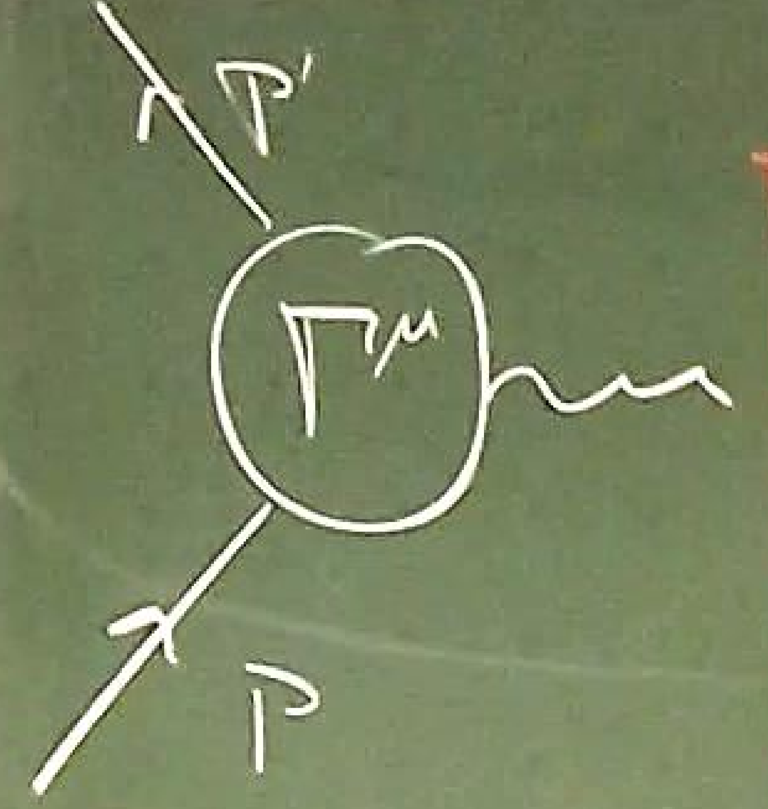
Recap

6.3 The Electron Vertex Function



$$\boxed{\Gamma^\mu(p',p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2)}$$

$$\boxed{g = 2 + 2F_2(0)}$$

Anomalous magnetic moment: $a_e = \frac{g-2}{2} = F_2(0) = \underbrace{\frac{\alpha}{2\pi}}_{\text{Schwinger term}} + O(\alpha^2)$

- $F_1(q^2) = 1 + \alpha F_1^{(1)}(q^2) + O(\alpha^2)$

$$F_1^{(1)}(q^2) = \frac{1}{2\pi}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1)\Big[\log\Big(\frac{m^2(1-z)^2}{m^2(1-z)^2 - q^2xy}\Big) + \frac{m^2(1-4z+z^2)+q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy + z\mu^2} - \frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + z\mu^2}\Big]$$

- $F_2(q^2) = \alpha F_2^{(1)}(q^2) + O(\alpha^2)$

$$F_2^{(1)}(q^2) = \underbrace{\frac{1}{2\pi}\int_0^1 dx\int_0^1 dy\int_0^1 dz\,\delta(x+y+z-1)\Big[\frac{2m^2 z(1-z)}{m^2(1-z)^2 - q^2xy}\Big]}_{\text{No divergence!}}$$

Small photon mass $\mu \to 0$ to regulate IR divergence

Problem: $F_1^{(1)} \xrightarrow{\mu\to 0} \infty$

How to solve this?

6.3.4 The Infrared Divergence

1] Goal: $|F_1(q^2)| \xrightarrow{\mu \to 0} \infty$?

2] Show: $\{q^2, m^2\}$

$$F_1(q^2) = 1 - \frac{\alpha}{2\pi} f_{IR}(q^2) \log\left(\frac{\Lambda}{\mu^2}\right) + O(\alpha^2)$$

$$f_{IR}(q^2) = \int_0^1 d\xi \, \frac{m^2 - q^2/2}{m^2 - q^2 \xi(1-\xi)} - 1 \geq 0$$

3] Cross section:

