Recap:

6.4. Field Strength Renormalization

6.4.1 Structure of Two-Point Correlators in interacting theories

7] Källén-Lehmann spectral representation:

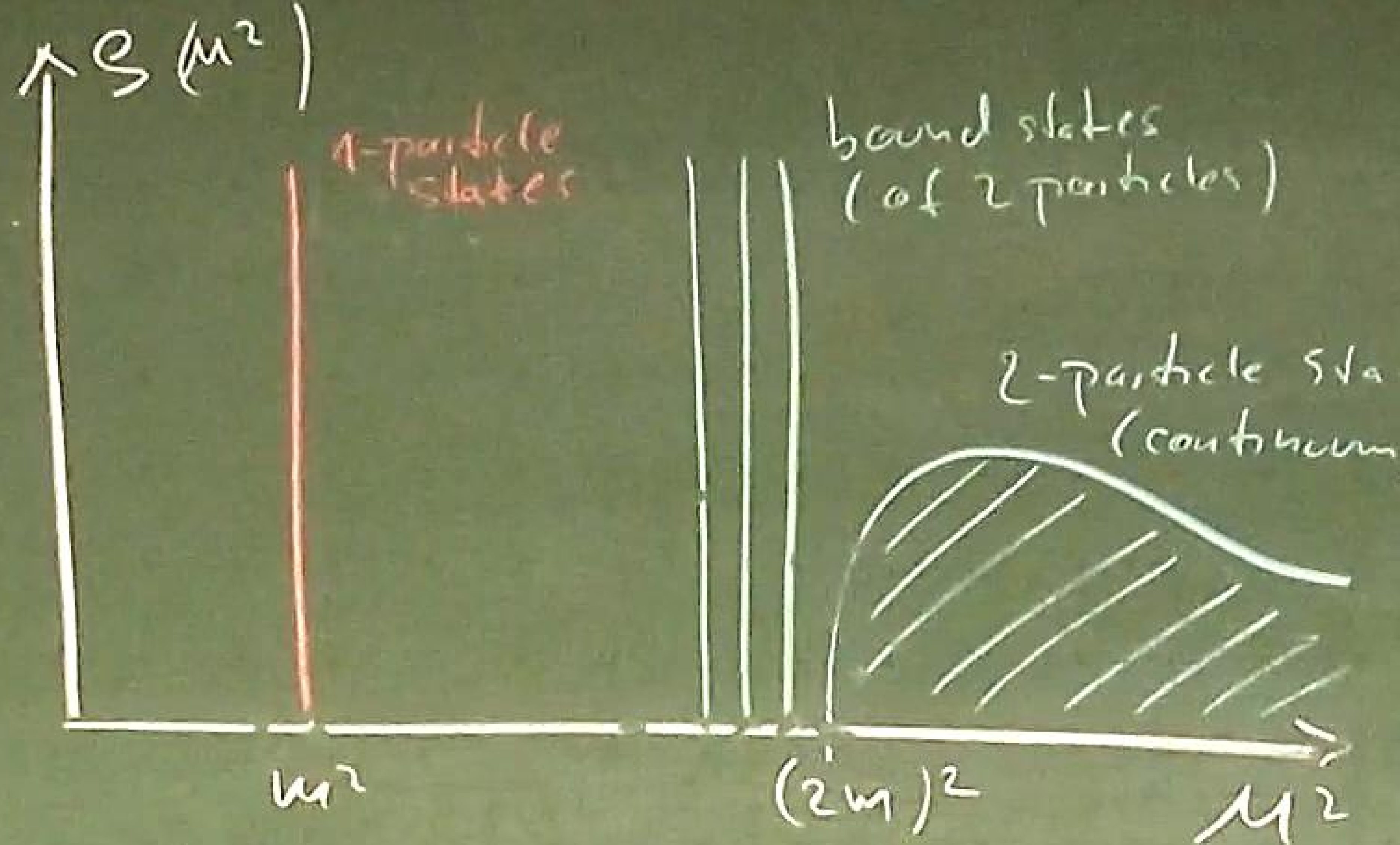
$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \frac{dM^2}{2\pi}\,\rho(M^2)\,D_F(x-y,M^2)$$

