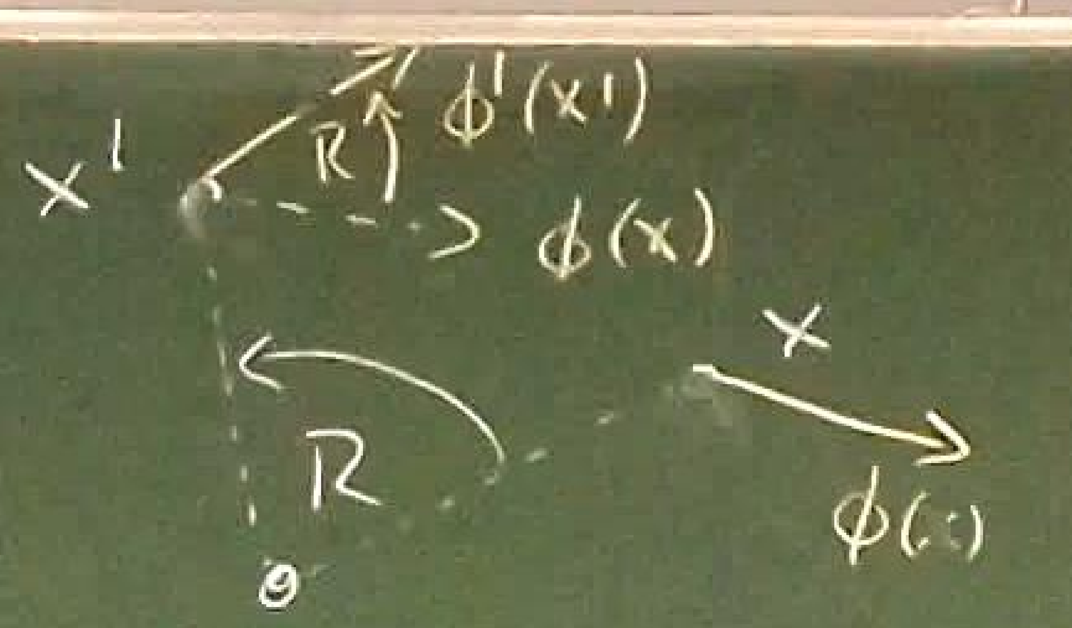
Recap:



• Transformations:

$x \mapsto x' = x'(x)$

$\phi(x) \mapsto \phi'(x') = \hat{\mathcal{F}}(\phi(x))$

• Action:

$$S' = S[\phi'] = \int d^dx \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}\left(\hat{\mathcal{F}}(\phi(x)), \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu \mathcal{F}(\phi(x))\right) \quad (*)$$

Infinitesimal Transformations

1. IT:

$$x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a}(x)$$

$$\phi'(x') = \phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta\omega_a}(x)$$

2. Generator of IT:

$$\delta_\omega \phi(x) = \phi'(x) - \phi(x) \equiv -i\omega_a G_a \phi(x)$$

With:

$$\phi'(x) = \phi(x) - \omega_a \frac{\delta x^\mu}{\delta\omega_a}\partial_\mu\phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta\omega_a}(x) + \mathcal{O}(\omega_a^2)$$

$$\Rightarrow \boxed{iG_a\phi = \frac{\delta x^\mu}{\delta\omega_a}\partial_\mu\phi - \frac{\delta\mathcal{F}}{\delta\omega_a}}$$

Example 1.6. Translations

1. $x'^\mu = x^\mu + \omega^\mu$

$\quad = x^\mu + \omega^\nu \frac{\delta x^\mu}{\delta\omega^\nu}$ $\quad \delta^\mu_\nu$

2. $\frac{\delta\mathcal{F}}{\delta\omega^\nu} = 0$

3. $iG_\mu\phi = \delta^\nu_\mu \partial_\nu\phi - 0$

$\Downarrow$

$$\boxed{G_\mu = -i\partial_\mu = P_\mu}$$

Example 1.7. Scale Transformations

$$G = -ix^\mu \partial_\mu = D$$

Example 1.8. Spatial Rotations

$$G_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + S_{\mu\nu}$$

Noether's Theorem

1. $\cancel{\mathcal{L}}$

$$\boxed{\begin{array}{l}\text{Symmetry of the action} \\ \Longleftrightarrow \quad S[\phi] = S[\phi']\end{array}}$$

for ω_a independent of x (rigid transformations)