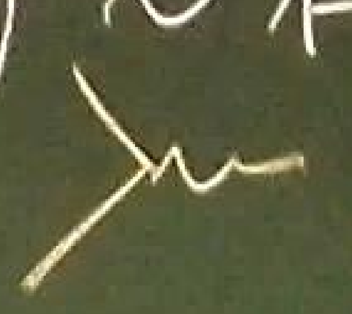
$$\frac{d\sigma(p \to p')}{d\Omega} \sim \left(\frac{d\sigma}{d\Omega}\right)_0 \cdot \left[1 - \frac{\alpha}{\pi} f_{IR}(q^2) \log\left(\frac{\Lambda}{\mu^2}\right) + O(\alpha^2)\right]$$

$$|M|^2 \sim |\Gamma^\mu|^2 \sim |F_1|^2$$

4] Limit $-q^2 \to \infty$

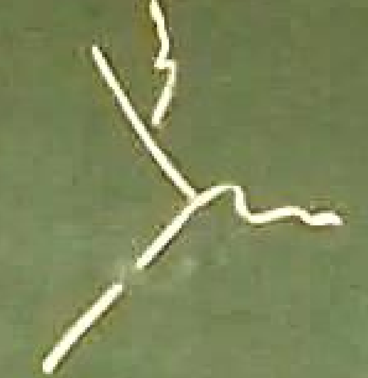
$$f_{IR}(q^2) \sim \int_0^1 d\xi \, \frac{-q^2/2}{-q^2 \xi(1-\xi) + m^2} \mathrel{\hat{\sim}} \log\left(\frac{-q^2}{m^2}\right)$$

$$\to F_1(-q^2 \to \infty) \overset{\mu \to 0}{\sim} 1 + \frac{\alpha}{2\pi} \underbrace{\log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right)} + O$$



5] $\cdot \dfrac{d\sigma(p \to p')}{d\Omega} \overset{\mu \to 0}{\sim} \left(\dfrac{d\sigma}{d\Omega}\right)_0 \left[1 - \dfrac{\alpha}{\pi} \log\left(\dfrac{-q^2}{m^2}\right) \log\left(\dfrac{-q^2}{\mu^2}\right) + O(\alpha^2)\right]$

$\cdot \dfrac{d\sigma(p \to p' + \gamma)}{d\Omega} \overset{\mu \to 0}{\sim} \left(\dfrac{d\sigma}{d\Omega}\right)_0 \left[+ \dfrac{\alpha}{\pi} \log\left(\dfrac{-q^2}{m^2}\right) \log\left(\dfrac{-q^2}{\mu^2}\right) + O(\alpha^2)\right]$



detection threshold of detector $\gamma(k < E_{min})$

6]

$$\left(\frac{d\sigma(p \to p')}{d\Omega}\right)_{\text{measured}} = \frac{d\sigma(p \to p')}{d\Omega} + \frac{d\sigma(p \to p' + \gamma)}{d\Omega}$$

7] General q:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} \overset{\mu\to 0}{=} \left(\frac{d\sigma}{d\Omega}\right)_0 \left[\underbrace{\boxed{1-\frac{\alpha}{\pi} f_{IR}(q^2)\log\left(\frac{A}{\mu^2}\right)}}_{\text{Elastic scattering}} + \underbrace{\boxed{\frac{\alpha}{2\pi}\,\mathcal{I}(p,p')\log\left(\frac{E_{min}^2}{\mu^2}\right)}}_{\text{Bremsstrahlung}} + O(\alpha^2)\right]$$

$$\mathcal{I}(p,p') = \int \frac{d\Omega_k}{4\pi}\sum_r \left|\frac{p'\varepsilon^r}{p'\hat{q}} - \frac{p\varepsilon^r}{p\hat{q}}\right|^2$$

8] $\mathcal{I}(p,p') \overset{*}{=} 2 f_{IR}(q^2)$

9] $$\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} \overset{\mu\to 0}{\sim} \left[1-\frac{\alpha}{\pi} f_{IR}(q^2)\log\left(\frac{A}{E_{min}^2}\right) + O(\alpha^2)\right]\left(\frac{d\sigma}{d\Omega}\right)_0$$

Independent of μ

$$\overset{q\to\infty}{\sim} \left[1-\frac{\alpha}{\pi}\boxed{\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{E_{min}^2}\right)} + O(\alpha^2)\right]\left(\frac{d\sigma}{d\Omega}\right)_0$$

Sudakov double logarithm

Correct expression:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}, p\to p'} \sim \left(\frac{d\sigma}{d\Omega}\right)_0 \underbrace{\exp\left[-\frac{\alpha}{\pi} f_{IR}(q^2)\log\left(\frac{-q^2}{E_{min}^2}\right)\right]}_{\text{Sudakov form factor}}$$

6.4. Field Strength Renormalization

6.4.1. Structure of Two-Point Correlators in Interacting Theories

$\measuredangle$ ϕ^4-theory.