Spectral density: $\rho(M^2)$; D_F: Feynman propagator for mass M

$$\rho(M^2) = 2\pi \sum_\lambda \delta(M^2-m_\lambda^2)\,|\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$

m_λ: eigenenergy of $|\lambda_0\rangle$, $\vec{p}=0$

8] Typical spectral density:



$$\rho(M^2) = 2\pi\,\delta(M^2-m^2)\cdot Z + \left\{\text{multi particle states for } M^2 \gtrsim (2m)^2\right\}$$

with: $\lambda = 1$

- **Field strength renormalization**
  $$Z = |\langle\Omega|\phi(0)|1_0\rangle|^2$$
- **Physical mass:**
  $m = m_1$ (given by $H|1_0\rangle = m_1|1_0\rangle$)
- **Bare mass:**
  m_0 (given by $H = \dots \frac{1}{2}m_0^2\phi^2 \dots$)

- **Free theory:**
  $$Z = |\langle 0|\phi(0)|\vec{p}=0\rangle|^2 = 1 \qquad \left(|\vec{p}\rangle = \sqrt{2E_p}\,a_p^\dagger|0\rangle\right)$$
  $m = m_0$

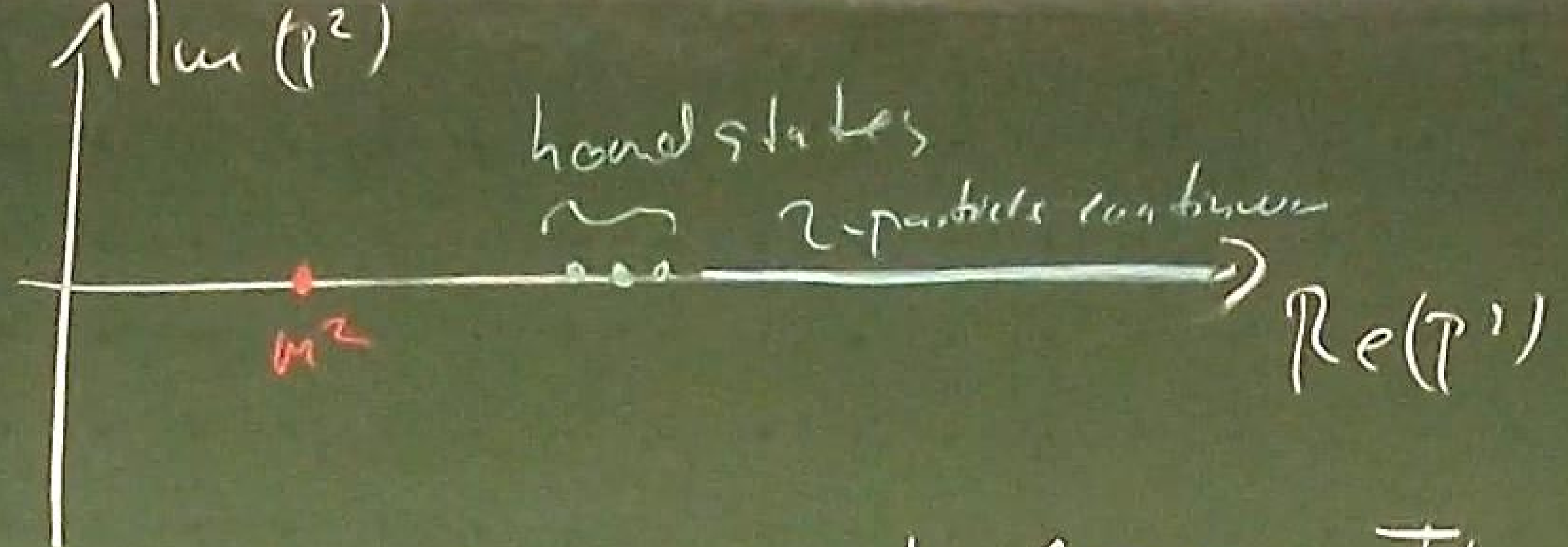
9| Fourier transform:

$$\int d^4x\, e^{ipx}\langle\Omega|T\phi(x)\phi(0)|\Omega\rangle$$

$$= \int \frac{dM^2}{2\pi}\,\frac{i\rho(M^2)}{p^2-M^2+i\epsilon}$$

$$= \frac{iZ}{p^2-m^2+i\epsilon} + \int_{\sim(2m)^2}^{\infty}\frac{dM^2}{2\pi}\,\frac{i\rho(M^2)}{p^2-M^2+i\epsilon}$$

