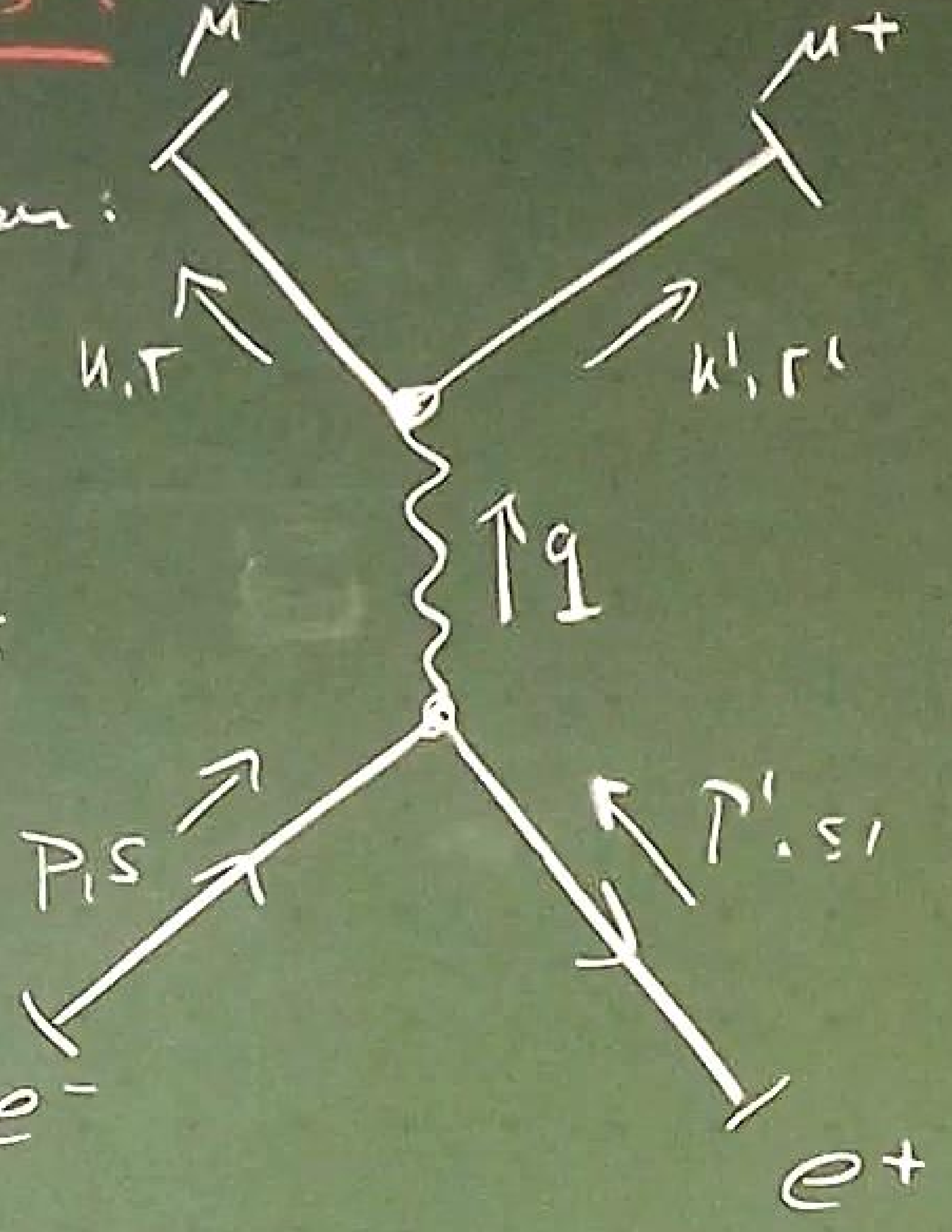
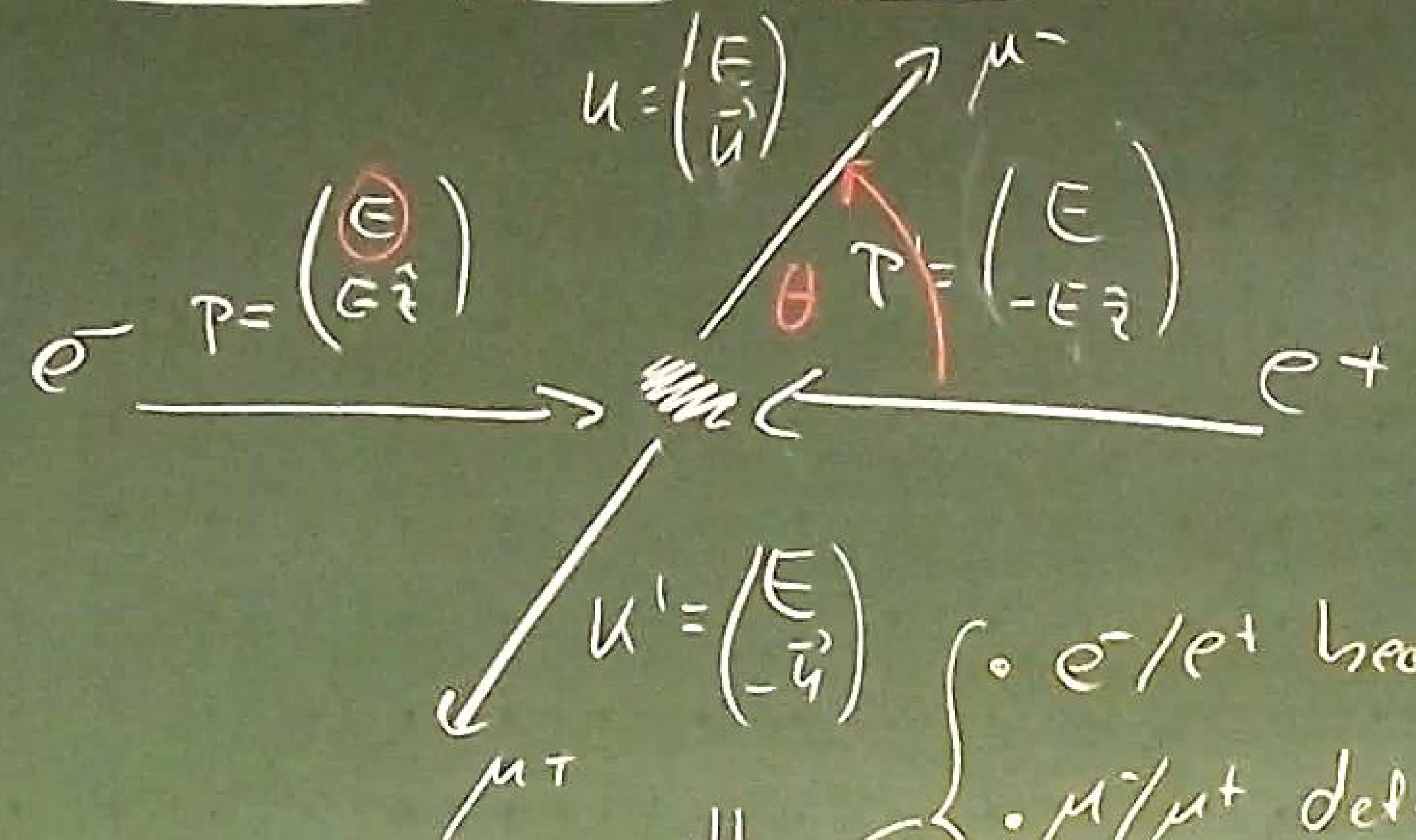
Recap:

Reaction:



$$iM = \text{[diagram]} + O(\alpha^2)$$

11| Center of mass frame:



- e^-/e^+ beam unpolarized
- μ^-/μ^+ detector cannot resolve spin

$\Downarrow$

$$\Longrightarrow \boxed{\overline{|M|}^2 = \frac{1}{4}\sum_{s,s',r,r'} |M|^2 = e^4\left[\left(1+\frac{m_\mu^2}{E^2}\right) + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]}$$

12| Differential scattering cross section

(special case for 2 outgoing particles, Eq. (4.122))

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{2E_p 2E_{p'}\underbrace{|v_p - v_{p'}|}_{2}} \frac{|\vec{k}|}{(2\pi)^2 4 \underbrace{E_{cm}}_{2E}} \overline{|M|}^2$$

$$= \frac{\alpha^2}{4E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left[\left(1+\frac{m_\mu^2}{E^2}\right) + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

13|

$$\int d\Omega \ldots = \boxed{\frac{4\pi\alpha^2}{3E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{m_\mu^2}{2E^2}\right) = \sigma_{total}}$$

1.4) Discussion.

- $2 m_\mu \lesssim E = E_{cm}$
- Prediction of QED:

  non-trivial energy dependence of $|\bar{M}|^2$

  $\rightarrow$ Experimentally verified
- Measure σ_{total} as function of E

  $\Longrightarrow$ muon mass m_μ

5.2. Summary of QED calculations

1. Draw diagrams
2. Use Feynman rules to compute M
3. $|\bar{M}|^2 = \sum_{spin} |M|^2$
4. Evaluate trace
5. Fix a frame (eg center of mass)

   $P = P(E, \theta, \varphi)$
6. Plug $|\bar{M}|^2$ in (4.120)

   integrate over momenta that are not measured.

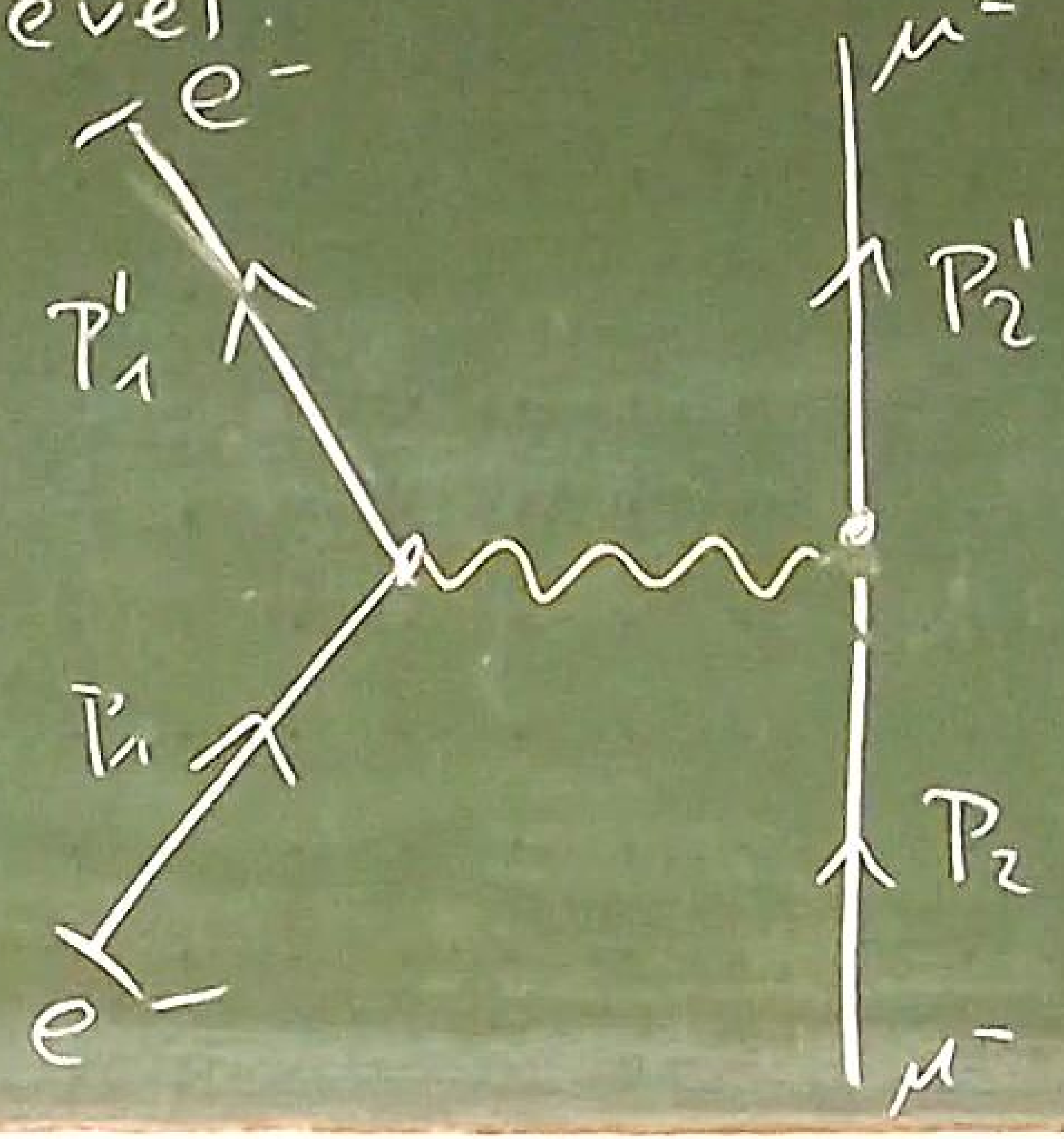
6. Radiative Corrections of QED

6.1. Overview

1) Process

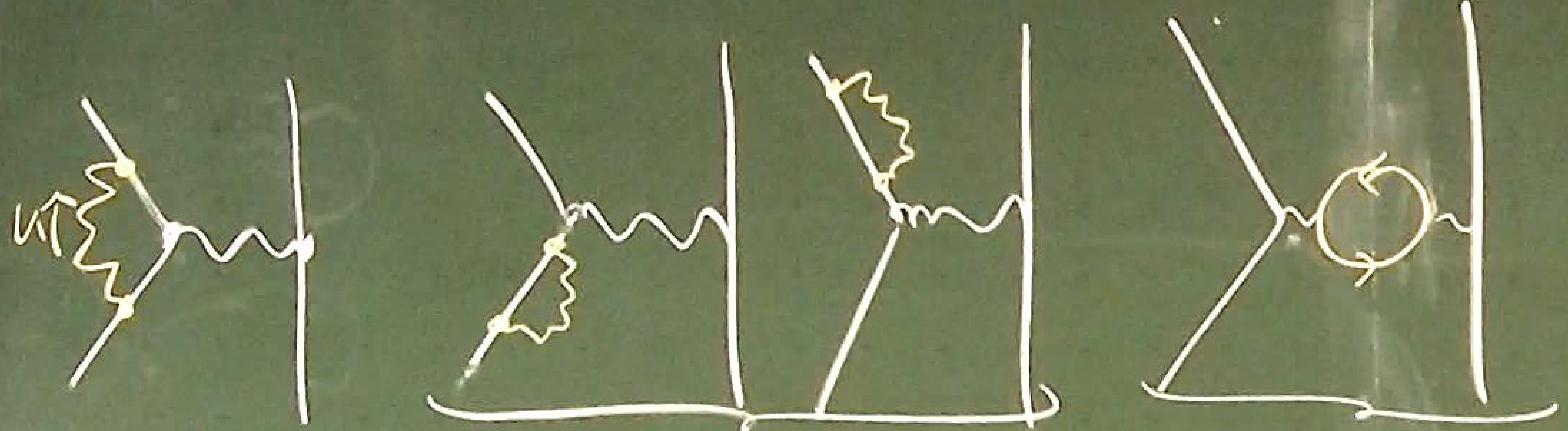
$$\lim_{m_\mu \to \infty} \{ e^- + \mu^- \longrightarrow e^- + \mu^- \}$$

2) Tree level.



3| Radiative corrections:

- loops:

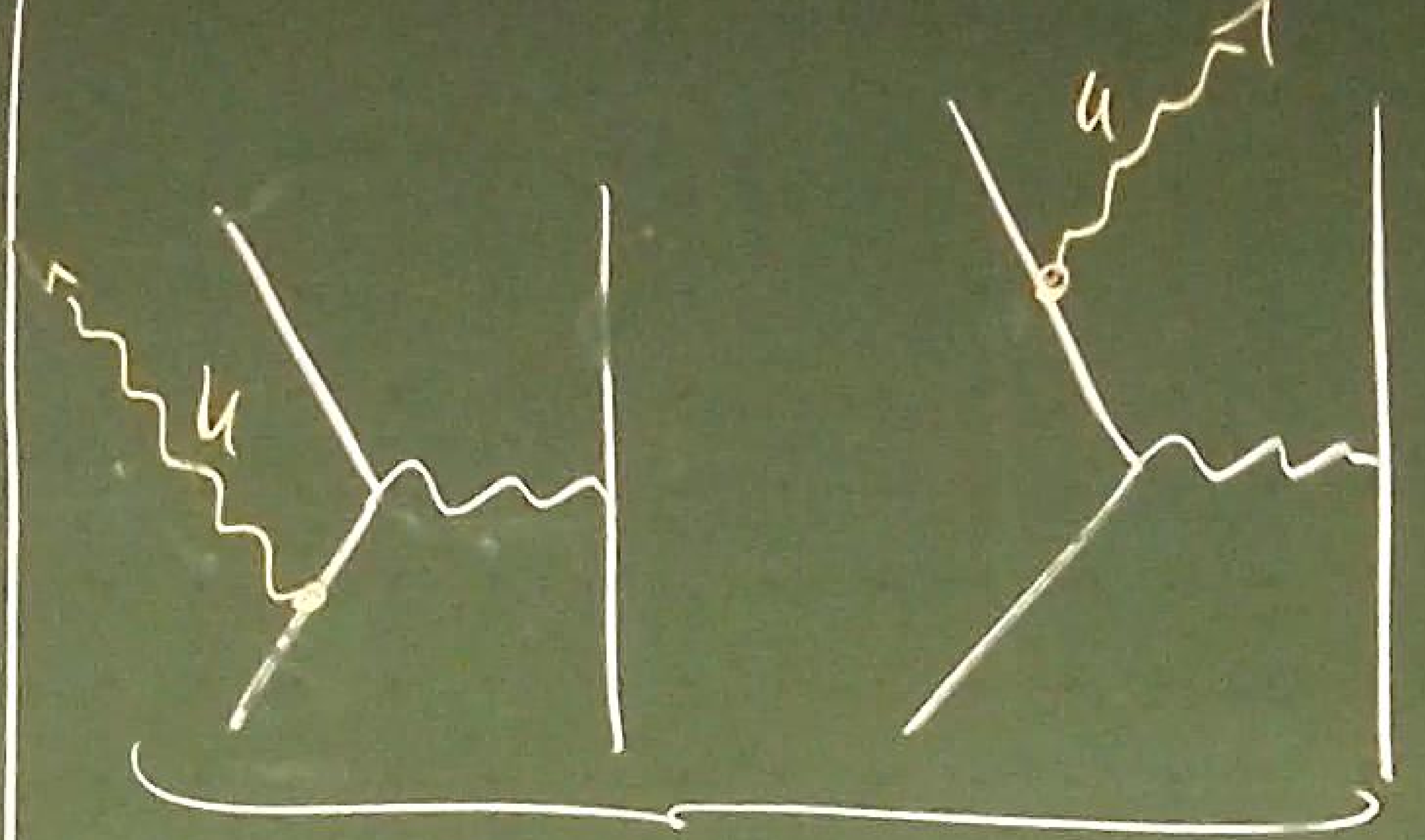


Vertex correction
- UV divergence
- IR divergence

External leg corrections
- UV div
- IR div

Vacuum polarization
- UV divergence

- external final state photons:



Bremsstrahlung
- IR-divergence

4| Spoiler!

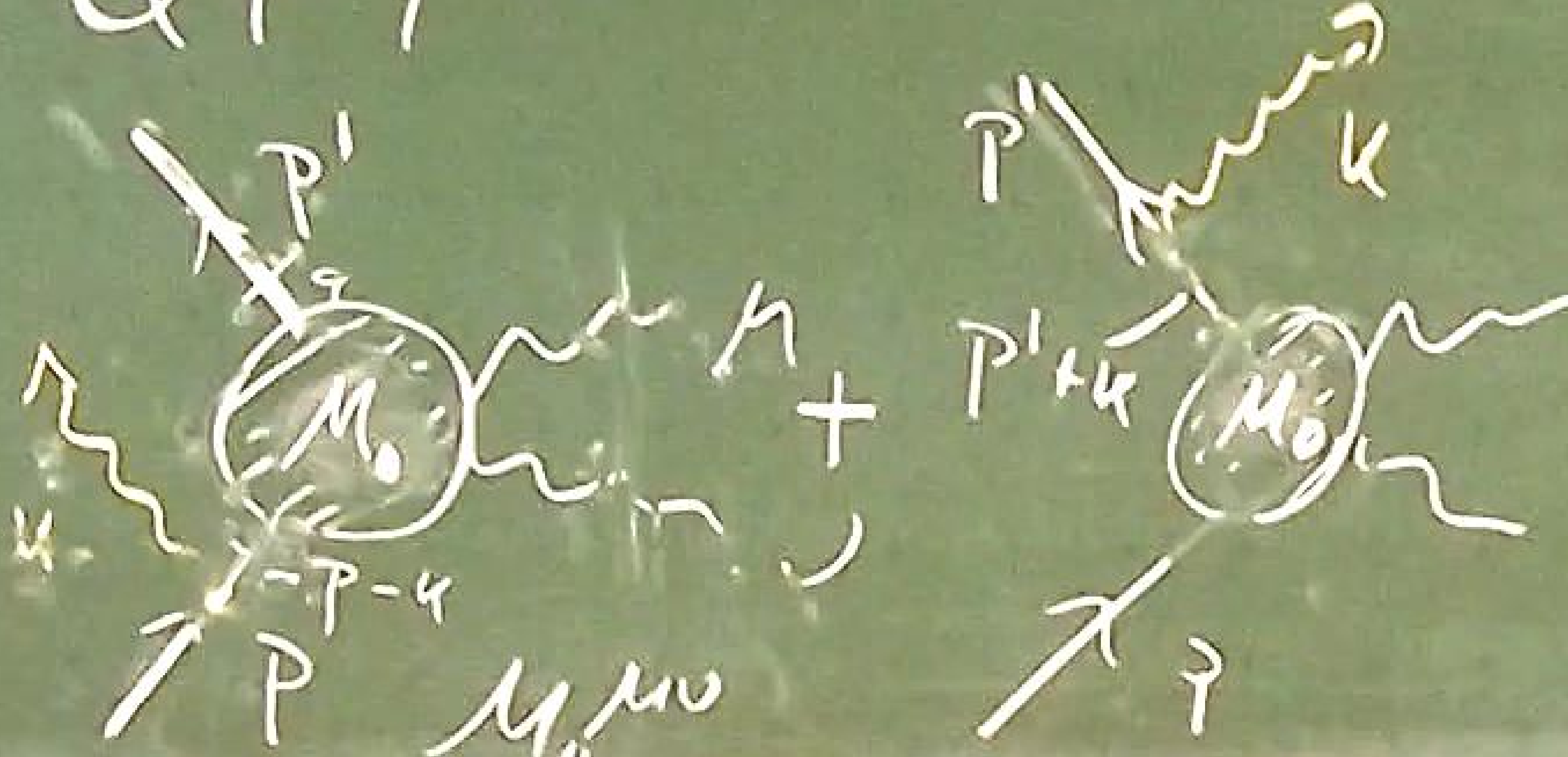
- UV div → cancel in observable quantities
- IR div → cancel with divergences of Bremsstrahlung diagrams

6.2 Soft Bremsstrahlung

1| Soft = Low-energy photons ($k \approx 0$)

2| Can be classically derived from Maxwell equations

3| & QFT



$$\mathcal{M} = \bar u(p') \mathcal{M}_0(p', p-k) \frac{i(\not p - \not k + m)}{(p-k)^2 - m^2 + i\varepsilon} (-ie\gamma^\mu) u(p)\, \varepsilon_\mu^*(k)$$

$$+ \bar u(p')(-ie\gamma^\mu) \frac{i(\not p' + \not k + m)}{(p'+k)^2 - m^2 + i\varepsilon} \mathcal{M}(p'+k, p) u(p)\, \varepsilon_\mu^*(k)$$

4| Simplifications:

$p^2 = m^2$, $k^2 = 0$

- $(p' \pm k)^2 - m^2 = \pm 2p'k$
- Soft photons: $|\vec k| \ll |\vec p - \vec p'|$

$\rightarrow \mathcal{M}_0(p', p-k) \approx \mathcal{M}(p', p) \approx \mathcal{M}_0(p'+k, p)$

$\rightarrow \not p - \not k \approx \not p$

- Dirac algebra:

$$(\not p + m)\gamma^\mu \varepsilon_\mu^* u(p) \doteq 2p^\mu \varepsilon_\mu^* u(p)$$

$$\bar u(p')\gamma^\mu \varepsilon_\mu^* (\not p' + m) \doteq \bar u(p') 2p'^\mu \varepsilon_\mu^*$$

$\overset{\text{use}}{(\not p - m)} u(p) = 0$

$\sum v \underbrace{\bar v u}_{=0}$

5|

$$i\mathcal{M} = \underbrace{\bar u(p') \mathcal{M}_0(p', p) u(p)}_{\text{elastic scattering}} \cdot \underbrace{\left[e\left(\frac{p'\varepsilon^*}{p'k} - \frac{p\varepsilon^*}{pk} \right) \right]}_{\text{bremsstrahlung}}$$

6) Scattering cross section:

$$d\sigma(p \to p' + \gamma) = d\sigma(p \to p') \cdot \underbrace{\int \frac{d^3k}{(2\pi)^3} \sum_r \frac{e^2}{2|\vec{k}|} \left| \frac{p'\varepsilon^r}{p'k} - \frac{p\varepsilon^r}{pk} \right|^2}_{dP_k(p \to p')}$$

7) Evaluate:

$$\int dP_k = \frac{\alpha}{\pi} \int_0^\infty dk \frac{1}{k} \underbrace{\boxed{\left(\frac{d\Omega_k}{4\pi} \sum_r \left| \frac{p'\varepsilon^r}{p'\tilde{k}} - \frac{p\varepsilon^r}{p\tilde{k}} \right|^2 \quad \tilde{k} = \frac{k}{|\vec{k}|} = \begin{pmatrix} 1 \\ \hat{k} \end{pmatrix}\right.}}_{\mathcal{I}(p,p')}$$

$$= \frac{\alpha}{\pi} \mathcal{I}(p,p') \left[\log(\infty) - \log(0) \right]$$

8) Approximations:

i) Problem 1: $|\vec{k}| \ll |\underbrace{\vec{p} - \vec{p}'}_{q}|$

$|\vec{k}| \propto |\vec{q}|$

$\to$ Introduce upper cutoff $|\vec{q}|$

ii) Problem 2: IR-divergence

"Solution": Regularisation with finite photon mass $\mu > 0$

$$\frac{1}{k} = \frac{1}{E_k} \longmapsto \frac{1}{\sqrt{\mu^2 + \vec{k}^2}}$$

$$\to \int_0^{|\vec{q}|} dk \frac{1}{\sqrt{\mu^2 + k^2}} = \log\left(\frac{\sqrt{\mu^2 + |\vec{q}|^2} + |\vec{q}|}{\mu} \right) \overset{\mu \to 0}{\sim} \log\left(2 \frac{|\vec{q}|}{\mu}\right) \sim \log\left(\frac{|\vec{q}|}{\mu}\right) = \frac{1}{2} \log\left(\frac{|\vec{q}|^2}{\mu^2}\right)$$

2) General form:

$$\Gamma^\mu(p',p) = f(p^\mu, p'^\mu, \gamma^\mu, m, e, 1)$$

3) Restrictions:

1) Lorentz covariance: Γ^μ transforms like γ^μ

$$\boxed{\Gamma^\mu = A \cdot \gamma^\mu + \tilde{B} p^\mu + \tilde{C} p'^\mu}$$

iii] Relativistic limit

$E_p, E_{p'} \gg m$

$$\mathcal{I}(p,p') \overset{*}{=} 2\log\left(\frac{-q^2}{m^2}\right)$$

$$-q^2 = -\underbrace{(p'-p)^2}_{<0} \geq 0$$

$\sim |\vec{q}|^2$

9] Result

$$\boxed{d\sigma(p\to p'+\gamma) \simeq d\sigma(p\to p')\,\frac{\alpha}{\pi}\,\boxed{\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{\mu^2}\right)}}$$

$\boxed{\begin{array}{l}\mu\to 0\\ E_{p,p'} \gg m\end{array}}$

Sudakov double logarithm

10] Two problems.

- Depends on physical photon mass. ↯
- Logarithmic divergence for $-q^2\to\infty$

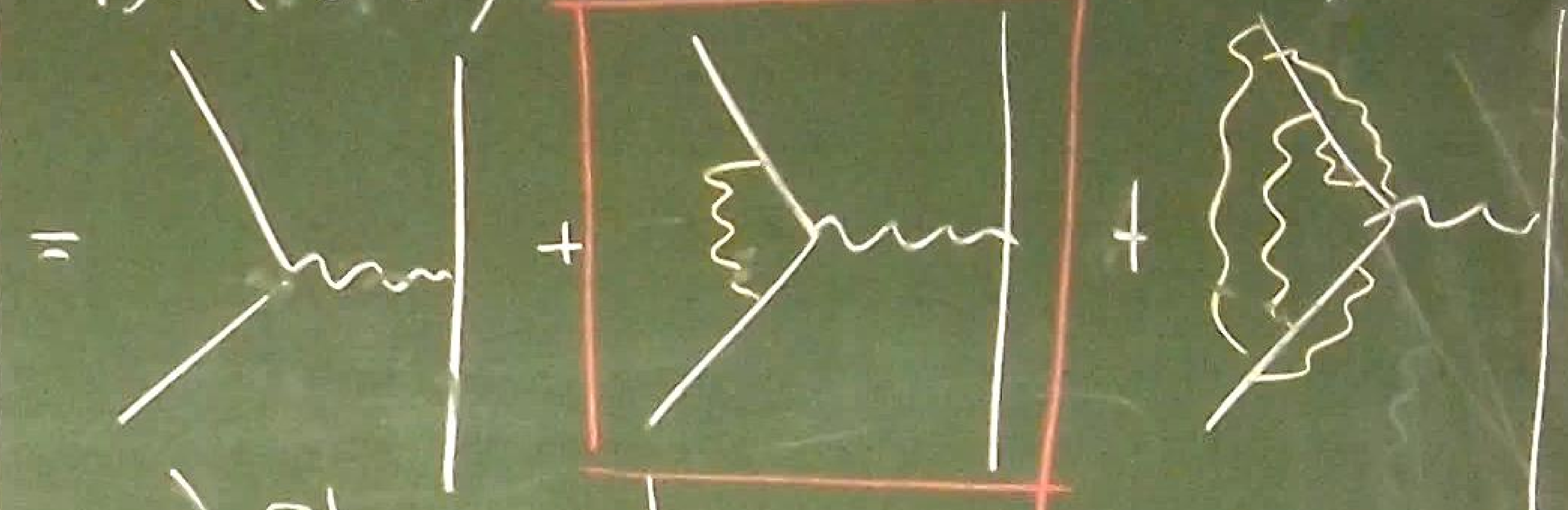
↯ Probability.

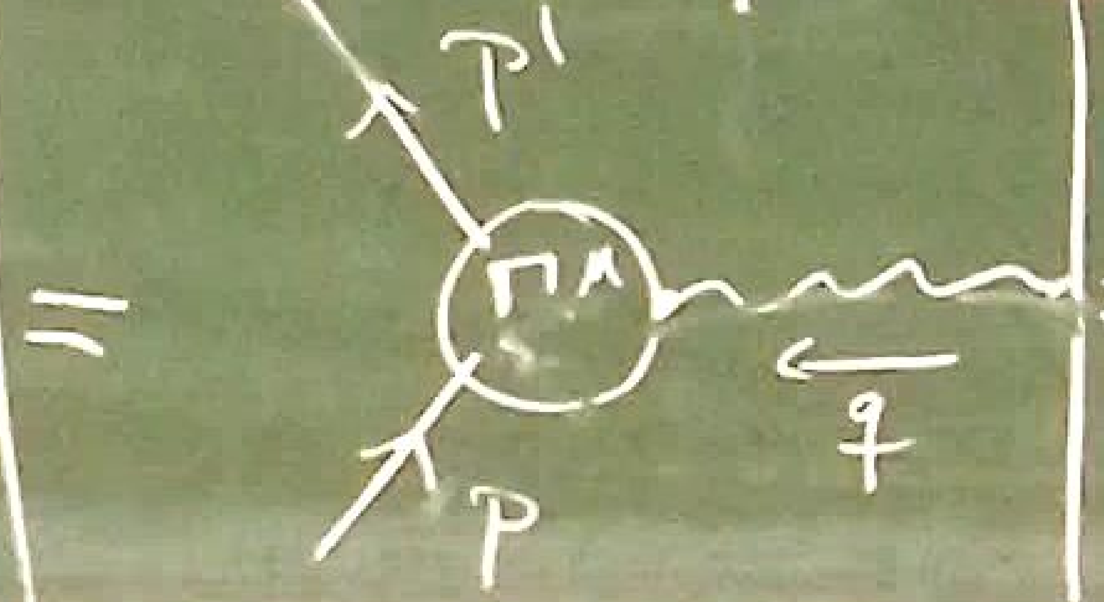
6.3 The Electron Vertex Function

6.3.1 Formal Structure

1] Scattering amplitude:

$$i\mathcal{M}(e^-(p)\mu^-(k) \to e^-(p')\mu^-(k'))$$





$$= ie^2\left(\bar{u}_e(p')\Gamma^\mu(p',p)u_e(p)\right)\frac{1}{q^2}\left(\bar{u}_m(k')\gamma_\mu u_m(k)\right)$$