1| Goal: $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$

2| Interpretation for free theory:

$\langle 0|T\phi(x)\phi(y)|0\rangle$, $x^0>y^0$, $\phi(x) \sim a^\dagger + a$, $\phi(y) \sim a^\dagger + a$

= Amplitude of particle to propagate from y to x

3| Mathematical preliminaries:

i| $\measuredangle$ Hilbert space $\mathcal{H}_{int}$ of interacting theory, H

ii| Basis of $\mathcal{H}_{int}$:

$[H,\vec{P}]=0 \rightarrow |\lambda_{\vec{p}}\rangle$ eigenstates of H with $E_{\vec{p}}(\lambda)$ and momentum $\vec{p}$

iii| $\measuredangle$ $P^\mu = \begin{pmatrix} H \\ \vec{P} \end{pmatrix}$, $P^\mu|\lambda_{\vec{p}}\rangle = \begin{pmatrix} E_{\vec{p}}(\lambda) \\ \vec{p} \end{pmatrix}|\lambda_{\vec{p}}\rangle$

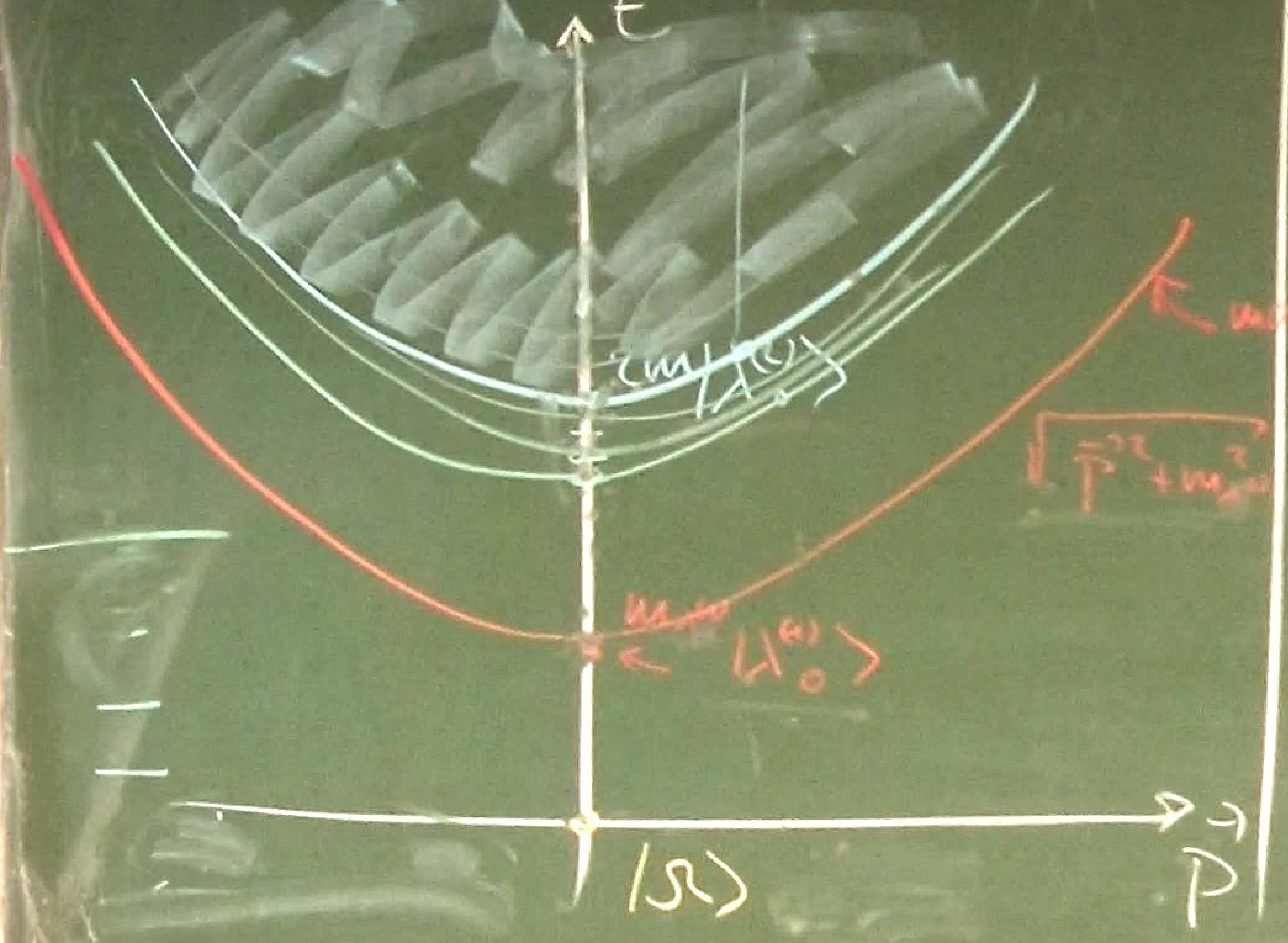
$\rightarrow$ Boost $\Lambda_{\vec{p}} \in SO^+(1,3)$