$$\overset{\text{free}}{=} \frac{i\cdot 1}{p^2-m_0^2+i\epsilon}$$



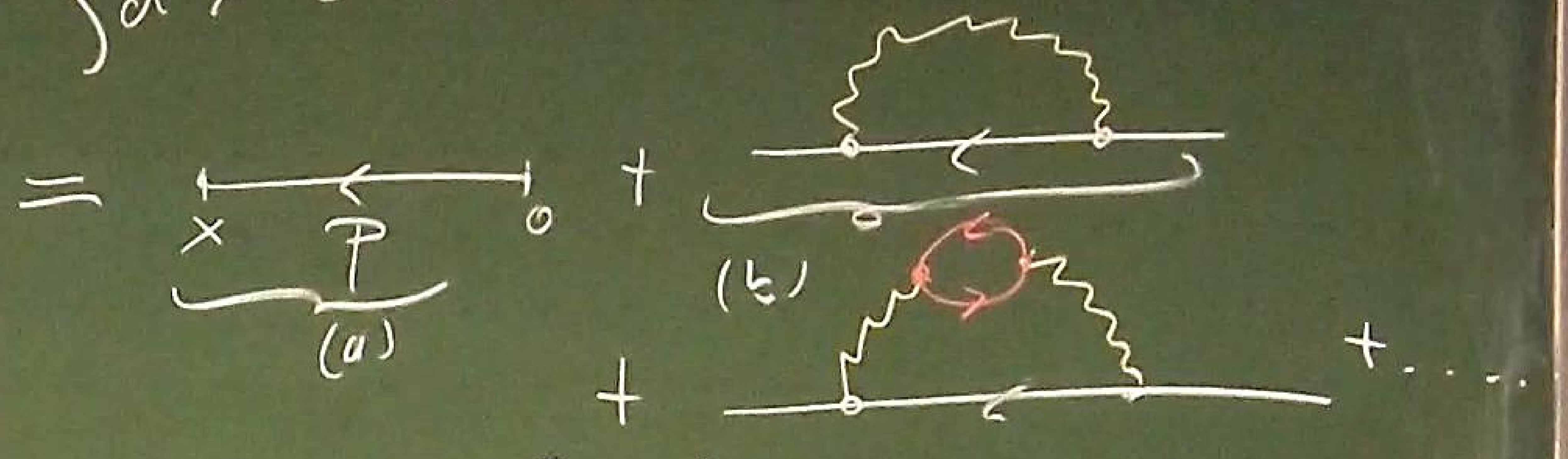
6.4.2 Application to QED: The Electron Self-Energy

1| ϕ^4 theory → QED

$$\int d^4x\, e^{ipx}\langle\Omega|T\psi(x)\bar\psi(0)|\Omega\rangle \overset{*}{=} \frac{iZ_2(\not p+m)}{p^2-m^2+i\epsilon} + \dots$$

2| On the other side:

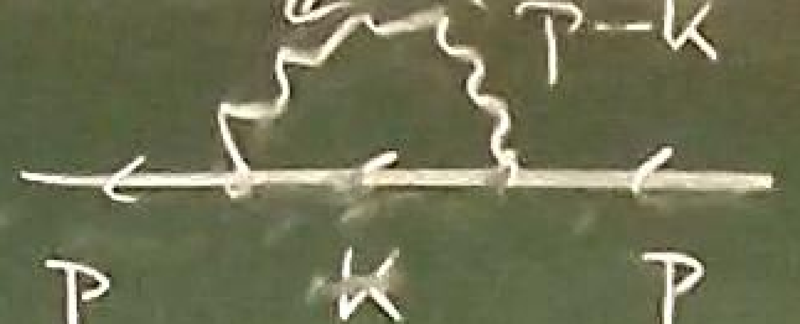
$$\int d^4x\, e^{ipx}\langle\Omega|T\psi(x)\bar\psi(0)|\Omega\rangle$$



3| α^0-order:

$$(a) = \frac{i(\not p+m_0)}{p^2-m_0^2+i\epsilon}$$

4| α^1-order.



$$(b) = \frac{i(\not p + m_0)}{p^2 - m_0^2 + i\epsilon}\underbrace{\left[(-ie)^2 \int \frac{d^4k}{(2\pi)^4}\, \gamma^\mu \frac{i(\not k + m_0)}{k^2 - m_0^2 + i\epsilon}\, \gamma_\mu \frac{-i}{(p-k)^2 + i\epsilon}\right]}_{-i\Sigma_2(p)} \frac{i(\not p + m_0)}{p^2 - m_0^2 + i\epsilon}$$

→ IR and UV divergence

$$\Sigma_2(p) \overset{\Lambda \to \infty}{\sim} \frac{\alpha}{2\pi} \int_0^1 dx\, (2m_0 - x\not p) \log\left[\frac{x\Lambda^2}{(1-x)m_0^2 + x\mu^2 - x(1-x)p^2}\right]$$

↑

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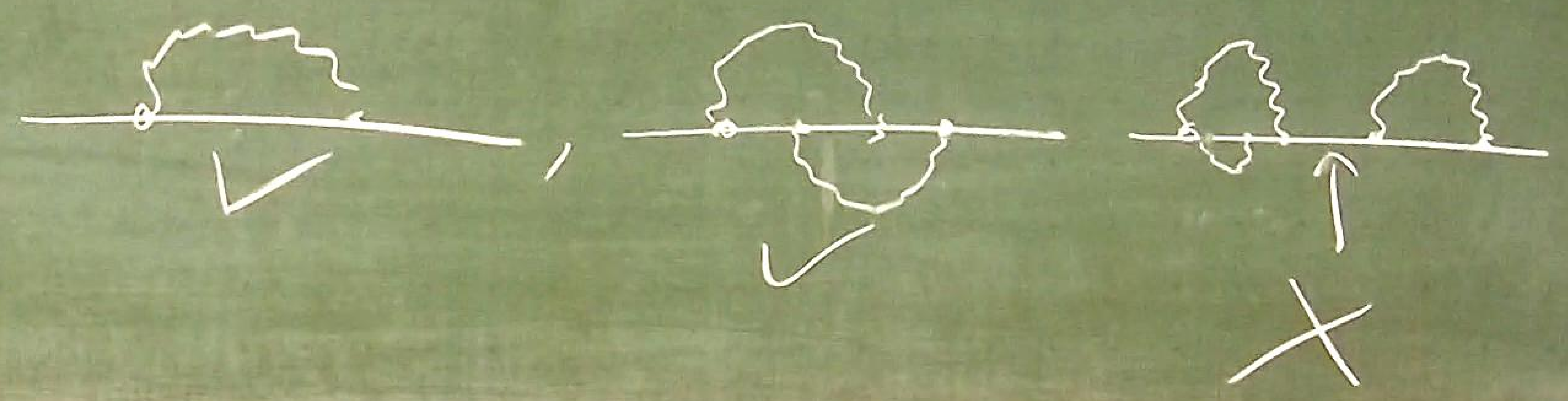
$p^2 \geq (m_0 + \mu)^2$

5| Summation to all orders in α:

i| Definition:

One-particle irreducible (1PI) diagrams = bridgeless graphs

Example:



Define

$$-i\Sigma(p) := \{\text{Sum of all 1PI diagrams}\}$$

$$= -i\Sigma_2(p) + \mathcal{O}(\alpha^2)$$

$$\Sigma(p) = f(\gamma^\mu p_\mu) + g(p^\mu p_\mu) + c(\gamma^\mu \gamma_\mu)$$

$$= \Sigma(\not p) \qquad \not p \qquad p^2 = \not p^2 \qquad 4$$

ii) $\int d^4x \, e^{ipx} \langle \Omega | T \phi(x) \phi(0) | \Omega \rangle$

$$= \{\text{Sum of all one-particle diagrams}\}$$

$$= \frac{i(\not p + m_0)}{p^2 - m_0^2} + \frac{i(\not p + m_0)}{p^2 - m_0^2}\left(-i\Sigma(p)\right)\frac{i(\not p + m_0)}{p^2 - m_0^2} + \dots$$

$$p^2 - m_0^2 = (\not p - m_0)(\not p + m_0) \qquad \not p^2 = p^2 \mathbb{1}$$

$$= \frac{i}{\not p - m_0} \sum_{n=0}^{\infty} \left(\frac{\Sigma(\not p)}{\not p - m_0} \right)^n = \frac{1}{\not p - m_0} \frac{1}{1 - \frac{\Sigma(\not p)}{\not p - m_0}}$$

$$= \frac{1}{\not p - m_0 - \Sigma(\not p)}$$

b) Laurent series:

$$\frac{i}{\not p - m_0 - \Sigma(\not p)} \overset{!}{=} \frac{i Z_2}{\not p - m} + \dots$$

→ Pole for $\not p = \mathbb{1} m = m$.

$$m - m_0 = \Sigma(\not p = m)$$

(implicit eq. for m)

→ Expand denominator around the root:

$$\not p - m_0 - \Sigma(\not p) = (\not p - m)\left(\mathbb{1} - \frac{d\Sigma}{d\not p}\Big|_{\not p = m}\right) + \mathcal{O}\left((\not p - m)^2\right)$$

6)

$$\Rightarrow \boxed{Z_2 = \left(1 - \frac{d\Sigma(\not p)}{d\not p}\Big|_{\not p = m}\right)^{-1}}$$

7) Results in leading order $O(\alpha)$, $d \to 4$

i) Physical mass:

$$\delta m = m - m_0 = \Sigma(\not p = m)$$

$$= \Sigma_2(\not p = m) + O(\alpha^2)$$

$$= \Sigma_2(\not p = m_0) + O(\alpha^2)$$

$$\overset{\Lambda \to \infty}{\sim} \frac{3\alpha}{4\pi} m_0 \log\left(\frac{\Lambda^2}{m_0^2}\right) \xrightarrow{\Lambda \to \infty} \infty$$

↑ PSet 10

$\Rightarrow$ Mass shift δm is UV divergent

ii) Field strength renormalization:

$$\delta Z_2 = Z_2 - 1 = \frac{d\Sigma_2}{d\not p}\Big|_{\not p = m_0} + O(\alpha^2)$$

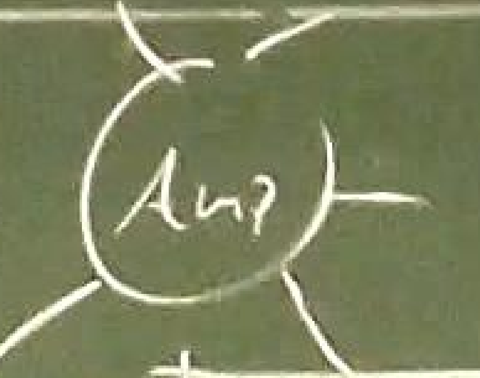
↑ $\frac{1}{1-x} = 1 + x + O(x^2)$

$$= \frac{\alpha}{2\pi}\int_0^1 dx \left\{ -x \log\left[\frac{x\Lambda^2}{(1-x)^2 m^2 + x\mu^2}\right] + 2(2-x)\frac{x(1-x)m^2}{(1-x)^2 m^2 + x\mu^2} \right\}$$

→ Field strength renormalization is UV divergent

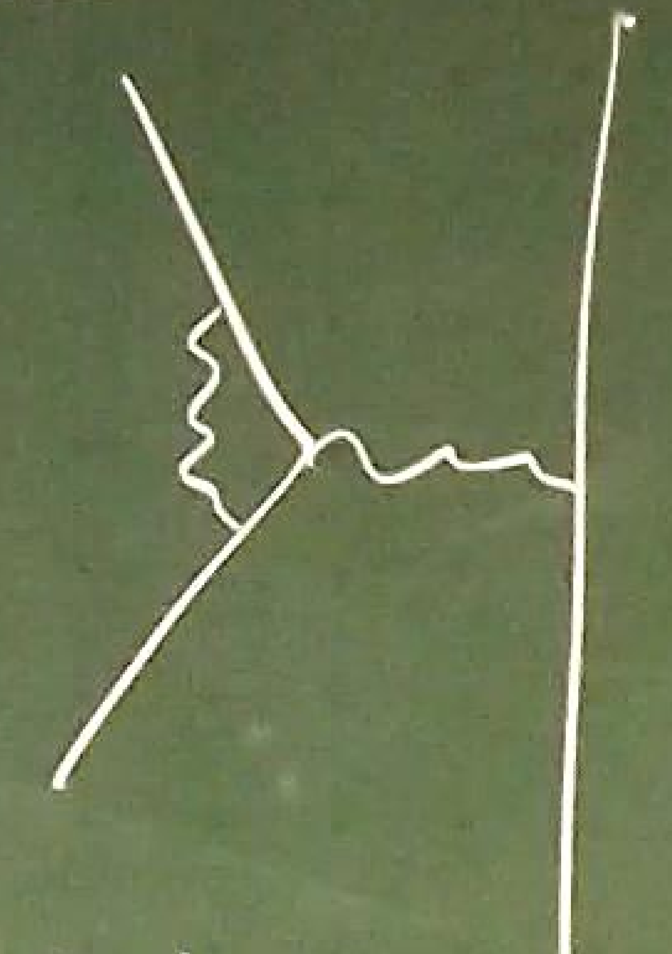
- $\delta Z_2 = -F_1^{(1)}(0)$

$\langle M \rangle = \sqrt{Z_2}$

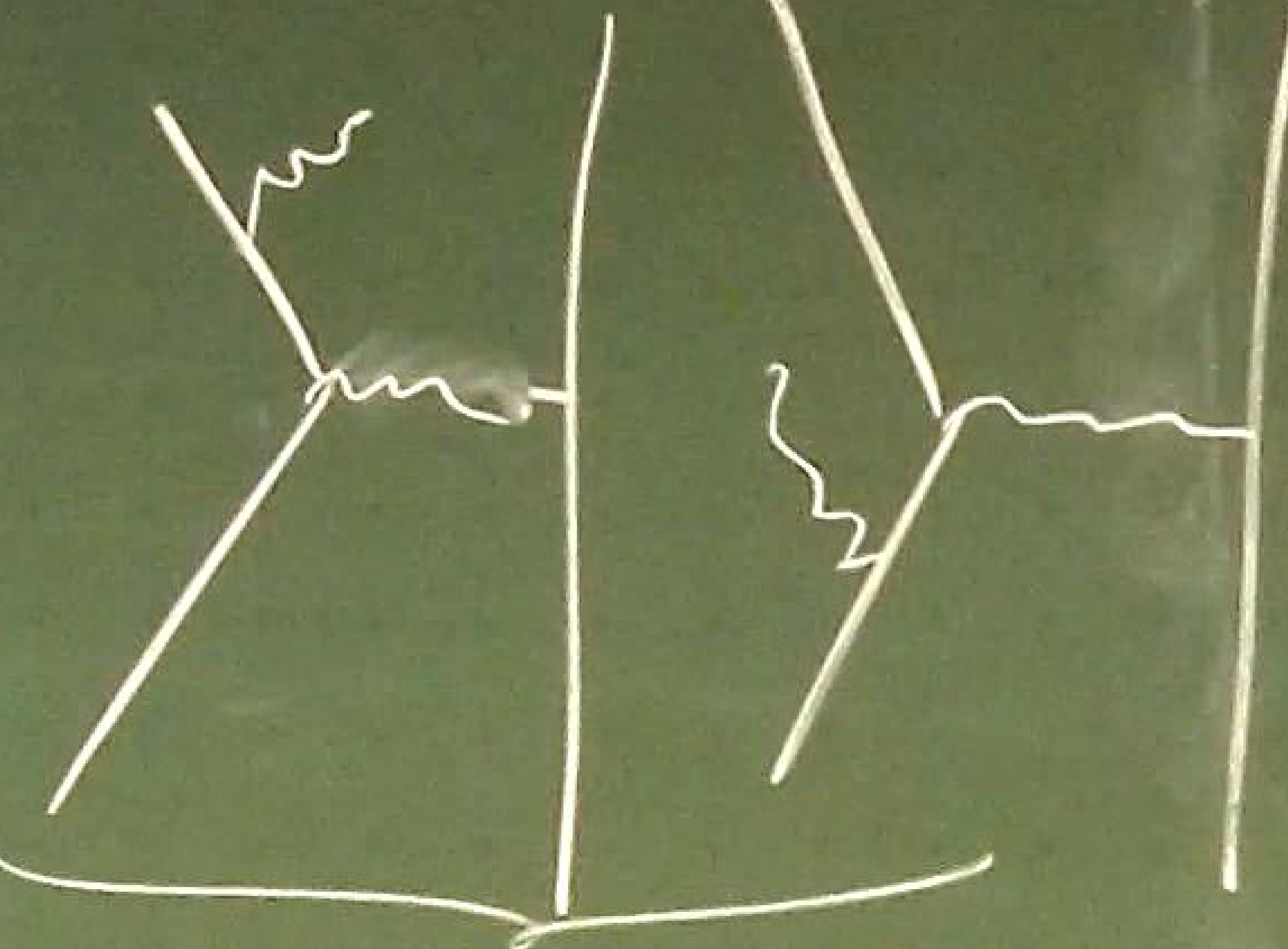


$$\Rightarrow F_1(q^2) = 1 + \boxed{F_1^{(1)}(q^2) + \delta Z_2 = F_1^{(1)}(q^2) - F_1^{(1)}(0)}$$

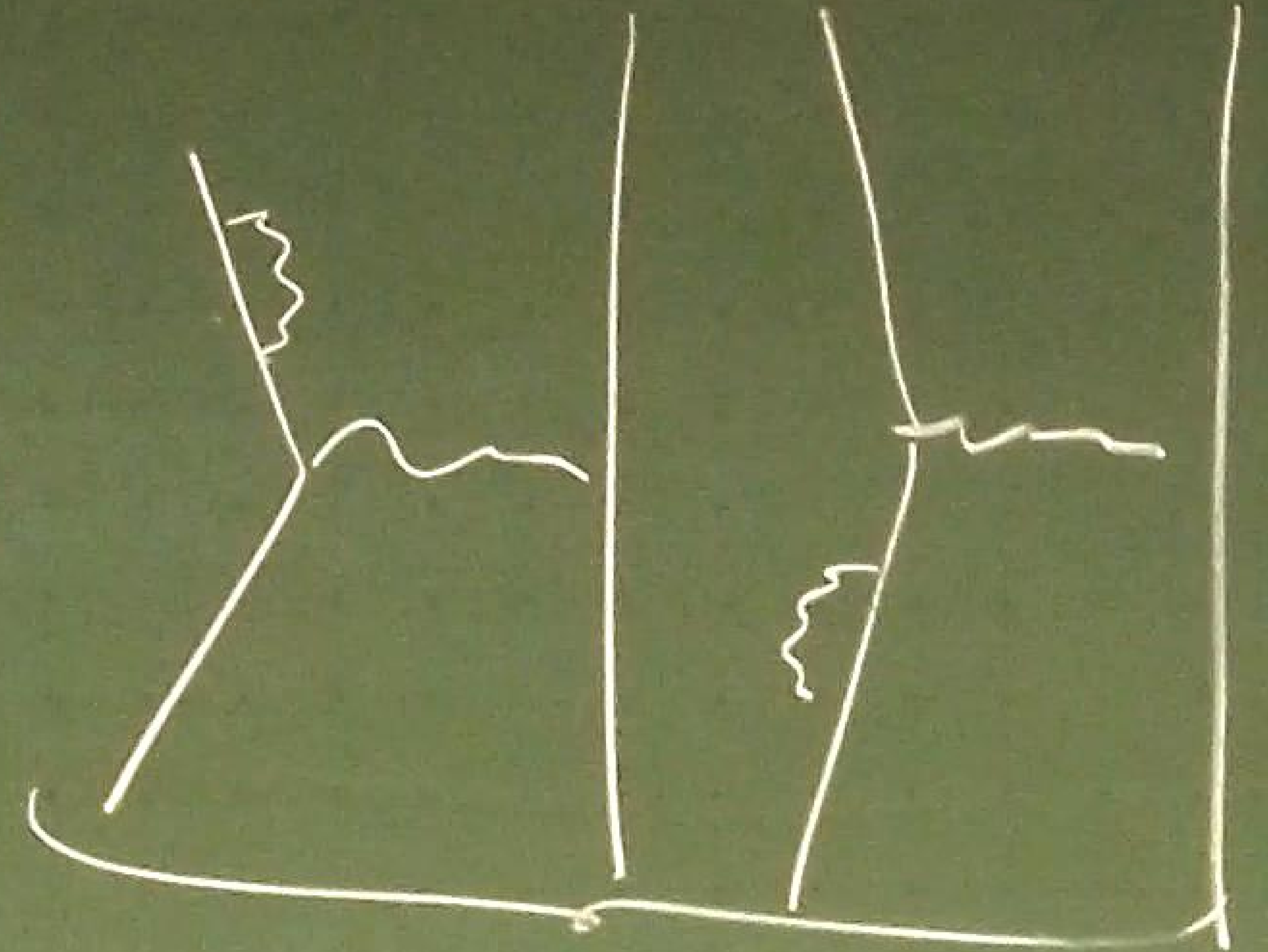
6.5. Electric Charge Renormalization



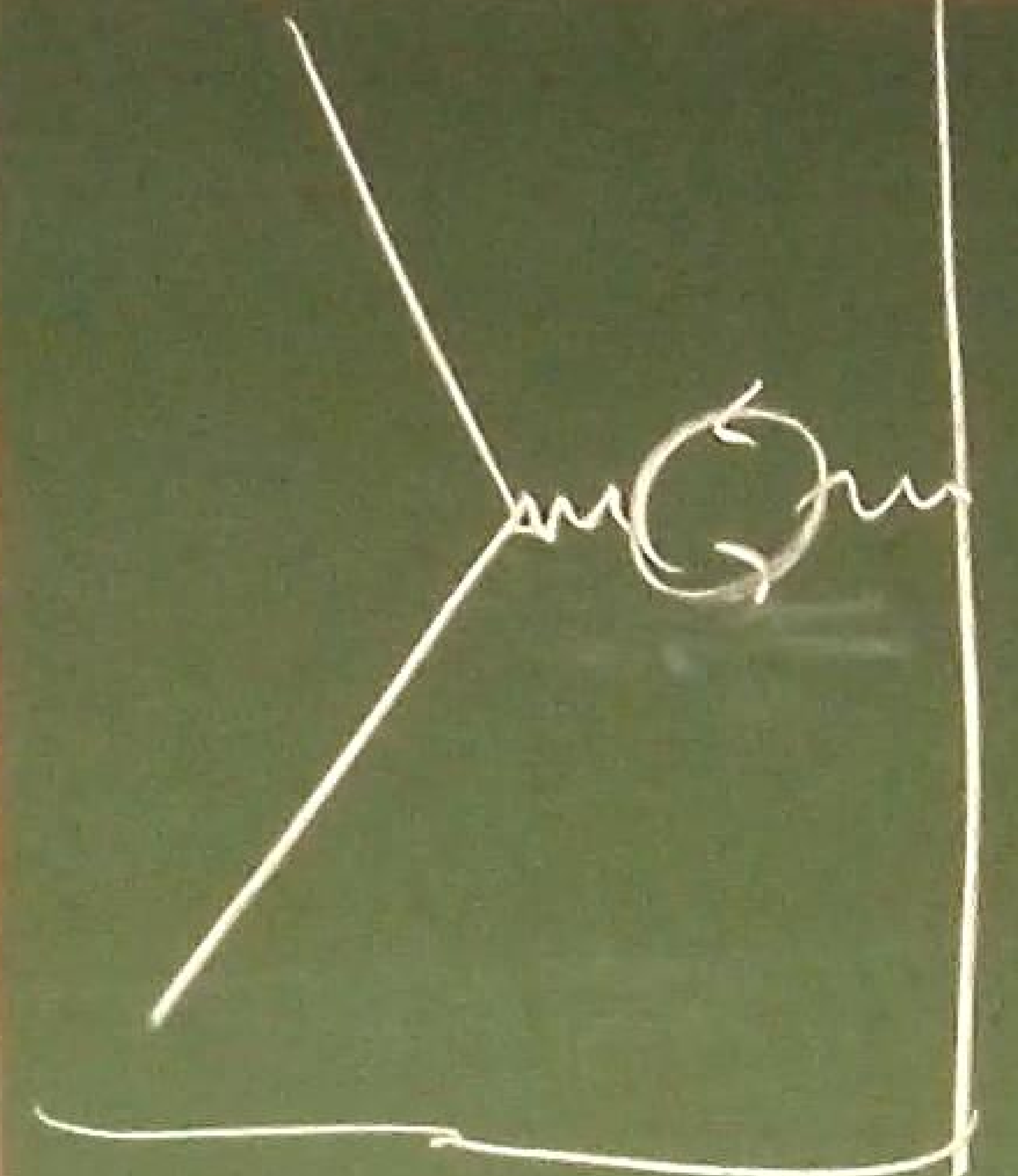
Vertex correction



Soft bremsstrahlung



Electron self-energy



- Photon self-energy
- Vacuum polarization diagram

$$-i\gamma_\nu \frac{g^{\mu\nu}}{q^2} \frac{e^2}{(\ldots q^2)} -i\gamma_\mu$$

$\alpha_{eff}(q^2)$

$S(\frac{1}{2})$ $P(\frac{1}{2})$