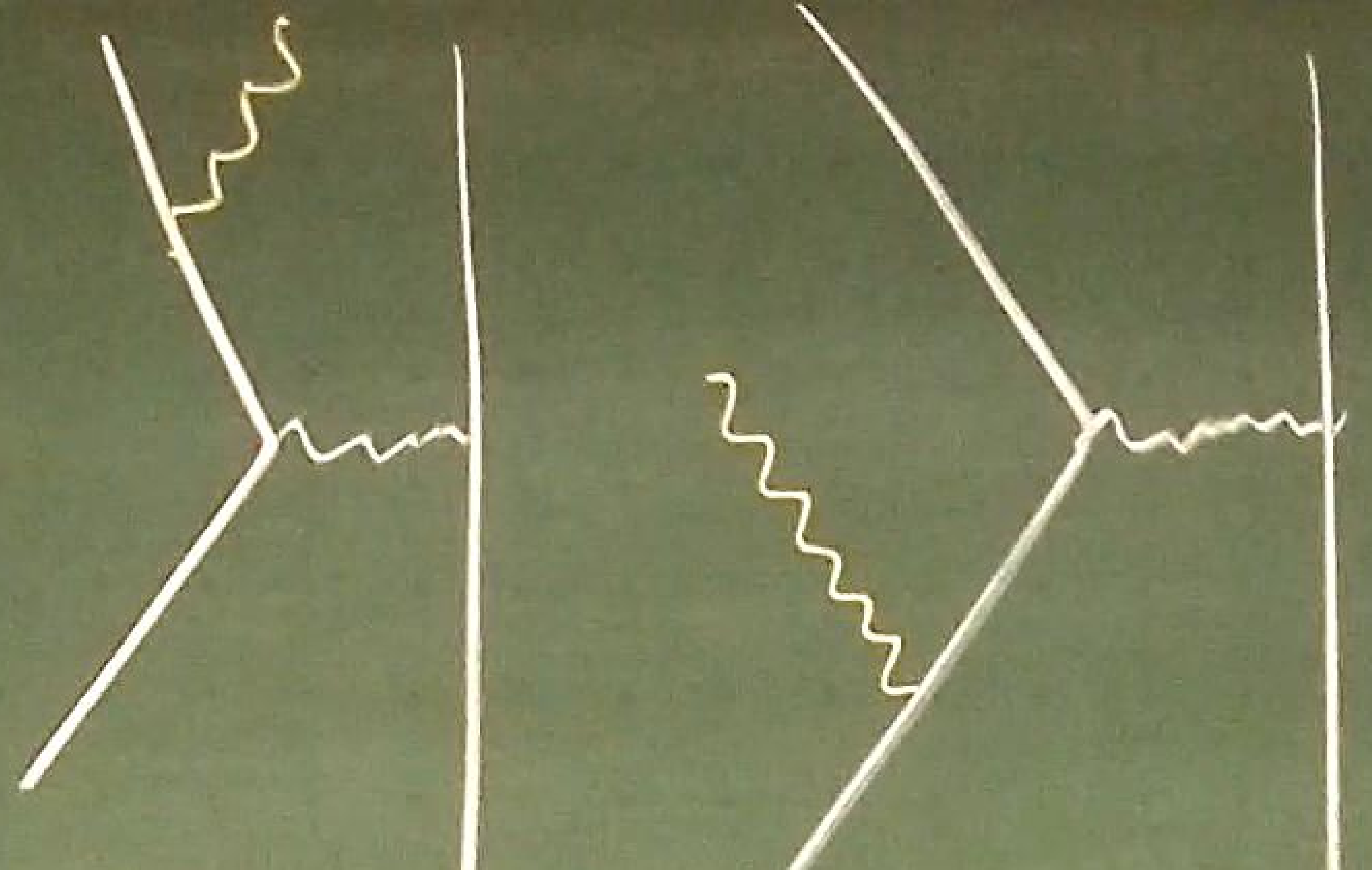
Radiative corrections:



Vertex correction

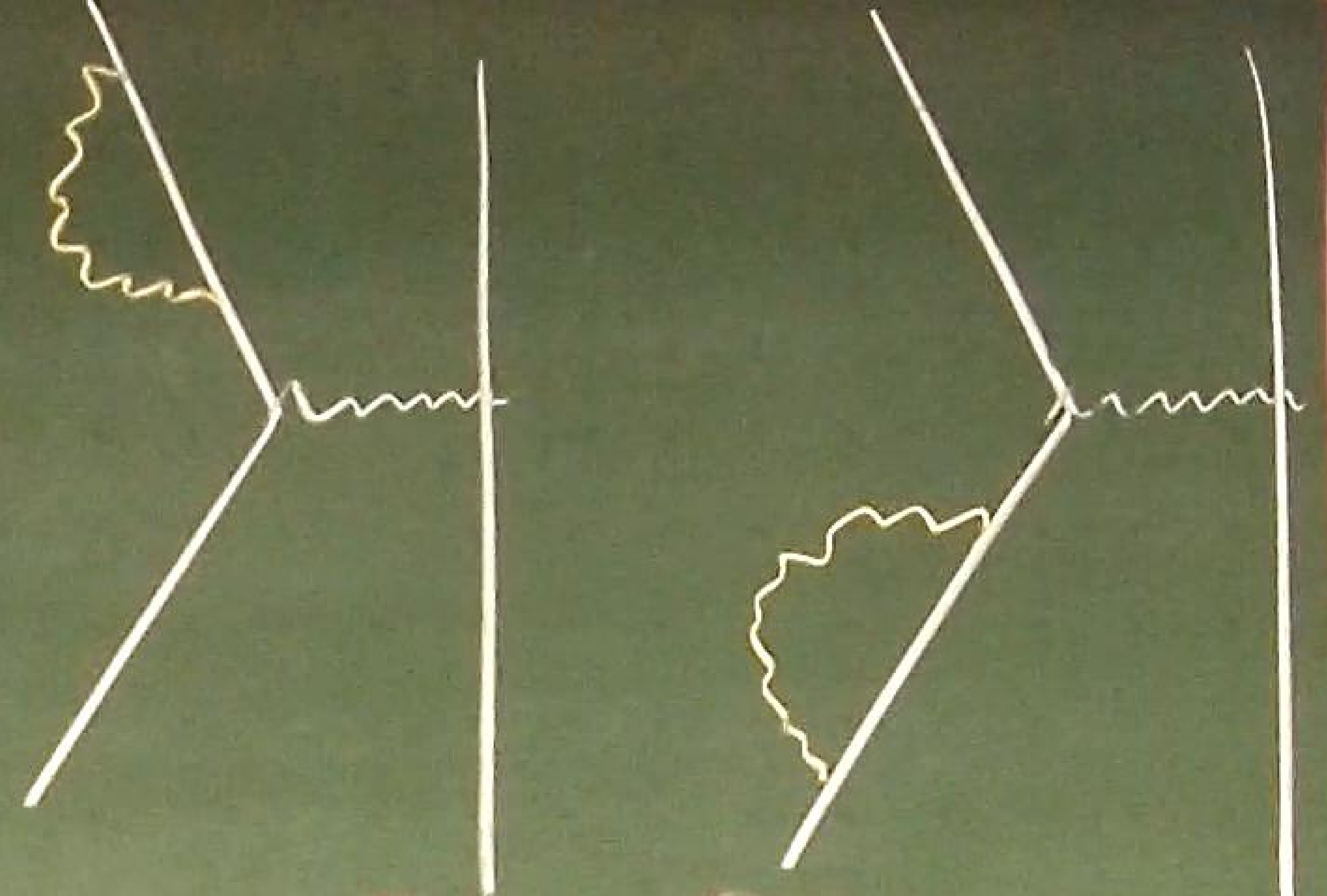
- Form factors F_1, F_2
- Anomalous magnetic moment:

$$\boxed{a = \frac{g-2}{2} = F_2(0) = \frac{\alpha}{2\pi} + \dots}$$



Soft Bremsstrahlung

- Sudakov Form factor
- Cancelled IR div. of vertex correction



Electron Self-Energy

- Field strength renormalization

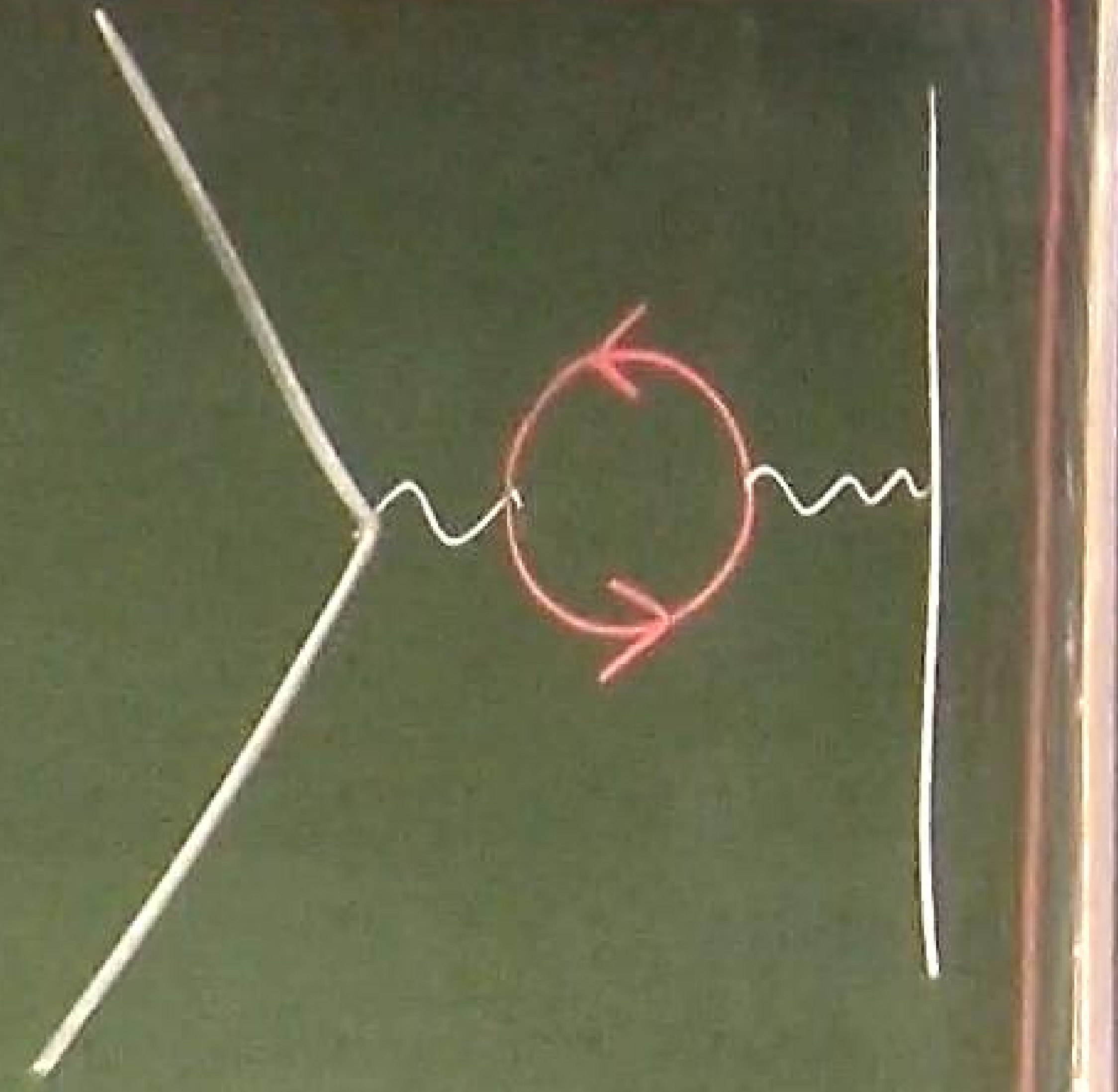
$$\boxed{Z_2 = \left(1 - \frac{d\Sigma}{d\not{p}}\bigg|_{\not{p}=m}\right)^{-1}}$$

$$-i\Sigma = \xrightarrow{p} (\mathrm{1PI}) \xrightarrow{p}$$

Sum of 1-particle irreducible diagrams

- Physical mass:

$$\boxed{m - m_0 = \Sigma(\not{p}=m)}$$



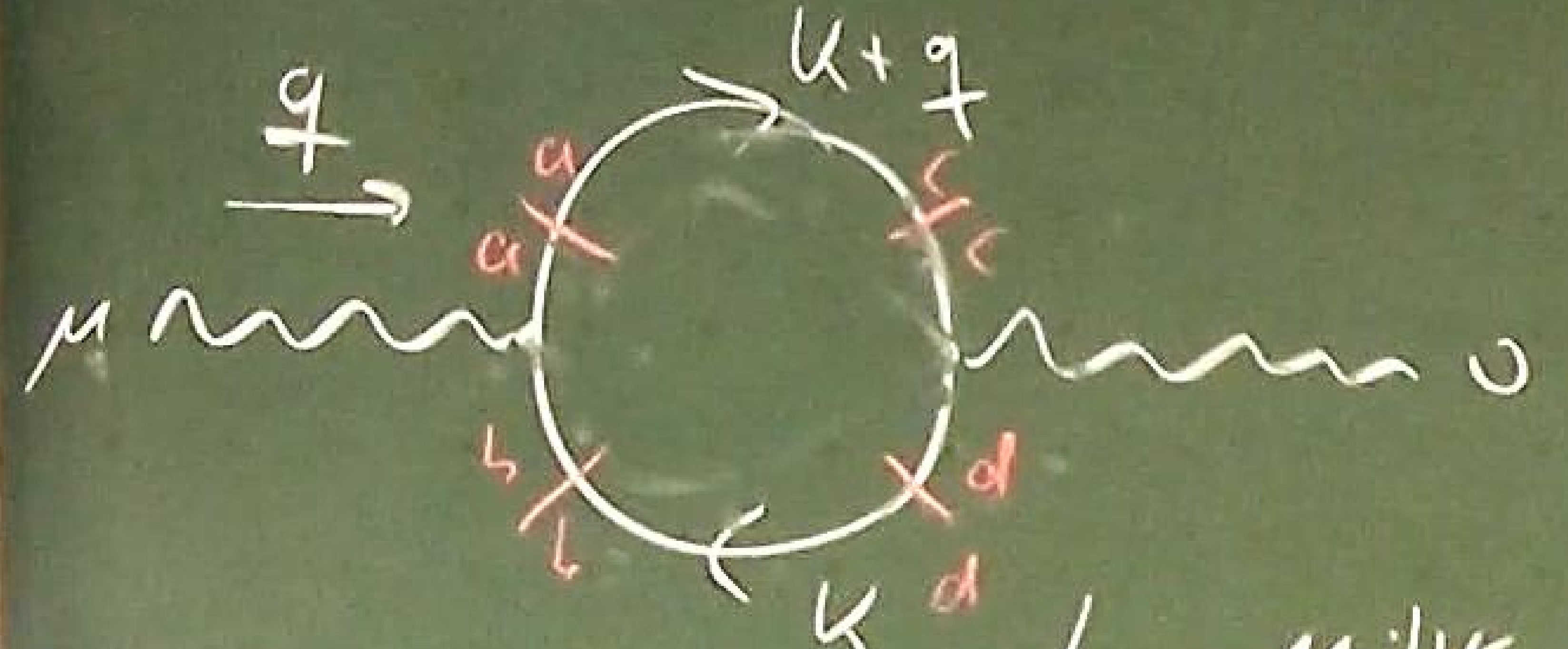
Vacuum Polarization / Photon Self-Energy

$\Downarrow$

Today

6.5 Electric Charge Renormalization

1] One-loop correction:

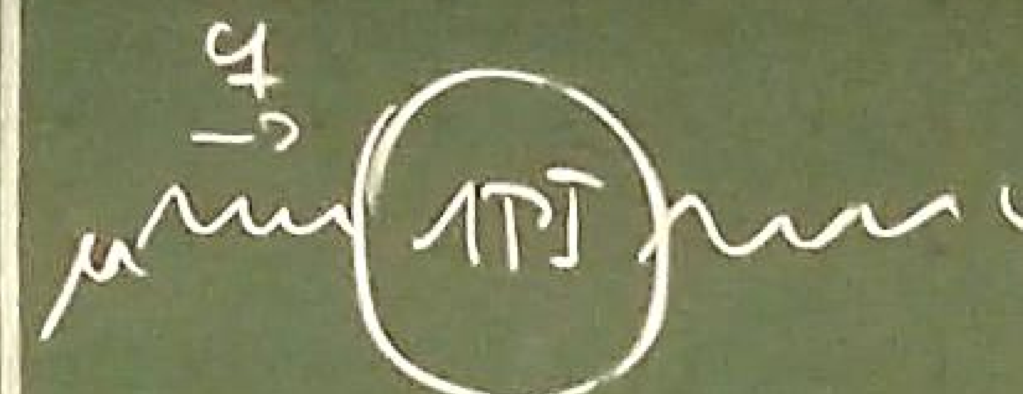


$$= (-1)(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \left(\gamma^\mu_{ab} \frac{i(\not k + m)_{bd}}{k^2-m^2} \gamma^\nu_{dc} \frac{i(\not k+\not q+m)_{ca}}{(k+q)^2-m^2} \right)$$

$$= (-1)(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^\mu \frac{i(\not k+m)}{k^2-m^2}\gamma^\nu \frac{i(\not k+\not q+m)}{(k+q)^2-m^2}\right] \equiv i\Pi_2^{\mu\nu}(q)$$

∘ $\bar\psi\psi\bar\psi\psi = \bar\psi\psi\psi\bar\psi = -\psi\bar\psi\bar\psi\psi$

2] ☆ Sum of all 1PI diagrams:



$$\equiv i\Pi^{\mu\nu}(q) = i\left[\Pi_2^{\mu\nu}(q) + O(\alpha^2)\right]$$

i] Only tensors available: $g^{\mu\nu}$; $q^\mu q^\nu$

$\longrightarrow \Pi^{\mu\nu}(q) = A(q^2)\, g^{\mu\nu} + B(q^2)\, q^\mu q^\nu$

ii] Ward identity: $q_\mu M^\mu \overset{*}{=} 0$

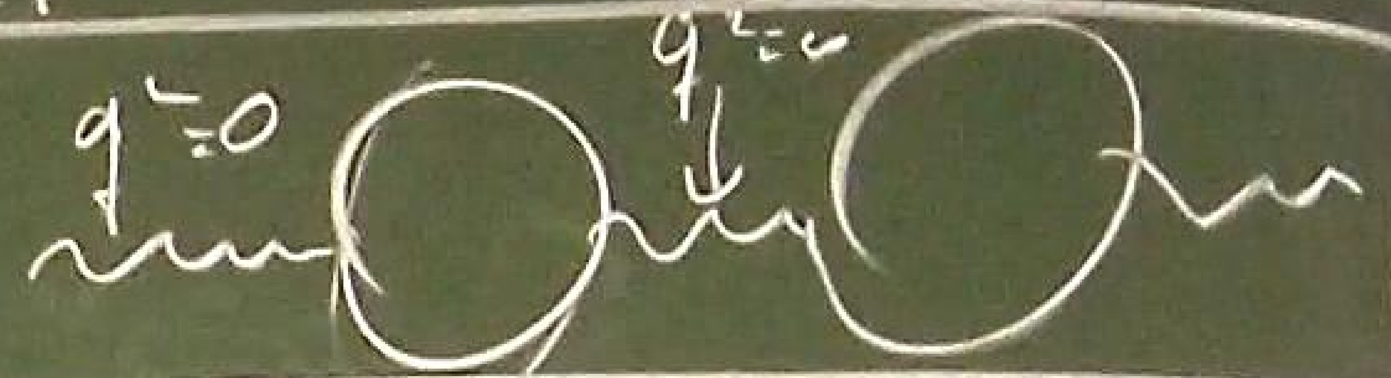
$$q_\mu \Pi^{\mu\nu}(q) \overset{*}{=} 0$$

$\longrightarrow B = -\frac{A}{q^2}$

$\longrightarrow \Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\frac{A}{q^2}$

iii] $\xrightarrow{*}$ $\Pi^{\mu\nu}(q)$ has no pole at $q^2=0$

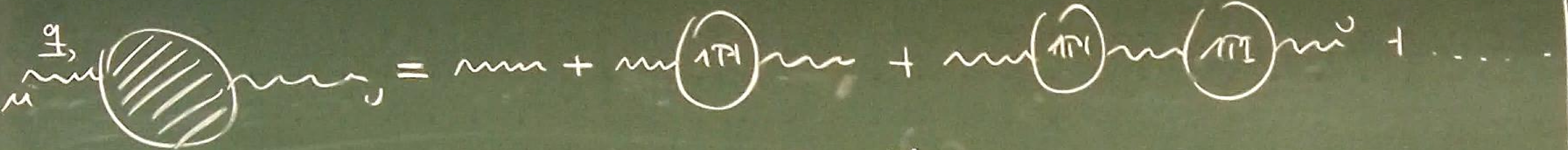
$$\frac{g^{\mu\nu}}{q^2+i\epsilon}$$



$\longrightarrow \Pi(q^2) \equiv \frac{A(q^2)}{q^2}$ regular at $q^2=0$

$\longrightarrow \boxed{\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\,\Pi(q^2)}$

3] X : Sum of all diagrams:



$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\left[i(q^2 g^{\rho\sigma} - q^\rho q^\sigma)\Pi(q^2)\right]\frac{-ig_{\sigma\nu}}{q^2} + \dots\dots$$

$$\Delta^\rho_\nu = \delta^\rho_\nu - \frac{q^\rho q_\nu}{q^2}, \quad g^{\rho\sigma} g_{\sigma\nu} = \delta^\rho_\nu$$

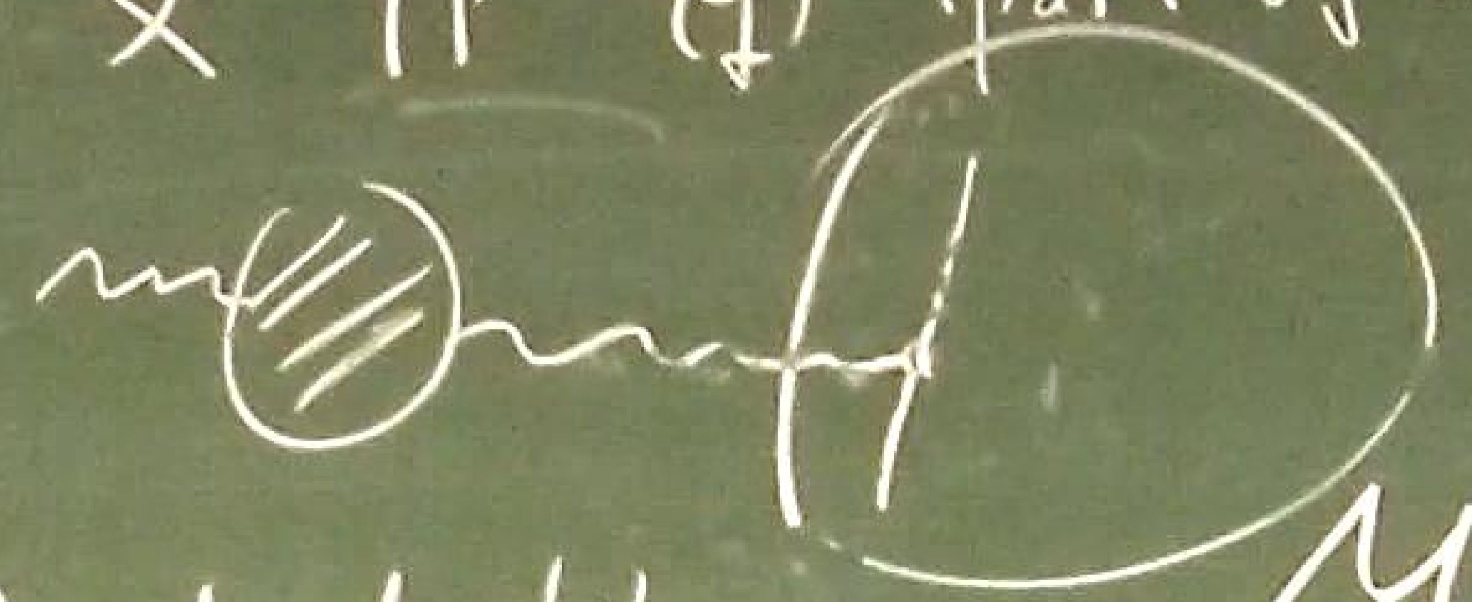
$$= -\frac{ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\Delta^\rho_\nu\,\Pi(q^2) + \frac{-ig_{\mu\rho}}{q^2}\Delta^\rho_\sigma\Delta^\sigma_\nu\,\Pi^2(q^2) + \dots$$

$$\Delta^\rho_\sigma\Delta^\sigma_\nu = \Delta^\rho_\nu$$

$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\Delta^\rho_\nu \boxed{\sum_{n=1}^{\infty}\Pi^n(q^2)} \to \left(\frac{1}{1-\Pi(q^2)} - 1\right)$$

$$= \frac{-i}{q^2(1-\Pi(q^2))}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i}{q^2}\frac{q_\mu q_\nu}{q^2}$$

4] X $\Pi^{\mu\nu}(q)$ part of S-matrix computation.



$\mathcal{M}^\mu$

$$q_\mu \mathcal{M}^\mu \overset{*}{=} 0$$

Ward identity

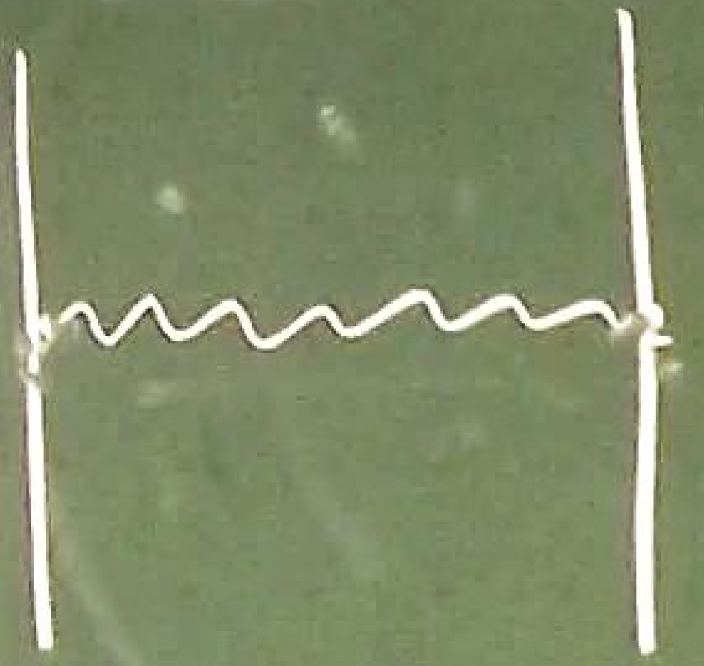
$*$ $q \to 0$

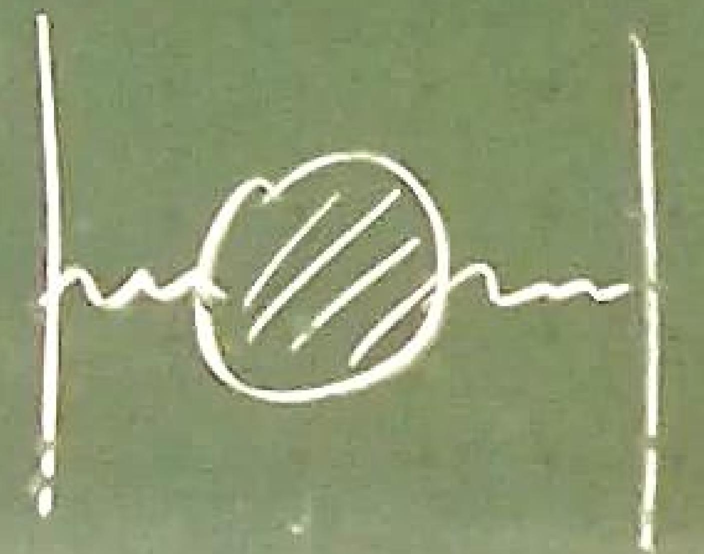
$$\hat{=} \frac{-ig_{\mu\nu}}{q^2[1-\Pi(q^2)]}$$

5) Charge renormalization.

i) Define $\boxed{Z_3 \equiv \frac{1}{1-\Pi(0)}}$

& $q^2 \to 0$

 $= \frac{e^2 g_{\mu\nu}}{q^2}$

$\to$  $= \frac{Z_3 e_0^2 g_{\mu\nu}}{q^2}$

ii) $\to$ Charge renormalization:

Bare charge e_0 (given by $H = e_0 \bar\psi \gamma^\mu \psi A_\mu$)

Physical charge $e \equiv \sqrt{Z_3}\, e_0$

Fine structure constant: $\frac{e^2}{4\pi} = \alpha = Z_3 \alpha_0 = Z_3 \frac{e_0^2}{4\pi}$

Note. $\alpha = \alpha_0 + O(\alpha_0^2) \to O(\alpha^2) = O(\alpha_0^2)$

iii) $\frac{-i g_{\mu\nu}}{q^2} \frac{e_0^2}{1-\Pi(q^2)} = \frac{-i g_{\mu\nu}}{q^2} \frac{e^2[1-\Pi(0)]}{1-\Pi(q^2)}$

$= \frac{-i g_{\mu\nu}}{q^2} \frac{e^2[1-\Pi_2(0)]}{1-\Pi_2(q^2)} + O(\alpha^2)$

$(1-x) = (1+x)^{-1} + O(x^2)$

$= \frac{-i g_{\mu\nu}}{q^2} \frac{e^2}{[1-\Pi_2(q^2)][1+\Pi_2(0)]} + O(\alpha^2)$

$= \frac{-i g_{\mu\nu}}{q^2} \frac{e^2}{1-[\Pi_2(q^2)-\Pi_2(0)]} + O(\alpha^2)$

q^2-dependent fine structure constant.

$\to \boxed{\alpha_{eff}(q^2) = \frac{e_0^2/4\pi}{1-\Pi(q^2)} = \frac{\alpha}{1-[\Pi_2(q^2)-\Pi_2(0)]} + O(\alpha^2)}$

6| Computation of Π_2:

i| $i\Pi_2^{\mu\nu}(q) = -(-ie)^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}[\;\ldots\;]$

Trace identities

$$= -4e^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu(k+q)^\nu + k^\nu(k+q)^\mu - g^{\mu\nu}(k(k+q) - m^2)}{(k^2-m^2)\,((k+q)^2 - m^2)}$$

Feynman parameters, $l = k + xq$, Wick rotation $l^0 = i l_E^0$

$$\hat{=} -4ie^2 \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{-\frac{2}{d} g^{\mu\nu} l_E^2 + g^{\mu\nu} l_E^2 - 2x(1-x)\, q^\mu q^\nu + g^{\mu\nu}(m^2 + x(1-x) q^2)}{(l_E^2 + \Delta)^2}$$

$$\Delta = m^2 - x(1-x) q^2$$

$$\boxed{\begin{aligned} & l^\mu l^\nu \hat{=} \frac{1}{d} l^2 g^{\mu\nu} \\ & g^{\mu\nu} g_{\mu\nu} = d \end{aligned}}$$

ii| Strong UV-divergence: & $|l_E| < \Lambda$

$$i\Pi_2^{\mu\nu}(q) \sim e^2 g^{\mu\nu} \Lambda^2$$

$\rightarrow$ Regularization needed!

iii| Dimensional regularization:

1. Lower spacetime dimension $d \in \mathbb{N}$ until UV div. vanishes
2. Generalize expressions to $d \in \mathbb{R}$
3. Take limit $d \nearrow 4$

$$\int \frac{d^d l_E}{(2\pi)^d}\,\frac{1}{(l_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\,\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}$$

$$\int \frac{d^d l_E}{(2\pi)^d}\,\frac{l_E^2}{(l_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\,\frac{d}{2}\,\frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1}$$

IV] $i\Pi_2^{\mu\nu}(q) \doteq (q^2 g^{\mu\nu} - q^\mu q^\nu)\, i\Pi_2(q^2)$

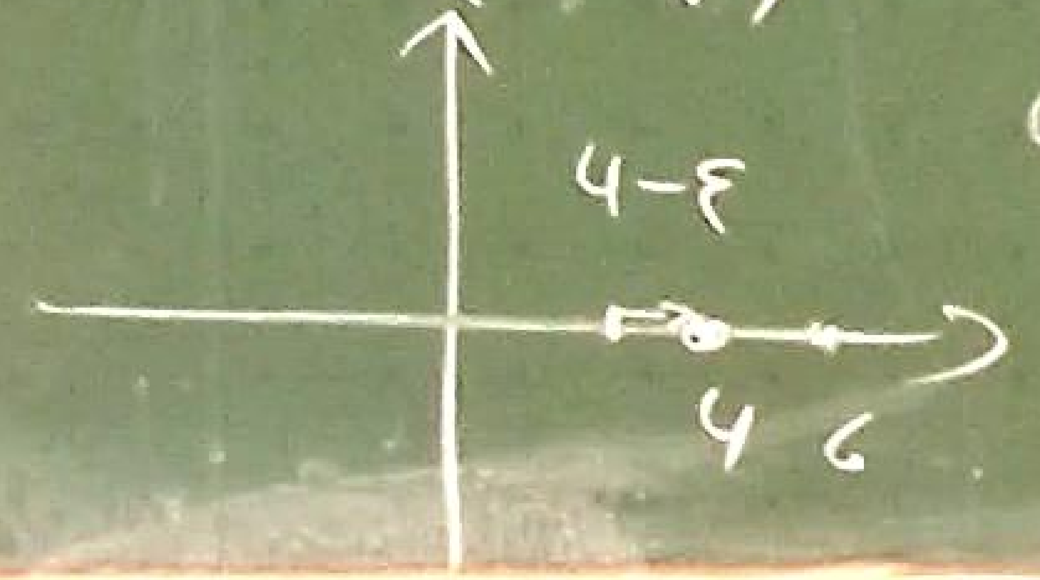
with

$$\Pi_2(q^2) = \frac{-8e^2}{(4\pi)^{d/2}}\int_0^1 dx\,\frac{x(1-x)\,\Gamma(2-\frac{d}{2})}{(m^2-x(1-x)q^2)^{2-\frac{d}{2}}}$$

$\measuredangle$ $n=2$:

$\Gamma(z)$ has poles $z = 0, -1, -2, \dots$

$\rightarrow$ $\Gamma(2-\frac{d}{2})$ has isolated poles at $d = 4, 6, 8, \dots$



$\measuredangle$ $d = 4-\varepsilon$

$$\Gamma\left(2-\frac{d}{2}\right) = \Gamma\left(\frac{\varepsilon}{2}\right) = \frac{2}{\varepsilon} - \gamma + O(\varepsilon)$$

Euler-Mascheroni constant