$$\Lambda_{\vec{p}}\begin{pmatrix} m_\lambda \\ \vec{0} \end{pmatrix} = \begin{pmatrix} E_{\vec{p}}(\lambda) \\ \vec{p} \end{pmatrix} \Rightarrow E_{\vec{p}}(\lambda) = \sqrt{|\vec{p}|^2 + m_\lambda^2}$$

$\rightarrow \forall |\lambda_{\vec{p}}\rangle\ \exists \Lambda_{\vec{p}}\ \exists |\lambda_0\rangle$:

$$\boxed{\begin{aligned} &\bullet\ |\lambda_{\vec{p}}\rangle = U(\Lambda_{\vec{p}})|\lambda_0\rangle \\ &\bullet \begin{cases} H|\lambda_0\rangle = m_\lambda|\lambda_0\rangle \\ \vec{P}|\lambda_0\rangle = 0 \end{cases} \\ &\bullet \begin{cases} H|\lambda_{\vec{p}}\rangle = E_{\vec{p}}(\lambda)|\lambda_{\vec{p}}\rangle \\ \vec{P}|\lambda_{\vec{p}}\rangle = \vec{p}|\lambda_{\vec{p}}\rangle \end{cases} \end{aligned}}$$

IV) Typical spectrum of $P^\mu = (H, \vec{P})$



VI) $$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda{}' \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}(\lambda)} |\lambda_{\vec p}\rangle\langle\lambda_{\vec p}|$$

$(x^0 > y^0)$

4) $$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} \underline{\langle\Omega|\phi(x)|\lambda_{\vec p}\rangle} \langle\lambda_{\vec p}|\phi(y)|\Omega\rangle + \text{const}$$

5) $$\langle\Omega|\phi(x)|\lambda_{\vec p}\rangle$$

$$= \langle\Omega| e^{iPx} \phi(0) e^{-iPx} |\lambda_{\vec p}\rangle$$

$$= \langle\Omega|\phi(0)|\lambda_{\vec p}\rangle \, e^{-ipx}\big|_{p^0 = E_{\vec p}(\lambda)}$$

ϕ scalar

$$= \langle\Omega|\phi(0)|\lambda_0\rangle \, e^{-ipx}\big|_{p^0 = E_{\vec p}(\lambda)}$$

$$U^\dagger(\Lambda)\phi(0)U(\Lambda) = \phi(\Lambda^\mu{}_\nu 0^\nu) = \phi(0)$$

6) $$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle$$

$$= \sum_\lambda |\langle\Omega|\phi(0)|\lambda_0\rangle|^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} e^{-ip(x-y)}\Big|_{p^0 = E_p(\lambda)}$$

$$\overset{x^0 > y^0}{=} \sum_\lambda - \,|\, - \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_\lambda^2 + i\epsilon} e^{-ip(x-y)}$$

$$\overset{x^0 < y^0}{=} \langle\Omega|\phi(y)\phi(x)|\Omega\rangle$$

7) → Källén-Lehmann spectral representation

$$\boxed{\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int \frac{dM^2}{2\pi}\,\rho(M^2)\,D_F(x-y;M^2)}$$

Spectral density:

$$\rho(M^2) = 2\pi\sum_\lambda \delta(M^2-m_\lambda^2)\,|\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$

8) Typical spectral density: