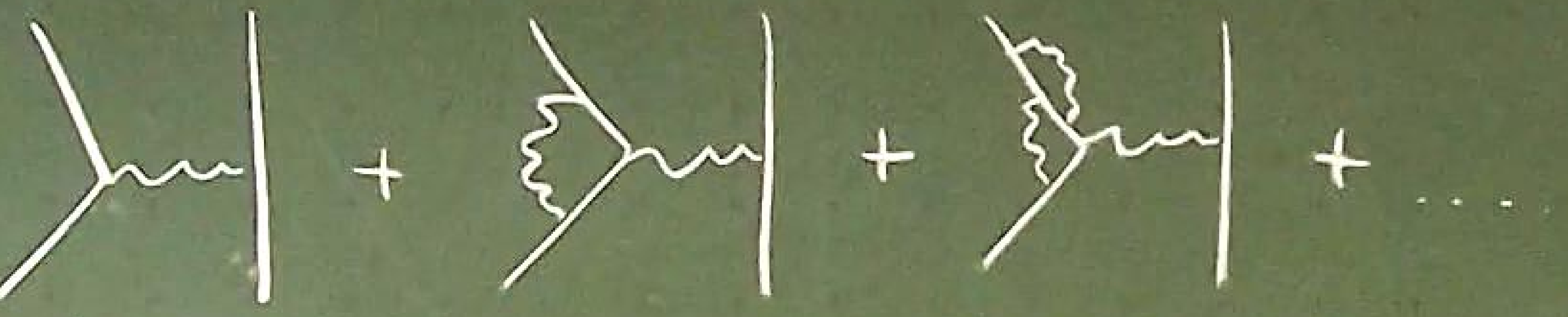
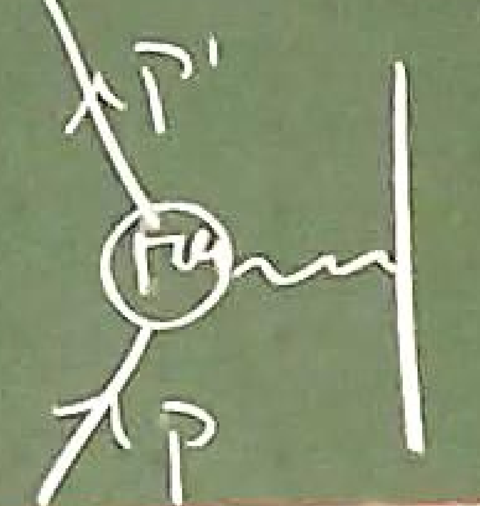
Recap:

6.3. The Electron Vertex Function

6.3.1. Formal Structure

$i\mathcal{M} =$  $+ \dots$

$\equiv$ 

Form factors

$$\boxed{\Gamma^\mu(p,p') = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2)}$$

F_1: $1 + O(\alpha)$ $\qquad$ F_2: $0 + O(\alpha)$

6.3.2. The Landé g-factor

- Response of electric charge $\Rightarrow$ $\boxed{F_1(0) = 1 = \underbrace{F_1^{(0)}(0)}_{1} + \alpha \underbrace{F_1^{(1)}(0)}_{0} + \underbrace{\alpha^2}_{0}}$

- Response of magnetic moment

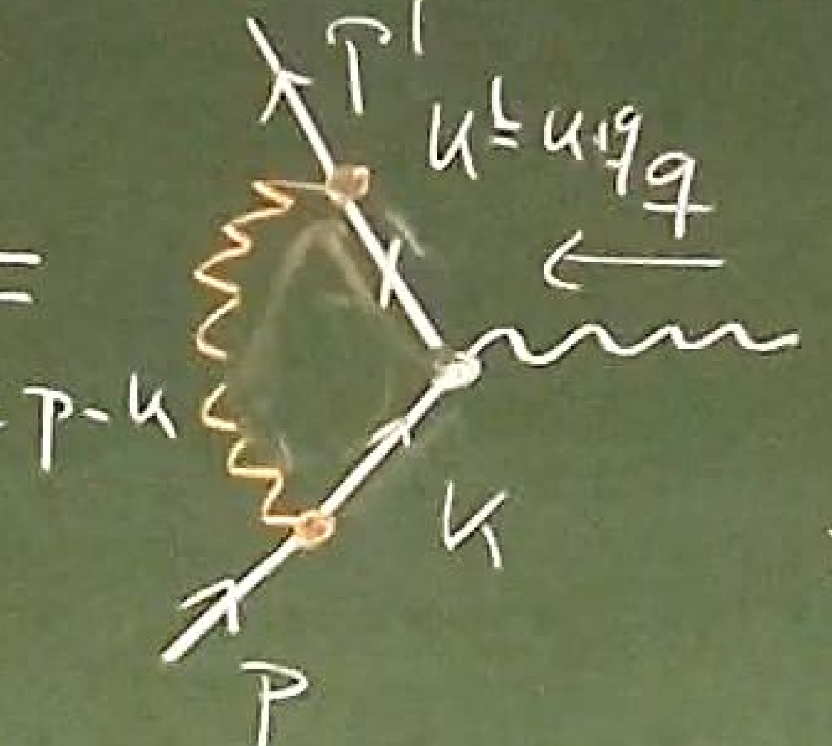
$\langle\vec{\mu}\rangle = g \cdot \mu_B \langle\vec{S}\rangle$