2. Assume that $\omega_a = \omega_a(x)$ is not rigid
($O(\omega_a) = O(\partial\omega_a)$)

3. Jacobian: $\frac{\partial x'^\nu}{\partial x^\mu} = \delta^\nu_\mu + \partial_\mu\left(\omega_a \frac{\delta x^\nu}{\delta\omega_a}\right)$

$$\det(\mathbb{1}+A) = 1 + \mathrm{tr}[A] + O(A^2)$$

$$\left|\frac{\partial x'}{\partial x}\right| = 1 + \partial_\mu\left(\omega_a \frac{\delta x^\mu}{\delta\omega_a}\right)$$

4. Inverse Jacobian matrix.

$$\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu - \partial_\mu\left(\omega_a \frac{\delta x^\nu}{\delta\omega_a}\right)$$

5. Use (*)

$$S' = \int d^dx \left(1 + \partial_\mu \omega_a \frac{\delta x^\mu}{\delta\omega_a}\right)$$
$$\times \mathcal{L}\left(\phi + \omega_a \frac{\delta\mathcal{F}}{\delta\omega_a}, \left[\delta^\nu_\mu - \partial_\mu\left(\omega_a \frac{\delta x^\nu}{\delta\omega_a}\right)\right]\right.$$
$$\left.\times\left[\partial_\nu\phi + \partial_\nu\left(\omega_a \frac{\delta\mathcal{F}}{\delta\omega_a}\right)\right]\right)$$

6. Expand in 1st order of ω_a, $\partial_\mu\omega^a$

7. $\delta S \equiv \underset{S[\phi']}{S'} - \underset{S[\phi]}{S}$

$\hookrightarrow$ Only terms $\propto \frac{\partial\omega_a}{\partial x^\mu}$ remain

8. For generic (non-rigid) transformation:

$$S[\phi'] \,''$$

$$\delta S = S' - S = -\int d^d x \, j^\mu_a \, \partial_\mu \omega_a$$

with the current:

$$\boxed{j^\mu_a = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \right) \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta \mathcal{F}}{\delta \omega_a}}$$

9. Integration by parts:

$$\delta S = \int d^d x \, \omega_a (\partial_\mu j^\mu_a) = 0 \quad \forall \omega_a(x)$$

10. $\forall \phi$ that obey the EOMs

$\longrightarrow$ $\delta S = 0$ for arbitrary variations

$$\phi' = \phi + \delta\phi$$

$\longrightarrow$ In particular true for non-rigid inf. transformations

$$\Rightarrow \quad \boxed{\partial_\mu j^\mu_a(x) = 0 \quad \forall x, a}$$

Noether's theorem

Conserved charge:

$$\boxed{Q_a = \int_{\text{Space}} d^{d-1}x \, j^0_a}$$

Proof:

$$\frac{dQ_a}{dt} = \int_{S_T} d^{d-1}x \, \partial_0 j^0_a \overset{\text{Noether}}{=} -\int_{S_T} d^{d-1}x \, \partial_u j^u_a, \quad u = 1,2,3$$

$$\overset{\text{Gauss}}{=} -\int_{\text{Surface}} d\sigma_u \, j^u_a \overset{!}{=} 0$$

Note 1.1

$$\tilde{j}^\mu_a := j^\mu_a + \partial_\nu B^{\mu\nu}_a \quad \text{with } B^{\mu\nu}_a = -B^{\nu\mu}_a \text{ arbitrary}$$

$$\Rightarrow \partial_\mu \tilde{j}^\mu_a = 0$$

Note 1.2

Symmetric Lagrangian

$\Downarrow$

Symmetric Action

$\Rightarrow$ Conserved currents.

$\Downarrow$

Symmetric EOM:

$\mathcal{L} = \lambda^2 \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}(\phi)^2 \lambda^2$

$\rightarrow \quad (\partial^2 + m^2)\phi = 0 \Rightarrow \phi' = \lambda\phi$

Application: The Energy-Momentum Tensor (EMT)

1. $\mathbb{1}$ spacetime T $\quad P = SO(1,3) \ltimes \mathbb{R}^4$

$x'^\mu = x^\mu + \varepsilon^\mu \rightarrow \frac{\delta x^\mu}{\delta \varepsilon^\nu} = \delta^\mu_\nu$

2. $\mathcal{L}$ translation invariant action. $S' = S$ $\quad \frac{\delta \mathcal{F}}{\delta \varepsilon^\rho} = 0$

3. Conserved currents:

$$\underbrace{j^\mu_a = T^\mu{}_\nu}_{} = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\rho \phi - \delta^\mu_\rho \mathcal{L}\right) \underbrace{\frac{\delta x^\rho}{\delta \varepsilon^\nu}}_{\delta^\rho_\nu}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}$$

$$\boxed{\begin{aligned} T^{\mu\nu} &= g^{\nu\rho} T^\mu{}_\rho \\ &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu - g^{\mu\nu} \mathcal{L} \end{aligned}}$$

Energy-Momentum-Tensor

with $\boxed{\partial_\mu T^{\mu\nu} = 0}$

and four conserved charges:

$$\boxed{P^\nu = \int d^3x \, T^{0\nu}}$$

4. Energy $\nu = 0$

$$P^0 = \int d^3x\, T^{00} = \int d^3x \underbrace{\left(\underbrace{\frac{\partial \mathcal{L}}{\partial \dot\phi}}_{\pi} \dot\phi - \mathcal{L} \right)}_{\mathcal{H}}$$

$= H$ (: Hamiltonian)

5. Kinetic momentum ($\nu = i$)

$$P^i = \int d^3x\, T^{0i} = \int d^3x \overbrace{\frac{\partial \mathcal{L}}{\partial \dot\phi}}^{\pi} (-\partial_i \phi)$$

$$= - \int d^3x\, \pi\, \partial_i \phi$$

Note 1.3

In general $T^{\mu\nu} \neq T^{\nu\mu}$ for canonical EMT

$\tilde{T}^{\mu\nu} := T^{\mu\nu} + \partial_\rho K^{\rho\mu\nu}$ with $K^{\rho\mu\nu} = -K^{\mu\rho\nu}$

Choose $K^{\rho\mu\nu}$ that $\tilde{T}^{\mu\nu} = \tilde{T}^{\nu\mu}$ ($\Rightarrow$ Belinfante EMT)

Example 1.9. Electromagnetism in vacuum

1. Four component gauge field $A^\mu = (\phi, A^1, A^2, A^3)$

2. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

3. Lagrangian $\mathcal{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

4. Action: $S_{em} = \int d^4x\, \mathcal{L}_{em}$

5. EOM: $\boxed{\partial_\mu F^{\mu\nu} = 0}$

6. S_{em}: Lorentz invariant + translational invariant

$\hookrightarrow$ EMT = conserved quantities

7. Canonical EMT:

$$T^{\mu\nu}_{em} = \frac{\partial \mathcal{L}_{em}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda - g^{\mu\nu} \mathcal{L}_{em}$$

8. $K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu$

$$\Rightarrow \boxed{\tilde{T}^{\mu\nu}_{em} \hat{=} \frac{1}{4} g^{\mu\nu} F_{\rho\lambda} F^{\rho\lambda} - F^{\mu\rho} F^{\nu}{}_{\rho}}$$

- $T^{00} = \frac{1}{2}(E^2 + B^2)$
- $T^{0i} = (\vec{E} \times \vec{B})_i$
- $\tilde{T}^{ij} = \sigma_{ij}$ (Maxwell stress tensor)

$$\tilde{T} = \left(\begin{array}{c|c} T^{00} & - \; T^{0n} \; - \\ \hline - & \\ - & \boxed{\sigma_{ij}} \\ - & \end{array}\right)$$

- $\underbrace{F = \int_V \rho \ldots\ldots}_{f}$