Landé factor $\quad \frac{e}{2m}$: Bohr magneton

$$\Rightarrow \boxed{g = 2 + 2F_2(0) = \underbrace{2}_{\text{Dirac eq.}} + \underbrace{2\alpha F_2^{(1)}(0)}_{\text{anomalous mag. moment}} + O(\alpha^2)}$$

6.3.3. Evaluation

1| $\bar{u}(p')\left[\alpha\Gamma^{(1)}(p',p)\right]^\mu u(p)$

$=$ 

$= \dots \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\nu \gamma^\mu \gamma_\nu}{A_1 A_2 A_3}$

Feynman rules

Dirac propagators

Simplify with contraction identities

2| Feynman parameters

3| $\dfrac{1}{\underbrace{(\tilde{q}^2+i\varepsilon)}_{A_1}\underbrace{(k'^2-m^2+i\varepsilon)}_{A_2}\underbrace{(k^2-m^2+i\varepsilon)}_{A_3}} = \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2}{D^3}$

$\boxed{D = \ell^2 - \Delta + i\varepsilon}$

$\boxed{\begin{aligned}\ell &= k + yq - zp \\ \Delta &= -xyq^2 + (1-z)^2 m^2\end{aligned}}$

4] Numerator $= \bar u(p')\left[\cancel{k}\gamma^\mu \cancel{k}' + m^2\gamma^\mu - 2m(k+k')^\mu\right]u(p) \overset{*}{=} \bar u(p')\Big\{-\tfrac{1}{2}\gamma^\mu l^2 + [-y\cancel{q}+z\cancel{p}]\gamma^\mu[(1-y)\cancel{q}+z\cancel{p}] + m^2\gamma^\mu - 2m[(1-2y)q^\mu + 2zp^\mu]\Big\}u(p)$

$\begin{cases} k' = k+q \\ k = l - yq + zp \end{cases}$

$* =$ only valid in $\int d^4l$ integral ! $\boxed{l^\rho l^\sigma}$

$$l'^{\mu\nu} = \int \frac{d^4l}{(2\pi)^4}\frac{l'^\mu l'^\nu}{D(l^2)} = \int\frac{d^4l'}{(2\pi)^4}\frac{l'^\mu l'^\nu}{D(l'^2)} = L^{\mu\nu}$$

$$\Rightarrow L^{\mu\nu} = g^{\mu\nu}C(l^2)$$

$$\Rightarrow g_{\mu\nu}L^{\mu\nu} = 4C(l^2) = \int\frac{d^4l}{(2\pi)^4}\frac{l^2}{D(l^2)}$$

$$\Rightarrow \boxed{L^{\mu\nu} = \int\frac{d^4l}{(2\pi)^4}\frac{g^{\mu\nu}l^2}{4D(l^2)}}$$

$x+y+z=1$

$$= \bar u(p')\Big\{\gamma^\mu \underbrace{\left[-\tfrac{1}{2}l^2 + (1-x)(1-y)q^2 + (1-2z-z^2)m^2\right]}_{A} + \underbrace{(p'+p)^\mu}_{\gamma^\mu + \ldots\ \sigma^{\mu\nu}} \underbrace{[mz(z-1)]}_{B} + \underbrace{q^\mu}_{(p'-p)^\mu}\cancel{[m(z-2)(x-y)]}\Big\}u(p)$$

5] $C = 0$

6] Gordon identity: $\longrightarrow$

$$\bar u(p')\,\alpha\Gamma^{(\mu)}u = 2ie^2\int\frac{d^4l}{(2\pi)^4}\int_0^1 dx\int_0^1 dy\int_0^1 dz\;\delta(x+y+z-1)\frac{2}{D(l^2)^3}$$

$$\bar u(p')\Big\{\gamma^\mu\left[-\tfrac{1}{2}l^2 + (1-x)(1-y)q^2 + (1-4z+z^2)m^2\right] + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\left[2m^2z(1-z)\right]\Big\}u(p)$$

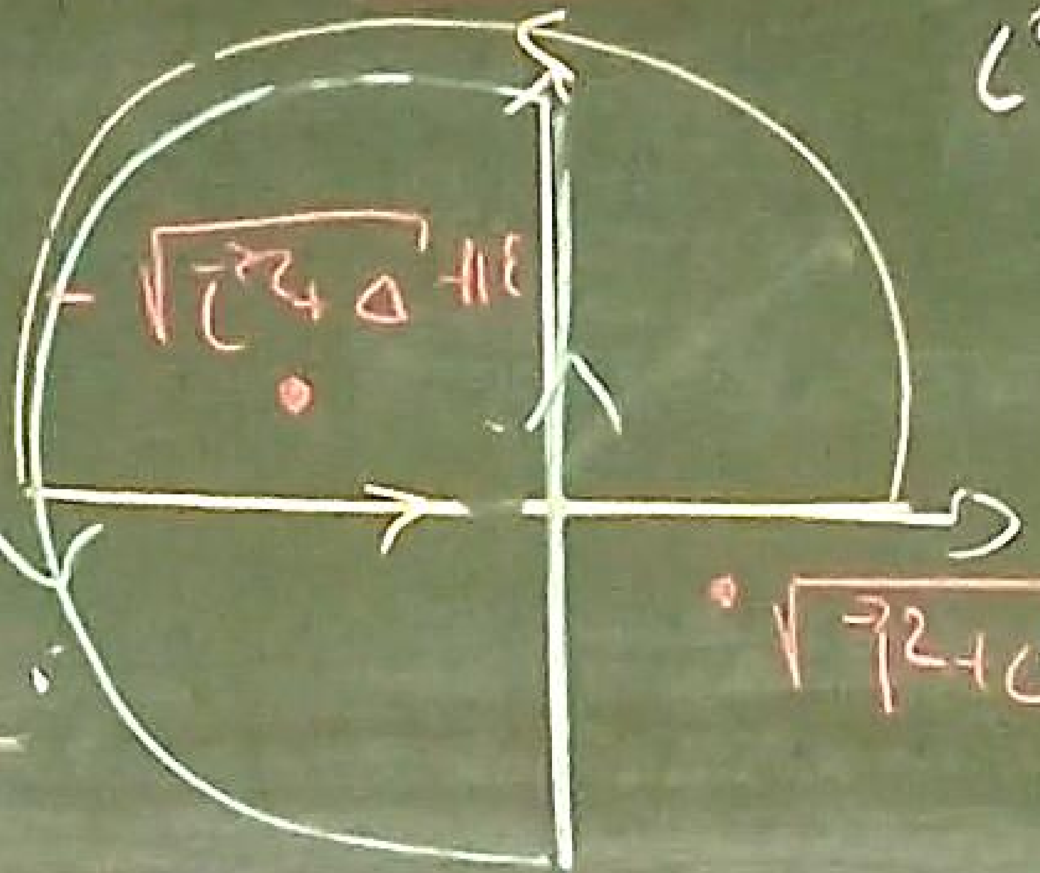
$l \in \mathbb{R}^{1,3}$

$l_E \in \mathbb{R}^4$

7] Momentum integral:

i] $l^2 = l_0^2 - \vec l^{\,2}$

Solution: Wick rotation:



$l^0 \in \mathbb{C}$

$$\boxed{\begin{gathered} l^0 = i l_E^0,\quad \vec l = \vec l_E \\ \Rightarrow l^2 = -l_E^{0\,2} - \vec l^{\,2} = -l_E^2 \end{gathered}}$$

ii) Then $m>2$.

$$\lim_{\varepsilon\to 0}\int\frac{d^4l}{(2\pi)^4}\frac{l^2\,1}{(l^2-\Delta+i\varepsilon)^m}\overset{WR}{=}\frac{i}{(-1)^m}\frac{1}{(2\pi)^4}\int d^4l_E\frac{1}{(l_E^2+\Delta)^m}$$

$$=\frac{i(-1)^m}{(2\pi)^4}\underbrace{\int d\Omega_4}_{2\pi^2}\underbrace{\int_0^\infty dl_E\frac{l_E^3}{(l_E^2+\Delta)^m}}_{=\frac{1}{2(m-1)(m-2)\Delta^{m-2}}}$$

$$=\frac{i(-1)^m}{(4\pi)^4}\frac{1}{(m-1)(m-2)\Delta^{m-2}}\qquad(*)$$

$$=\frac{i(-1)^{m-1}}{(4\pi)^4}\frac{2}{(m-1)(m-2)(m-3)\Delta^{m-3}}\qquad(**)$$

Problem: for $m=3$ I_m diverges!

$\to$ UV divergence

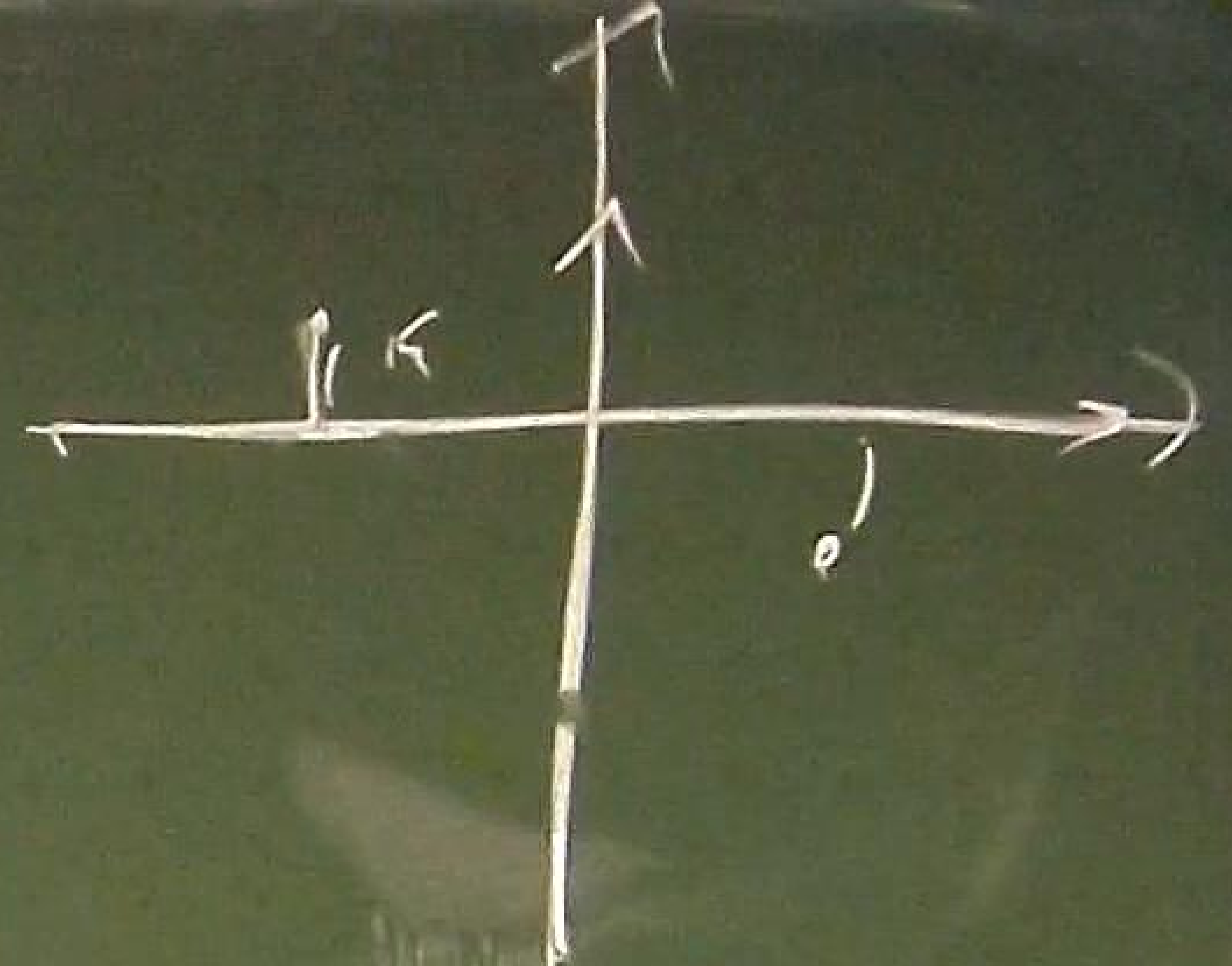
iii) Fix: Pauli-Villars regularization

$$\frac{-ig_{\mu\nu}}{q^2+i\varepsilon}\longmapsto\frac{-ig_{\mu\nu}}{q^2+i\varepsilon}-\boxed{\frac{ig_{\mu\nu}}{q^2-\Lambda^2+i\varepsilon}}$$

for $\Lambda\to\infty$.

Hope: Λ should drop out in physical quantities

$$\xrightarrow{0}\Delta_\Lambda=-xyq^2+(1-z)^2m^2+z\Lambda^2$$



iv] $m=3$

- Eq (*) $\longmapsto$ Eq (*) $- \underbrace{O(\Lambda^{-2})}_{\to 0 \text{ for } \Lambda \to \infty}$
- Eq (**) $\longmapsto$

$$\lim_{\varepsilon \to 0} \int \frac{d^4 l}{(2\pi)^4} \left[\frac{l^2}{(l^2 - \Delta + i\varepsilon)^3} - \frac{l^2}{(l^2 - \Delta_\Lambda + i\varepsilon)^3} \right]$$

$$\overset{G}{=} \frac{i}{(4\pi)^2} \log\left(\frac{\Delta_\Lambda}{\Delta}\right) \xrightarrow{\Lambda \to \infty} \frac{i}{(4\pi)^2} \log\left(\frac{z\Lambda^2}{\Delta}\right)$$

8] Result

$$\bar u(p') \, \delta\Gamma^{(1)\mu} \, u(p) = \frac{\alpha}{2\pi} \int_0^1 dx \int_0^1 dy \int_0^1 dz \, \delta(x+y+z-1)$$

$$\bar u(p') \Big\{ \gamma^\mu \underbrace{\left[\log\left(\frac{z\Lambda^2}{\Delta}\right) + \frac{(1-x)(1-y)q^2}{\Delta} + \frac{(1-4z+z^2)m^2}{\Delta} \right]}_{\text{"}F_1^{(1)}(q^2)\text{"}} + \frac{i\sigma^{\mu\nu} q_\nu}{2m} \underbrace{\left[\frac{2m^2 z(1-z)}{\Delta} \right]}_{\text{"}F_2^{(1)}(q^2)\text{"}} \Big\} u(p)$$

9] δF_1:

i] Problem 1: $F_1(0) = 1$, $F_1^{(0)}(0) = 1$, $\underline{F_1^{(1)}(0) = 0}$

But: $F_1^{(1)} \neq 0$ ↯

Fix 1: $F^{(1)}(q^2) \longmapsto F_1^{(1)}(q^2) - F_1^{(1)}(0)$

ii] Problem 2: IR divergence from $q \to 0$

$\delta\, q^2 = 0$: $\displaystyle \int_0^1 dx \int_0^1 dy \int_0^1 dz \, \delta(x+y+z-1) \frac{1-4z+z^2}{(1-z)^2} = \int_0^1 dz \left(\int_0^{1-z} dy \right) \frac{-2+(1-z)(3-z)}{(1-z)^2} = \overbrace{\int_0^1 dz \frac{-2}{1-z}}^{\infty} + \text{finite}$

Fix: Add small photon mass $\mu \to 0$

$$\Delta \mapsto \Delta_\mu = -xyq^2 + (1-z)^2 m^2 + z\mu^2$$

iii) Fix 1 + Fix 2:

$$F_1(q^2) = 1 + \alpha F_1^{(1)}(q^2) + O(\alpha^2)$$

$$F_1^{(1)}(q^2) = \frac{1}{2\pi}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1)$$

$$\times \Bigg[\underbrace{\log\left(\frac{m^2(1-z)^2}{m^2(1-z)^2 - q^2xy}\right)}_{\text{Fix 1}} + \underbrace{\frac{m^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy + \boxed{z\mu^2}}}_{\text{Fix 2}} - \underbrace{\frac{m^2(1-4z+z^2)}{m^2(1-z)^2 + \boxed{z\mu^2}}}_{\text{Fix 1} \quad \text{Fix 2}} \Bigg]$$

10) αF_2:

$$F_2(q^2) = \alpha F_2^{(1)}(q^2) + O(\alpha^2) \quad \text{with} \quad F_2^{(1)}(q^2) = \frac{1}{2\pi}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(x+y+z-1) \left[\frac{2m^2 z(1-z)}{m^2(1-z)^2 - q^2xy}\right]$$

$$F_2(0) = \frac{\alpha}{\pi}\int \dots \delta\, \frac{2z}{1-z} + O(\alpha^2)$$

$$= \frac{\alpha}{2\pi}\int_0^1 dz \int_0^{1-z} dy\, \frac{2z}{1-z} + O(\alpha^2)$$

$$= \frac{\alpha}{2\pi} + O(\alpha^2)$$

Anomalous magnetic moment:

$$g = 2 + 2F_2(0)$$

$$\alpha^2 = \frac{1}{137^2} \approx 0.5 \cdot 10^{-4}$$

$$a_0^{QED} = \frac{g-2}{2} = F_2(0) = \frac{\alpha}{2\pi} + O(\alpha^2)$$

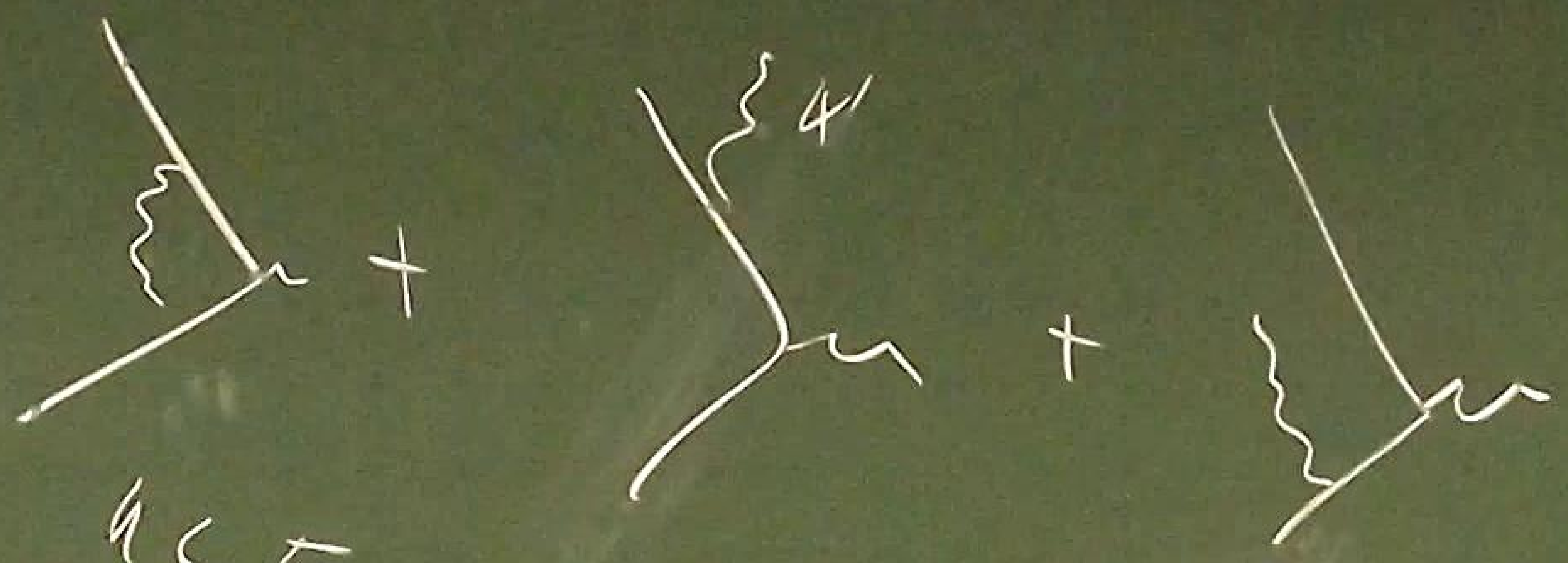
$$\approx 0.0011614$$

$$a_0^{exp} \approx 0.0011597$$

$$a_e^{SM} = 0.00115965218 \,|\, 2031$$

$$a_e^{exp} = 0.00115965218 \,|\, 073$$

6.3.4. The Infrared Divergencies



$k < E_{min}$

$e p + \gamma$ $\quad$ $e p'$

$e p$ $\quad$ $e p$

$$\underbrace{\alpha \log\frac{q^2}{m^2} \log\frac{q^2}{\mu} - - - - -}$$

$$\frac{d\sigma}{d\Omega} \sim \left(\frac{d\sigma}{d\Omega}\right)_0 \frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{E_{min}}\right)$$